Differential Equations & Applications
Volume 8, Number 3 (2016), 393–410 doi:10.7153/dea-08-22

EXAMPLES OF SHOCKS IN POPULATION DYNAMICS

ELINA DÉVOUÉ AND VICTOR DÉVOUÉ

(*Communicated by Pavel I. Naumkin*)

Abstract. The aim of this paper is to study some shocks in population dynamics by means of the recent theories of generalized functions in order to have a better understanding of the long-term behavior of these phenomena. The shocks analyzed can be wars, genocides, epidemics, natural disasters, cancers... Population dynamics is represented by the transport equation and shocks by initial data which are distributions. This justifies to search for solutions in Colombeau algebra. Moreover we build well-posed problems. We study two models, the genesis model and a top hat condition.

1. Introduction

The purpose of this paper is to study shocks (which can be wars, genocides, natural disasters, epidemics, significant immigrations, cancers...) in population dynamics, by means of the recent theories of algebras of generalized functions in order to have a better understanding of the long-term behavior of these phenomena. The meeting of mathematics and economy allows us to understand facts, sometimes to anticipate them, or even to change them. The facts show the persistent effect of the shocks in population dynamics whose amplitude diminishes with time for wars [8], [6], genocides [25], [29], natural disasters [3], [4], [28], cancers [17] and in various cases. The impact of shocks, in terms of deaths, is heavier in low income countries than in high income countries. The natural disasters have considerable repercussions on the vulnerable populations and have long-term consequences for human capital in the poorest countries. Rich countries rank ahead poor countries in terms of economic losses owed to natural disasters because of a higher value of guaranteed goods. Nevertheless, when numbers are presented in proportion to Gross Domestic Product, the developing countries rank at the top in terms of economic impact [3].

Transport equation is used as model for some phenomena in economy: health, finance, taxation, insurance, economic growth, economy of transport, demography. The solution of the transport equation is well known in various classical cases with initial data in the space of C^k functions. However, some situations cannot be interpreted by equations with regular data. To study them we must use data which are distributions or other generalized functions [13, 14]. The Lotka-McKendrick equation is the common model describing the dynamics of age structured population. It is formulated by the

Mathematics subject classification (2010): 35D99, 35L05, 35L70, 46F30.
Keywords and phrases: Colombeau algebra, generalized functions, transport equation, nonlinear partial differential equations, population dynamics, shocks.

Paper DEA-08-22

transport equation and shock by initial data which are distributions. Since we investigate solutions with distributions [27], [18], or other generalized functions as initial data, that necessitates a non-linear theory of distributions like that of Colombeau. Thus we must search for solutions in the Colombeau algebra, [9, 10]. This algebra gives an efficient algebraic framework which allows for a precise study of solutions. Introducing it in economy will permit to build well-posed problems and to have a fine interpretation of the results. We try to show that the theory of nonlinear generalized functions, which is a natural continuation of the classical theory of smooth functions and distributions, gives a better and more rigorous approach of some economic and demographic problems.

Moreover we must use the results of H. A. Biagioni [5] because the functions are defined on Ω, a subset of $\mathbb{R}^2$ which is not an open set, and we must use a specific family of mollifiers to stay in Ω.

The paper is structured as follows. Section 2 describes the model for regular data. In Section 3, we briefly recall Colombeau's construction obtained as a $(\mathscr{C},\mathscr{E},\mathscr{P})$-algebra of J.-A. Marti [20, 21, 22] and we study two models, the genesis model and a top hat initial condition. In both models we treat the case where birth distribution is a Dirac distribution.

2. The model for regular data

We use the classical age-structured population model usually formulated by the Lotka-McKendrick equation [19], [23], [16], which is a hyperbolic Partial Differential Equation where $u(t,x)$ is the age-density at position x at time t of the population

$$(P)\begin{cases} \dfrac{\partial u}{\partial t}(t,x)+\dfrac{\partial u}{\partial x}(t,x)=-\mu(x)u(t,x), \\ u\,|_{\{t=0\}}=\phi, \end{cases}$$

ϕ is the initial data, supplemented by a nonlocal boundary condition, the birth law,

$$u(t,0)=\int_0^{x_m}\beta(\zeta)u(t,\zeta)d\zeta$$

which gives the total number of newborn (individuals with age $x=0$) at time t. The maximal age is x_m, $(t,x)\in\mathbb{R}_+\times[0,x_m)$ and for $x>x_m$ we have $u(t,x)=0$.

The age-specific fertility $\beta(x)$ is the number of newborn, in one time unit, coming from a single individual whose age is in the infinitesimal age interval $[x;x+dx]$. Thus the number of newborn in one time unit, coming from individuals with age in $[a_1;a_2]$ is

$$\int_{a_1}^{a_2}\beta(\zeta)u(t,\zeta)d\zeta.$$

The age specific mortality $\mu(x)$ is the death rate of people having age in $[x;x+dx]$.

