Differential Equations & Applications
Volume 8, Number 3 (2016), 385–391 doi:10.7153/dea-08-21

EXISTENCE OF SOLUTIONS FOR AN ELLIPTIC BOUNDARY VALUE PROBLEM VIA A GLOBAL MINIMIZATION THEOREM ON HILBERT SPACES

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(*Communicated by Qihu Zhang*)

Abstract. We present a new global minimization theorem on Hilbert spaces which is different from the one in Hofer [7] using the notion of a nonexpansive potential operator. An example is given to illustrate our result.

1. Introduction

In [12] the link between a fixed point of a potential operator and a global minimum of some energy functional was established. Usually a minimization theory is based on the Palais-Smale condition (see, [6, 7]). In this paper we present a minimization theorem using nonexpansive potential operator theory in Hilbert spaces and our proof is based on a fixed point approach.

Let $(H,(.,.))$ be a real Hilbert space. An operator $T : H \longrightarrow H$ is called a potential operator (or gradient operator) on H, if there exists a Gâteaux differentiable functional $\varphi : H \longrightarrow \mathbb{R}$ such that $Grad\varphi(x) = T(x)$, for all $x \in H$ i.e.

$$\lim_{t \to 0} \frac{\varphi(x+th) - \varphi(x)}{t} = (T(x),h) \quad \forall x,h \in H.$$

Let $(.,.)$ denote the scalar product on H and $\|.\| = \sqrt{(.,.)}$ the norm. Consider the functional,

$$\varphi(x) = \frac{1}{2}\|x\|^2 - \int_0^1 (T(sx),x)\,ds$$

for all $x \in H$.

PROPOSITION 1. [12]. The fixed points of T agree with the global minima of the functional φ.

In our main result, we need the following fixed point theorem.

Mathematics subject classification (2010): 74G65, 46C05, 47H09, 47G40.
Keywords and phrases: minimization theorem, Hilbert spaces, nonexpansive potential operator, BVP.

THEOREM 1. [2] *(Browder Theorem) Let H be a Hilbert space and C a nonempty closed convex bounded subset of H. Then every nonexpansive mapping $F : C \longrightarrow C$ has a fixed point in C.*

THEOREM 2. [11] *(Leray-Schauder type theorem) Let U be an open bounded subset of a Hilbert space H, $0 \in U$ and $F : \overline{U} \longrightarrow H$ a nonexpansive map. Assume*

$$\lambda F(u) \neq u \text{ for all } u \in \partial U \text{ and } \lambda \in [0,1].$$

Then F has at least one fixed point in U.

Now, we recall some concepts from critical point theory.

DEFINITION 1. [8]. Let $\varphi \in C^1(H,\mathbb{R})$. If any sequence $(u_n) \subset H$ for which $(\varphi(u_n))$ is bounded in $\mathbb{R}$ and $\varphi'(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$ in H' possesses a convergent subsequence, then we say that φ satisfies the Palais-Smale condition ((PS) condition for short).

PROPOSITION 2. [6, 7]. Let H be a real Hilbert space and let $\varphi \in \mathscr{C}^1(H,\mathbb{R})$ satisfy the Palais-Smale condition. Let C be a closed convex subset of H. Suppose that $T = I - \varphi'$ maps C into C and that φ is bounded below in C. Then, there is a $u_0 \in C$ such that $\varphi'(u_0) = 0$, and $\inf_C \varphi = \varphi(u_0)$.

In [6] the authors considered the problem

$$\begin{cases} -\sum_{i,j=1}^{N} D_j(a_{ij}(x)D_i(u)) + c(x)u = f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded subset of $\mathbb{R}^N$, $a_{ij} \in L^\infty(\Omega)$, $c \in L^{\frac{N}{2}}$, $c(x) \geqslant 0$, and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that

$$|f(x,s)| \leqslant c|s|^\sigma + d(x), \tag{1.2}$$

where $\sigma = \frac{N+2}{N-2}$ and $d \in L^{\frac{2N}{N+2}}$ if $N \geqslant 3$, or $d \in L^p$ and $1 < p, \sigma < \infty$ if $N = 2$. The associated Dirichlet problem is

$$\begin{cases} a[u,v] = \int_\Omega f(x,u)v, & \forall v \in H_0^1(\Omega) \\ u \in H_0^1(\Omega) \end{cases} \tag{1.3}$$

with $a[u,v] = \int_\Omega \sum_{i,j=1}^{N} a_{ij}(x) D_i u D_j v + c(x)uv$.

By a subsolution and a supersolution of (1.3), we mean respectively $w, W \in H_0^1(\Omega)$ satisfying $a[w,v] \leqslant \int_\Omega f(x,w)v$, and $a[W,v] \geqslant \int_\Omega f(x,W)v$, $\forall v \in H_0^1(\Omega), v \geqslant 0$.