The functions μ, β, ϕ are, of course, non negative,

$$\mu\in\mathrm{C}^0(\mathbb{R}_+),\beta\in\mathrm{C}^1(\mathbb{R}_+),\phi\in\mathrm{L}^1_{loc}(\mathbb{R}_+),\int_0^{\infty}\mu(\zeta)d\zeta=+\infty.$$

The total population at time t is given by

$$N(t)=\int_0^{+\infty}u(t,\zeta)d\zeta.$$

In general

$$\phi(0)\neq\int_0^{x_m}\beta(\zeta)u(0,\zeta)d\zeta.$$

Under these conditions, as t increases, the boundary condition introduces discontinuities in the solutions.

Integrating the equation along characteristics, one obtains

$$u(t,x)=u(t_0,x_0)\exp(-\int_0^t\mu(s+x_0-t_0)ds);x-x_0=t-t_0$$

where x_0 is the age variable at time $t=t_0$.

Then the solution of the McKendrick equation is

$$u(t,x)=\begin{cases} v(t-x)e^{-\int_0^x\mu(\zeta)d\zeta},t\geqslant x,\\ \phi(x-t)e^{-\int_{x-t}^x\mu(\zeta)d\zeta},t<x\end{cases}$$

where $v(t)=u(t,0)$ is the birth rate.

Substituting u into the birth law one obtains the renewal equation

$$v(t)=\int_0^t v(t-a)\beta(a)e^{-\int_0^a\mu(\zeta)d\zeta}da+\int_t^{+\infty}\phi(a-t)\beta(a)e^{-\int_{a-t}^a\mu(\zeta)d\zeta}da.$$

for the birth rate $v(t)$.

When μ is constant, we have

$$u(t,x)=\begin{cases} g(t,x)=v(t-x)e^{-\mu x},t\geqslant x,\\ w(t,x)=\phi(x-t)e^{-\mu t},t<x\end{cases}$$

and

$$v(t)=\psi(t)+\int_0^t v(t-a)\beta(a)e^{-\mu a}da$$

with

$$\psi(t)=\int_t^{+\infty}\phi(a-t)\beta(a)e^{-\mu t}da.$$

The solution space is divided into two regions: for $t<x$, the solution is determined by initial density of individuals of age $x-t$; for $t\geqslant x$, the solution is determined, via the renewal condition, by the birth rate at time $t-x$.

Set

$$D=\{(t,x):t\geqslant 0,x\geqslant 0\},D_1=\{(t,x):t\geqslant x\geqslant 0\},$$
$$D_2=\{(t,x):0\leqslant t<x\}.$$

We consider $(g,w)\in C^\infty(D_1)\times C^\infty(D_2)$ as solution of the problem.

Now consider a population which has only one fertile age class. This correspond to a birth function of the form $\beta(x)=B\delta(x-x_0)$ where B is the number of offspring [15], [7], [11].

THEOREM 1. *Assume that* $\phi \in L^1_{loc}(\mathbb{R}_+)$ *is a locally integrable initial condition for the McKendrick equation,* $\beta(x) = B\delta(x - x_0)$ *where* δ *is the Dirac delta function,* $x_0 > 0$ *and* $B > 0$*, then*

$$u(t,0) = Bu(t,x_0)$$

and if μ *is constant, the solution of the McKendrick equation is*

$$u(t,x) = \begin{cases} g(t,x) = B^m \phi(mx_0 + x - t)e^{-\mu t}, t \geqslant x, \\ w(t,x) = \phi(x-t)e^{-\mu t}, t < x \end{cases}$$

where $m = Ent((t-x)/x_0 + 1)$*,* $Ent(x) =$ *integer part of* x*. The solution of the renewal equation is*

$$v(t) = B^s \phi(sx_0 - t)e^{-\mu t}$$

where $s = Ent(t/x_0 + 1)$*.*

Proof. Take the sets

$$\Delta_m = \{(t,x) : x + (m-1)x_0 \leqslant t \leqslant x + mx_0; x \geqslant 0; t \geqslant 0\}$$

where $m \in \mathbb{N}^*$ and $\Delta_0 = \{(t,x) : 0 \leqslant t < x\}$. So, for $m \geqslant 1$, $(t,x) \in \Delta_m$ if and only if $t \geqslant x$ and $m = Ent((t-x)/x_0 + 1)$. Take $m = 1$, $(t,x) \in \overset{o}{\Delta}_1$, take the point (t',x') on the line $t = t'$, therefore $x' < t' < x' + x_0$. The characteristic line that passes by (t',x') and crosses the line $x = 0$ at $t'_1 = t' - x'$. According to the method of characteristic we have

$$u\left(t',x'\right) = u\left(t'-x',0\right)e^{-\mu x'}.$$

As $\beta(x) = B\delta(x - x_0)$ then the boundary condition becomes $u(t,0) = Bu(t,x_0)$, then

$$u\left(t',x'\right) = u\left(t'-x',x_0\right)e^{-\mu x'}.$$

As $t'_1 = t' - x' < x_0$ then $(t'-x',x_0) \in \Delta_0$, so

$$u\left(t'-x',x_0\right) = \phi\left(x_0 - t' + x'\right)e^{-\mu\left(t'-x'\right)}$$

and

$$\begin{aligned} u\left(t',x'\right) &= B\phi\left(x_0 - t' + x'\right)e^{-\mu\left(t'-x'\right)}e^{-\mu x'} \\ &= B\phi\left(x_0 + x' - t'\right)e^{-\mu t'}. \end{aligned}$$

Now we proceed by induction. Suppose that, for some $k \geqslant 1$, the solution can be written in the form

$$u(t,x) = B^m \phi(mx_0 + x - t)e^{-\mu t} \tag{A1}$$

where $m = Ent((t-x)/x_0 + 1)$, $(t,x) \in \Delta_m$, $1 \leqslant m \leqslant k$. It is true for $m = 1$. Suppose now that $(t',x') \in \Delta_{k+1}$. We have

$$u\left(t',x'\right) = u\left(t'-x',0\right)e^{-\mu x'} = u\left(t'-x',x_0\right)e^{-\mu x'}.$$

As $(t',x') \in \Delta_{k+1}$ we have $x' \leqslant t'$ and $k+1 = Ent((t'-x')/x_0+1)$ is equivalent to

$$k+1 < 1+(t'-x')/x_0 < k+2$$

then

$$x_0\,(k+1) < x_0 + (t'-x') < x_0\,(k+2)\,.$$

As

$$Ent((t'-x'-x_0)/x_0+1) = Ent((t'-x')/x_0) = k,$$

we have $(t'-x',x_0) \in \Delta_k$ then, according to ($A1$), we have

$$\begin{aligned} u\left(t',x'\right) &= B\left(B^k\phi\left(kx_0+x_0-(t'-x')\right)e^{-\mu\left(t'-x'\right)}\right)e^{-\mu x'} \\ &= B^{k+1}\phi\left((k+1)x_0+x'-t'\right)\right)e^{-\mu t'}. \end{aligned}$$

Thus ($A1$) is true for $m = k+1$. This gives the result.

3. Models for irregular data

It is convenient to describe shocks with help of discontinuous functions or distributions as data, thus we use the framework of generalized algebras to obtain a well-posed problem.

3.1. The Colombeau algebra

First we briefly recall Colombeau's construction obtained as a $(\mathscr{C},\mathscr{E},\mathscr{P})$-algebra of J.-A. Marti [20, 21, 22]. Set $\Lambda = (0;1]$. Set

$$A = \left\{(m_\varepsilon)_\varepsilon \in \mathbb{R}^\Lambda : \exists p \in \mathbb{R}_+^*, \exists C \in \mathbb{R}_+^*, \exists \mu \in (0;1], \forall \varepsilon \in (0;\mu], |m_\varepsilon| \leqslant C\varepsilon^{-p}\right\}$$

and

$$I_A = \left\{(m_\varepsilon)_\varepsilon \in \mathbb{R}^\Lambda : \forall q \in \mathbb{R}_+^*, \exists D \in \mathbb{R}_+^*, \exists \mu \in (0;1], \forall \varepsilon \in (0;\mu], |m_\varepsilon| \leqslant D\varepsilon^{q}\right\}$$

an ideal of A. Thanks to the results of H. A. Biagioni [5], J. Aragona [1], [2], A. Delcroix [12] and J.-A. Marti [20, 21, 22], we can define Colombeau spaces on Ω an subset of $\mathbb{R}^n$ such that

$$O \subset \Omega \subset \overline{O}, \tag{$H1$}$$

where O is an open subset of $\mathbb{R}^n$ and $\overline{O}$ the closure of O.

Define for $n \in \mathbb{N}$ and Ω an subset of $\mathbb{R}^n$ verifying ($H1$),

$$\begin{aligned} \mathscr{X}(\Omega) &= \{(u_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N}, \left(P_{K,l}(u_\varepsilon)\right)_\varepsilon \in |A|\}, \\ \mathscr{N}(\Omega) &= \{(u_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N}, \left(P_{K,l}(u_\varepsilon)\right)_\varepsilon \in |I_A|\} \end{aligned}$$

where

$$P_{K,l}(u_\varepsilon) = \sup_{|\alpha|\leqslant l} P_K(u_\varepsilon) \text{ with } P_K(u_\varepsilon) = \sup_{x\in K} |D^\alpha u_\varepsilon(x)|,\ K \Subset \Omega,$$

the notation $K \Subset \mathbb{R}^n$ means that K is a compact subset of $\mathbb{R}^n$ and

$$D^\alpha = \frac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial z_1^{\alpha_1}\ldots\partial z_n^{\alpha_n}} \text{ for } z=(z_1,\ldots,z_n)\in\Omega,\ \alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n.$$

The pointwise product, the addition and the generalized derivation $D^\alpha : u(=[u_\varepsilon]) \mapsto D^\alpha u = [D^\alpha u_\varepsilon]$ provide $\mathscr{G}(\Omega)$ with a differential algebraic structure.

The sheaf of factor algebras $\mathscr{G}(\cdot) = \mathscr{X}(\cdot)/\mathscr{N}(\cdot)$ is called the sheaf of simplified Colombeau algebras. $\mathscr{G}(\Omega)$ is the simplified Colombeau algebra of generalized functions.