Let $\varphi : H_0^1(\Omega) \longrightarrow \mathbb{R}$ be defined by

$$\varphi(u) = \frac{1}{2}a[u,u] - \int_\Omega F(x,u)dx \tag{1.4}$$

with $F(x,s) = \int_0^s f(x,\xi)d\xi$.

THEOREM 3. *(Theorem 6 in [6]). Assume conditions on f that guarantee that φ defined in (1.4) satisfies the Palais-Smale condition. Suppose that there exist a subsolution $w \in H_0^1$ and a supersolution $W \in H_0^1$ of (1.3) such that $w \leqslant W$. Assume also for each fixed $x \in \Omega$, $f(x,s)$ is a nondecreasing function of s for $w(x) \leqslant s \leqslant W(x)$. Then there exists a $u_0 \in H_0^1(\Omega)$ such that*

$$u_0 \in [w,\, W],\ \ \varphi(u_0) = \inf_{[w,\, W]} \varphi \text{ and } \varphi'(u_0) = 0,$$

where $[w,\, W]$ is the segment defined by $[w,\, W] = \{tw + (1-t)W,\, t \in [0,1]\}$. Consequently u_0 is a solution of (1.3).

In this paper we remove the Palais-Smale condition in Proposition 2 and Theorem 3 and replace it with easy verifiable conditions on the functional φ.

2. Main Result

Let H a real Hilbert space and $(.,.)$ the scalar product.

THEOREM 4. *Let $\varphi : H \longrightarrow \mathbb{R}$ be a functional such that:*

1. *φ is twice Gateaux differentiable on H.*
2. *$\|(I' - \varphi'')(u)\| \leqslant 1, \quad \forall u \in H$.*
3. *$(I - \varphi')(C) \subset C \quad$ for some convex nonempty closed and bounded subset C of H.*

Then, φ has a global minimum on H. Indeed there exists a $u_0 \in C$ such that

$$\varphi(u_0) = \inf_H \varphi.$$

In particular, $\varphi'(u_0) = 0$.

Proof. Let $T : H \longrightarrow H$ with $\varphi' = I - T$ (i.e. $T = I - \varphi'$). Note that T is a potential operator. To show T is nonexpansive, note from the mean value theorem (see [13] pp. 122) that for all $u, v \in H$ there exists $\tau_0 \in [0\ \ 1]$ such that:

$$\begin{aligned} \|Tu - Tv\| &\leqslant \|DT(\tau_0 u + (1-\tau_0)v).(u-v)\| \\ &\leqslant \|DT(\tau_0 u + (1-\tau_0)v)\|\|u-v\| \\ &= \|(I' - \varphi'')(\tau_0 u + (1-\tau_0)v)\|\|u-v\|. \end{aligned}$$

Here, DT is the differential of the operator T. Using assumption (2) we infer that

$$\|Tu - Tv\| \leqslant \|u - v\|, \quad \forall u, v \in H.$$

From Theorem 1, the operator T has a fixed point in C and, by Proposition 1, such a fixed point is a global minimizer of φ on H.

Now, one can prove an analogue of the above theorem.

THEOREM 5. *Let $\varphi : H \longrightarrow \mathbb{R}$ be a functional as in Theorem 4 and assume conditions 1 and 2 in the statement of Theorem 4 hold. Also suppose there is an open bounded subset U of H with $0 \in U$ and*

$$\varphi'(u) \neq \frac{\lambda-1}{\lambda} u \ ,\forall u \in \partial U,\ \forall \lambda \in [0,1].$$

Then, φ has a global minimum on U. Indeed there exists a $u_0 \in U$ such that

$$\varphi(u_0) = \inf_H \varphi.$$

In particular, $\varphi'(u_0) = 0$.

Proof. The result follows from Theorem 2, since the above condition guarantees that $\lambda(I-\varphi')(u) \neq u$ for all $u \in \partial U$ and $\lambda \in [0,1]$.

3. Application

Consider the problem

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$. Here f and $f' : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions where f' is the derivative of f with respect to its second variable. Also assume

$$|f(x,s)| \leqslant c_1|s|^{\sigma_1} + d_1, \quad \text{and} \quad |f'(x,s)| \leqslant c_2|s|^{\sigma_2} + d_2 \tag{3.2}$$

for some positive constants c_1, c_2, d_1, d_2 and $0 \leqslant \sigma_1, \sigma_2 < \frac{N+2}{N-2}$ if $N \geqslant 3$ $(0 \leqslant \sigma_1, \sigma_2 < \infty$ if $N = 1,2)$
A weak solution of (3.1) is a solution of the problem,

$$\begin{cases} \int_\Omega \nabla u . \nabla v dx - \int_\Omega f(x,u).v dx = 0, & \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \tag{3.3}$$

Let $w, W \in H_0^1(\Omega)$ be respectively a subsolution and a supersolution of (3.3) and let λ_1 be the first eigenvalue of the linear Dirichlet problem

$$\begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

THEOREM 6. *Assume that*

$$|f(x,s)| \leqslant c_1|s|^{\sigma_1} + d_1, \quad \textit{and} \quad |f'(x,s)| \leqslant c_2|s|^{\sigma_2} + d_2$$

for some positive constants c_1, c_2, d_1, d_2 and $0 \leqslant \sigma_1, \sigma_2 < \frac{N+2}{N-2}$ if $N \geqslant 3$ $(0 \leqslant \sigma_1, \sigma_2 < \infty$ if $N = 1,2)$ and that

1. *for each fixed* $x \in \Omega$, $f(x,y)$ *is a nondecreasing function of* y *for* $w(x) \leqslant y \leqslant W(x)$,
2. $|f'(x,s)| \leqslant \lambda_1$, $\forall x \in \Omega$, $\forall s \in \mathbb{R}$.