We denote by $[u_\varepsilon]$ the class in $\mathscr{G}(\Omega)$ defined by the representative $(u_\varepsilon)_{\varepsilon\in\Lambda} \in \mathscr{X}(\Omega)$.

The set of generalized real numbers is defined as $\overline{\mathbb{R}} = A/I_A$.

Relationship with distribution theory.

Let Ω an subset of $\mathbb{R}^n$ verifying $(H1)$. If $(\varphi_\varepsilon)_{\varepsilon\in(0;1]}$ is a family of mollifiers $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, $x\in\mathbb{R}^n$, $\int\varphi(x)\,dx = 1$ and if $T\in\mathscr{D}'(\Omega)$, the convolution product family $(T*\varphi_\varepsilon)_\varepsilon$ is a family of smooth functions slowly increasing in $1/\varepsilon$. The space of distributions $\mathscr{D}'(\Omega)$ is embedded into $\mathscr{G}(\Omega)$ by $T\mapsto[T*\varphi_\varepsilon]$, [12].

We choose a special kind of mollifiers which moments of higher order vanish.

The association process. The association relation identifies elements of $\mathscr{G}(\Omega)$ if they coincide in the weak limit. That is, $u=[u_\varepsilon]$ and $v=[v_\varepsilon]\in\mathscr{G}(\Omega)$ are called associated if

$$\lim_{\varepsilon\to 0}\int (u_\varepsilon(x)-v_\varepsilon(x))\psi(x)dx = 0$$

for all test functions ψ. We write $u\sim v$. We can also define an association process between $u=[u_\varepsilon]$ and $T\in\mathscr{D}'(\Omega)$ by writing simply $u\sim T \iff \lim\limits_{\mathscr{D}'(\Omega),\Lambda} u_\varepsilon = T$, then u is said to admit T as associated distribution.

So, here we take the data in $\mathscr{G}([0;+\infty[)$ and we search for solution $(g,w)\in\mathscr{G}(D_1)\times\mathscr{G}(D_2)$.

3.2. The genesis model

In the genesis model we assume that $\phi = k\delta$ where δ the Dirac distribution and $k>0$. Thus all the initial population is at age zero. Let us assume also that the age-specific fertility β and the age specific mortality μ are both constantes. We have the problem formally written as

$$(P)\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial x} = -\mu u, \\ u\,|_{\{t=0\}} = k\delta. \end{cases}$$

For example, to model the growth of tumors, some assumptions are made about the initial distribution of cell ages. The simplest case is when all the cells start at age zero, so that $\phi = k\delta$ with k is the initial population of cells and δ the Dirac distribution [17].

3.2.1. Well formulated problem

We take some fixed smooth function φ,

$$\varphi(x) = 0 \text{ for } x \notin [0;1], \int \varphi(x)dx = 1, \sup_{x\in[0;1]} \varphi(x) = \varphi(0).$$

We consider the family of mollifiers $(\varphi_\varepsilon)_\varepsilon$ given by $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ then $\operatorname{supp}\varphi_\varepsilon = [0;\varepsilon]$. We approach problem (P) by a parametric family of (P_ε) ones

$$(P_\varepsilon)\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + \dfrac{\partial u_\varepsilon}{\partial x} = -\mu u_\varepsilon, \\ u_{\varepsilon,\eta}(0,x) = k\varphi_\varepsilon(x), \end{cases}$$

with $k > 0$, $\varphi_\varepsilon \in \mathrm{C}^\infty([0;+\infty[)$.

The generalized problem is well formulated as

$$(P_{gen})\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = -\mu u, \\ u\,|_{\{t=0\}} = [k\varphi_\varepsilon], \end{cases}$$

where $[k\varphi_\varepsilon] \in \mathscr{G}([0;+\infty[)$ and (g,w) is searched in $\mathscr{G}(D_1)\times\mathscr{G}(D_2)$.

3.2.2. Solution to (P_{gen})

THEOREM 2. *The solution to (P_ε) is $(g_\varepsilon, w_\varepsilon)$ with*

$$g_\varepsilon(t,x) = k\beta e^{t(\beta-\mu)}e^{-\beta x}, \quad \text{if} \quad t \geqslant x,$$
$$w_\varepsilon(t,x) = k\varphi_\varepsilon(x-t)e^{-\mu t} = \frac{k}{\varepsilon}\varphi\left(\frac{x-t}{\varepsilon}\right)e^{-\mu t}, \quad \text{if} \quad t < x.$$

Proof. We have

$$u_\varepsilon(t,x) = \begin{cases} g_\varepsilon(t,x) = v_\varepsilon(t-x)e^{-\mu x}, & \text{if} \quad t \geqslant x, \\ w_\varepsilon(t,x) = k\varphi_\varepsilon(x-t)e^{-\mu t} = \frac{k}{\varepsilon}\varphi\left(\frac{x-t}{\varepsilon}\right)e^{-\mu t}, & \text{if } t < x \end{cases}$$

where $v_\varepsilon(t) = u_\varepsilon(t,0)$. Thus the renewal equation takes the form

$$v_\varepsilon(t) = \psi_\varepsilon(t) + \int_0^t v_\varepsilon(t-a)\beta e^{-\mu a}da,$$

with

$$\psi_\varepsilon(t) = \int_t^{+\infty} k\varphi_\varepsilon(a-t)\beta e^{-\mu t}da = \int_0^{+\infty} k\varphi_\varepsilon(s)\beta e^{-\mu t}ds$$
$$= k\beta e^{-\mu t}\int_0^{+\infty}\varphi_\varepsilon(s)ds = k\beta e^{-\mu t}.$$

Thus we have

$$v_\varepsilon(t) = k\beta e^{-\mu t} + \beta\int_0^t v_\varepsilon(t-a)e^{-\mu a}da = k\beta e^{-\mu t} + \beta\int_0^t v_\varepsilon(s)e^{-\mu(t-s)}ds$$

$$= k\beta e^{-\mu t} + \beta e^{-\mu t} \int_0^t v_\varepsilon(s) e^{\mu s} ds.$$

Differentiation gives

$$\begin{aligned}
(v_\varepsilon)'(t) &= -\mu k\beta e^{-\mu t} + \beta e^{-\mu t}\left(v_\varepsilon(t)e^{\mu t}\right) - \mu\beta e^{-\mu t}\int_0^t v_\varepsilon(s)e^{\mu s}ds \\
&= -\mu\left(k\beta e^{-\mu t} + \beta e^{-\mu t}\int_0^t v_\varepsilon(s)e^{\mu s}ds\right) + \beta v_\varepsilon(t) \\
&= -\mu v_\varepsilon(t) + \beta v_\varepsilon(t) = (\beta-\mu)v_\varepsilon(t).
\end{aligned}$$

Then $(v_\varepsilon)'(t) = (\beta-\mu)v_\varepsilon(t)$. From the renewal equation with $t=0$ we have $v_\varepsilon(0) = k\beta$. Then $v_\varepsilon(t) = k\beta e^{(\beta-\mu)t}$ and $g_\varepsilon(t,x) = k\beta e^{(\beta-\mu)(t-x)}e^{-\mu x} = k\beta e^{t(\beta-\mu)}e^{-\beta x}$. This gives the result.

REMARK 1. For any ε, $g_\varepsilon(t,x) = k\beta e^{t(\beta-\mu)}e^{-\beta x}$ is independent of ε. Moreover $g_\varepsilon \in \mathrm{C}^\infty(D_1)$, then $(g_\varepsilon)_\varepsilon \in \mathscr{X}(D_1)$, we denote by $g = [g_\varepsilon]_{\mathscr{G}(D_1)}$ it class in $\mathscr{G}(D_1)$.

Recall

$$D_1 = \{(t,x) : t \geqslant x \geqslant 0\} \text{ and } D_2 = \{(t,x) : 0 \leqslant t < x\}.$$

Take $(w_\varepsilon)_\varepsilon \in \mathscr{X}(D_2)$, we denote by $[w_\varepsilon]_{\mathscr{G}(D_2)}$ it class in $\mathscr{G}(D_2)$.

THEOREM 3. *Taking $(g_\varepsilon, w_\varepsilon)$ the solution to (P_ε), thus the family $(g_\varepsilon)_\varepsilon$ lies in $\mathscr{X}(D_1)$, the family $(w_\varepsilon)_\varepsilon$ lies in $\mathscr{X}(D_2)$. Set $g = [w_\varepsilon]_{\mathscr{G}(D_1)}$, $w = [w_\varepsilon]_{\mathscr{G}(D_2)}$, then (g,w) is solution to (P_{gen}) in $\mathscr{G}(D_1)\times\mathscr{G}(D_2)$.*

Proof. Set $\operatorname{supp}\varphi = [0;1] = K$. Then $\forall L \Subset D_2$,

$$\begin{aligned}
(P_{L,0}(w_\varepsilon))_\varepsilon &= k\left(\sup_{(t,x)\in L}\left|\varphi_\varepsilon(x-t)e^{-\mu t}\right|\right)_\varepsilon \\
&\leqslant k\left(\sup_{\xi\in K}|\varphi_\varepsilon(\xi)|\right)_\varepsilon = k\left(P_{K,0}(\varphi_\varepsilon)\right)_\varepsilon \in |A|.
\end{aligned}$$

As the derivatives with respect to x have the same support, we obtain similar estimates. Set $n \in \mathbb{N}$,

$$\left(\sup_{(t,x)\in L}\left(\left|\frac{\partial^n}{\partial x^n}\varphi_\varepsilon(x-t)e^{-\mu t}\right|\right)\right)_\varepsilon \leqslant (P_{K,n}(\varphi_\varepsilon))_\varepsilon \in |A|.$$