Then, there exists a $u_0 \in H_0^1(\Omega)$ *which is a weak solution of problem (3.1) and* $u_0 \in [w, W]$.

Proof. Consider the problem,

$$\begin{cases} \int_\Omega \nabla u.\nabla v dx - \int_\Omega f(x,u).v dx = 0, & \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases}$$

Let $\varphi : H_0^1(\Omega) \longrightarrow \mathbb{R}$ be such that

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)dx \text{ with } F(x,u) = \int_0^u f(x,\xi)d\xi.$$

From assumption (3.2) (see [3], [5]), φ is twice differentiable and the derivatives are given by:

$$\varphi'(u).v = \int_\Omega \nabla u.\nabla v - \int_\Omega f(x,u).v dx, \quad \forall v \in H_0^1(\Omega), \tag{3.4}$$

$$(\varphi''(u).v).\omega = \int_\Omega \nabla v.\nabla \omega - \int_\Omega f'(x,u).v.\omega dx, \quad \forall\, v,\omega \in H_0^1(\Omega). \tag{3.5}$$

To prove that problem (3.3) has a solution we show that φ satisfies the assumptions in Theorem 4.

To show

$$\|(I' - \varphi'')(u)\| \leqslant 1, \forall u \in H_0^1(\Omega),$$

we use the Cauchy-Schwarz and the Poincaré inequalities, and we have

$$\begin{aligned} \|(I' - \varphi'')(u)\| &= \sup_{\|v\|\leqslant 1, \|\omega\|\leqslant 1} |(I' - \varphi'')(u).v.\omega|, \\ &= \sup_{\|v\|\leqslant 1, \|\omega\|\leqslant 1} \Big| \int_\Omega f'(x,u)v(x)\omega(x)dx \Big| \\ &\leqslant \sup_{\|v\|\leqslant 1, \|\omega\|\leqslant 1} \int_\Omega |f'(x,u)||v(x)||\omega(x)|dx \\ &\leqslant \lambda_1 \sup_{\|v\|\leqslant 1, \|\omega\|\leqslant 1} \int_\Omega |v(x)||\omega(x)|dx \\ &\leqslant \lambda_1 \sup_{\|v\|\leqslant 1, \|\omega\|\leqslant 1} \left(\int_\Omega |v(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\omega(x)|^2 dx\right)^{\frac{1}{2}} \\ &\leqslant \lambda_1 \sup_{\|v\|\leqslant 1, \|\omega\|\leqslant 1} \|v\|_{L^2}.\|\omega\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\leqslant \lambda_1 \sup_{\|v\|\leqslant 1, \|\omega\|\leqslant 1} \left(\frac{1}{\sqrt{\lambda_1}}\|v\|\right)\cdot\left(\frac{1}{\sqrt{\lambda_1}}\|\omega\|\right)\\
&= \sup_{\|v\|\leqslant 1, \|\omega\|\leqslant 1} \|v\|.\|\omega\|\\
&\leqslant 1.
\end{aligned}$$

Let $C=[w,\,W]=\left\{u\in H_0^1(\Omega): w(x)\leqslant u(x)\leqslant W(x),\ \forall x\in\Omega\right\}$. Note C is a closed convex subset of $H_0^1(\Omega)$ and it is bounded since if $u\in C=[w,\,W]$, then there exists $t\in[0,1]$ such that $u=tw+(1-t)W$, and so,

$$\|u\|=\|tw+(1-t)W\|\leqslant\|w\|+\|W\|.$$

The argument in $[1,\ p.\,712]$ shows that $(I-\varphi')(C)\subset C$.

The existence of a weak solution of (3.1) follows from Theorem 4.

EXAMPLE 1. Consider the boundary value problem

$$\begin{cases} -u''(x)=\frac{\pi^2}{4}\left(\ln(\exp(u(x))+10)-\dfrac{1}{2}\right), & x\in(-1,1),\\ u(-1)=u(+1)=0. \end{cases} \tag{3.6}$$

It is easy to see that w and W which are defined by $w(x)=0$ and $W(x)=\frac{\pi^2}{4}\left(|x|-|x|^2\right)$ are respectively subsolution and supersolution of (3.6). Also note all the assumptions in the statement of Theorem 6 are satisfied.

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(Received January 7, 2016)

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