Set $m \in \mathbb{N}$, compute the derivatives with respect to t,

$$\frac{\partial^m}{\partial t^m}\left(\varphi_\varepsilon(x-t)e^{-\mu t}\right) = \sum\nolimits_{l=0}^m (-1)^m \binom{m}{l}\mu^{m-l}e^{-\mu t}\frac{\partial^l\varphi_\varepsilon}{\partial t^l}(x-t),$$

then

$$\sup_{(t,x)\in L}\left(\left|\frac{\partial^m}{\partial t^m}\varphi_\varepsilon(x-t)e^{-\mu t}\right|\right)\leqslant \sum_{l=0}^m\binom{m}{l}\mu^{m-l}P_{K,m}(\varphi_\varepsilon),$$

so $\left(P_{K,(l,0)}(w_\varepsilon)\right)_\varepsilon \in |A|$. Compute the cross derivatives with $n\in\mathbb{N}$, $m\in\mathbb{N}$

$$\frac{\partial^{m+n}}{\partial t^m\partial x^n}\left(\varphi_\varepsilon(x-t)e^{-\mu t}\right)=\sum_{l=0}^m(-1)^m\binom{m}{l}\mu^{m-l}e^{-\mu t}\frac{\partial^{l+m}\varphi_\varepsilon}{\partial t^l\partial x^n}(x-t).$$

Then we have $(P_{K,n+m}(w_\varepsilon))_\varepsilon\in|A|$. It results from this that

$$\forall L\Subset\mathbb{R}^2,\forall l\in\mathbb{N},\left(P_{K,l}(w_\varepsilon)\right)_\varepsilon\in|A|.$$

Then $(w_\varepsilon)_\varepsilon\in\mathscr{X}^s(D_2)$. Put $w=[w_\varepsilon]_{\mathscr{G}(D_2)}$ then (g,w) is a solution to (P_{gen}) in $\mathscr{G}(D_1)\times\mathscr{G}(D_2)$.

REMARK 2. The total population size

$$\begin{aligned}
N_\varepsilon(t)&=\int_0^\infty u_\varepsilon(t,x)dx\\
&=\int_0^t u_\varepsilon(t,x)dx+\int_t^\infty u_\varepsilon(t,x)dx\\
&=\int_0^t k\beta e^{(\beta-\mu)t}e^{-\beta x}dx+\int_t^\infty k\varphi_\varepsilon(x-t)e^{-\mu t}dx\\
&=k\beta e^{(\beta-\mu)t}\int_0^t e^{-\beta x}dx+ke^{-\mu t}=k\beta e^{(\beta-\mu)t}\left[-(1/\beta)e^{-\beta x}\right]_0^t+ke^{-\mu t}\\
&=ke^{(\beta-\mu)t}\left(1-e^{-\beta t}\right)+ke^{-\mu t}=ke^{(\beta-\mu)t}.
\end{aligned}$$

3.2.3. Behavior in $\mathscr{D}'(D_2)$.

We have $\lim\limits_{\varepsilon\to 0}u_\varepsilon(t,x)=ke^{-\mu t}\delta(x-t)$ with

$$<\delta(x-t)),f(t,x)>=\int_{-\infty}^{+\infty}f(t,t))dt.$$

We have a decreasing wave which propagates the data along the characteristic curve $\Gamma=\{x=t\}$.

3.2.4. Case where β is a Dirac distribution δ_{x_0}

We assume that individuals give birth when they reach a certain age x_0. This corresponds to a birth distribution of the form $\beta=B\delta_{x_0}$, where B is the number of offspring and $x_0>1$ [7]. This model may be specialized to the case $\beta=2\delta_{x_0}$ thus expressing that mitosis (doubling of the cell, reproduction) arises at a given age x_0 [26]. It has been used and compared in in vitro experiments. We consider the case where the parent survives contributing to the total population.

The generalized problem is also well formulated as

$$(P_{gen})\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial x}=-\mu u, \\ u\,|_{\{t=0\}}=k\,[\varphi_\varepsilon]\,, \end{cases}$$

where $k>0$, $[\varphi_\varepsilon]\in\mathscr{G}([0;+\infty[)$ and (g,w) is searched in $\mathscr{G}\,(D_1)\times\mathscr{G}(D_2)$.

Recall $D=\{(t,x): t\geqslant 0, x\geqslant 0\}$.

THEOREM 4. *The solution* (u_ε) *to* (P_ε) *in* $\mathrm{C}^\infty(D)$ *is defined by*

$$u_\varepsilon(t,x)=\sum_{m=0}^{+\infty}B^m k\varphi_\varepsilon(mx_0+x-t)e^{-\mu t}.$$

Proof. According to Theorem 1, we have

$$u_\varepsilon(t,x)=\begin{cases} g_\varepsilon(t,x)=B^m\phi_\varepsilon(mx_0+x-t)e^{-\mu t}, t\geqslant x, \\ w_\varepsilon(t,x)=\phi_\varepsilon(x-t)e^{-\mu t}, \text{ if } t<x \end{cases}$$

where $m=Ent((t-x)/x_0+1)$. Then the solution $(g_\varepsilon,w_\varepsilon)$ to (P_ε) is defined by

$$g_\varepsilon(t,x)=B^m k\varphi_\varepsilon(mx_0+x-t)e^{-\mu t}, \text{ if } t\geqslant x,$$
$$w_\varepsilon(t,x)=k\varphi_\varepsilon(x-t)e^{-\mu t}, \text{ if } t<x.$$

As $\operatorname{supp}\varphi_\varepsilon=[0;\varepsilon]$, for ε small, $\varepsilon<1<x_0$, then we have the result.

REMARK 3. Behavior in $\mathscr{D}'(D_2)$. If $t<x$, $\lim\limits_{\varepsilon\to 0}u_\varepsilon(t,x)=ke^{-\mu t}\delta(x-t)$. We have a decreasing wave which propagates the data along the characteristic curve $\Gamma=\{x=t\}$.

REMARK 4. Set

$$T_m=\{(t,x): x+(m-1)x_0\leqslant t<x+mx_0; t\geqslant 0, x\geqslant 0\}.$$

For $(t,x)\in T_m$ we have $u_\varepsilon(t,x)=B^m\phi_\varepsilon(mx_0+x-t)e^{-\mu t}$ and

$$\lim_{\varepsilon\to 0}u_\varepsilon(t,x)=B^m ke^{-\mu t}\delta(mx_0+x-t).$$

The effect of the delta birth distribution is to create a new decreasing wave of individuals whenever the cohort reaches the reproduction age which propagates along the characteristic curve $\{t=x+mx_0\}$.

THEOREM 5. *Taking* $u=[u_\varepsilon]_{\mathscr{G}(D)}$ *then* u *is a solution to* (P_{gen}) *in* $\mathscr{G}(D)$.

For $(t,x)\in D$, we have $u_\varepsilon(t,x)=B^m\phi_\varepsilon(mx_0+x-t)e^{-\mu t}$ where $m=Ent((t-x)/x_0+1)$ if $t\geqslant x$, $m=0$ if $t<x$. Technical estimates prove that $\left(P_{K,l}(u_\varepsilon)\right)_\varepsilon\in|A|$ thus $(u_\varepsilon)_\varepsilon\in\mathscr{X}(D)$.

REMARK 5. The total population size

$$\begin{aligned}
N_\varepsilon(t) &= \int_0^\infty u_\varepsilon(t,x)dx \\
&= \sum_{m=0}^{m=Ent(t/x_0)} \int_0^\infty B^m k \varphi_\varepsilon(mx_0+x-t)e^{-\mu t}dx \\
&= ke^{-\mu t} \sum_{m=0}^{m=Ent(t/x_0)} B^m \int_0^\infty \varphi_\varepsilon(mx_0+x-t)dx \\
&= ke^{-\mu t} \sum_{m=0}^{m=Ent(t/x_0)} B^m = ke^{-\mu t}\frac{B^{Ent(t/x_0)+1}-1}{B-1}.
\end{aligned}$$

3.3. Top hat initial condition

A shock can correspond to a "top hat" initial condition [7], [24]. For example wars, genocides, epidemic or natural disasters. Take

$$\phi = L - d\left(H_a - H_b\right))$$

where L is the distribution associated with the function l, d lying in $\mathrm{C}^1(\mathbb{R}_+)$, a, $b\in\mathbb{R}_+$, $a<b<x_m$ and H is the Heaviside distribution. Let us assume also that the age-specific fertility β and the age specific mortality μ are both constantes. Let $I=[a;b]$ be the tested age bracket affected by the shock (for example we can take $I=[18;45]$ for a war).

Then ϕ is a distribution and the problem is ill-posed in the spaces of classical functions. To have a well-posed problem we must study the problem in an algebra of generalized functions, in this case the Colombeau algebra, and the data will be considered as generalized functions. We have the problem formally written as

$$(P)\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial x} = -\mu u, \\ u\mid_{\{t=0\}} = \phi. \end{cases}$$

3.3.1. Well formulated problem

We take some fixed smooth function φ,

$$\varphi(x) = 0 \text{ for } x \notin [0;1], \int \varphi(x)dx = 1.$$

We consider the family of mollifiers $(\varphi_\varepsilon)_\varepsilon$ given by $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ then $\operatorname{supp}\varphi_\varepsilon = [0;\varepsilon]$. We approach problem (P) by a parametric family of (P_ε) ones

$$(P_\varepsilon)\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}+\dfrac{\partial u_\varepsilon}{\partial x} = -\mu u_\varepsilon, \\ u_\varepsilon(0,x) = \phi_\varepsilon(x), \end{cases}$$

with $\phi_\varepsilon(x) = l(x) - d(x)(h_\varepsilon(x-a) - h_\varepsilon(x-b))$ and $h_\varepsilon = \varphi_\varepsilon * H \in C^\infty(\mathbb{R})$.

Assume that $l(x) = c$, a constant, and $d(x) = (1/2)\, l(x) = (1/2)\, c$, then

$$\phi_\varepsilon(x) = c - \frac{1}{2}c(h_\varepsilon(x-a) - h_\varepsilon(x-b)).$$

The generalized problem (P_{gen}) is well formulated as

$$(P_{gen}) \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = -\mu u, \\ u\,|_{\{t=0\}} = [\phi_\varepsilon], \end{cases}$$

where $[\phi_\varepsilon] \in \mathscr{G}([0;+\infty[)$ and (g,w) is searched in $\mathscr{G}(D_1) \times \mathscr{G}(D_2)$.

3.3.2. Solution to (P_{gen})

THEOREM 6. *The solution $(g_\varepsilon, w_\varepsilon)$ to (P_ε) is defined by*

$$g_\varepsilon(t,x) = K\beta e^{(\beta-\mu)t} e^{-\beta x}, \quad if \quad t \geqslant x,$$

$$\begin{aligned} w_\varepsilon(t,x) &= \phi_\varepsilon(x-t)e^{-\mu t} \\ &= \left[c - \frac{1}{2}c(h_\varepsilon(x-a) - h_\varepsilon(x-b))\right] e^{-\mu t}, \quad if \quad t < x, \end{aligned}$$

where $K = c\,(x_m - (1/2)\,(b-a))$.

Proof. We have

$$u_\varepsilon(t,x) = \begin{cases} g_\varepsilon(t,x) = v_\varepsilon(t-x)e^{-\mu x}, & \text{if} \quad t \geqslant x, \\ w_\varepsilon(t,x) = \phi_\varepsilon(x-t)e^{-\mu t}, & \text{if} \quad t < x \end{cases}$$

where $v_\varepsilon(t) = u_\varepsilon(t,0)$. Thus the renewal equation takes the form

$$v_\varepsilon(t) = \psi_\varepsilon(t) + \int_0^t v_\varepsilon(t-y)\beta e^{-\mu y} dy,$$

with

$$\begin{aligned} \psi_\varepsilon(t) &= \int_t^{+\infty} \phi_\varepsilon(y-t)\beta e^{-\mu t} dy = \int_0^{+\infty} \phi_\varepsilon(s)\beta e^{-\mu t} ds \\ &= \beta e^{-\mu t} \int_0^{+\infty} \phi_\varepsilon(s) ds = K\beta e^{-\mu t}, \end{aligned}$$

with

$$\begin{aligned} K &= \int_0^{+\infty} \phi_\varepsilon(s) ds = \int_0^{x_m} c(1 - 1/2(h_\varepsilon(s-a) - h_\varepsilon(s-b)))ds \\ &= cx_m - (1/2)\,c\,(b-a) = c\,(x_m - (1/2)\,(b-a)). \end{aligned}$$

Thus we have

$$v_\varepsilon(t) = K\beta e^{-\mu t} + \beta\int_0^t v_\varepsilon(t-y)e^{-\mu y}dy = K\beta e^{-\mu t} + \beta\int_0^t v_\varepsilon(s)e^{-\mu(t-s)}ds$$
$$= K\beta e^{-\mu t} + \beta e^{-\mu t}\int_0^t v_\varepsilon(s)e^{\mu s}ds.$$

Differentiation gives

$$\begin{aligned}(v_\varepsilon)'(t) &= -K\mu\beta e^{-\mu t} + \beta e^{-\mu t}\left(v_\varepsilon(t)e^{\mu t}\right) - \mu\beta e^{-\mu t}\int_0^t v_\varepsilon(s)e^{\mu s}ds\\ &= -\mu\left(K\beta e^{-\mu t} + \beta e^{-\mu t}\int_0^t v_\varepsilon(s)e^{\mu s}ds\right) + \beta v_\varepsilon(t)\\ &= -\mu v_\varepsilon(t) + \beta v_\varepsilon(t) = (\beta-\mu)\, v_\varepsilon(t).\end{aligned}$$

Then $(v_\varepsilon)'(t) = (\beta-\mu)\, v_\varepsilon(t)$. From the renewal equation with $t=0$ we have $v_\varepsilon(0) = K\beta$. Then $v_\varepsilon(t) = K\beta e^{(\beta-\mu)t}$. Thus

$$u_\varepsilon(t,x) = K\beta e^{(\beta-\mu)(t-x)}e^{-\mu x} = K\beta e^{(\beta-\mu)t}e^{-\beta x} = g_\varepsilon(t,x).$$

This gives the result.

REMARK 6. For any ε, $g_\varepsilon(t,x) = k\beta e^{t(\beta-\mu)}e^{-\beta x}$ is independent of ε. Moreover $g_\varepsilon \in \mathrm{C}^\infty(D_1)$, then $(g_\varepsilon)_\varepsilon \in \mathscr{X}(D_1)$, we denote by $g = [g_\varepsilon]_{\mathscr{G}(D_1)}$ it class in $\mathscr{G}(D_1)$.

THEOREM 7. *The family $(w_\varepsilon)_\varepsilon$ lies in $\mathscr{X}(D_2)$. Taking $w = [w_\varepsilon]_{\mathscr{G}(D_2)}$ then (g,w) is solution to (P_{gen}) in $\mathscr{G}(D_1)\times\mathscr{G}(D_2)$.*

Technical estimates proving that $\left(P_{K,l}(w_\varepsilon)\right)_\varepsilon \in |A|$, then $(w_\varepsilon)_\varepsilon$ lies in $\mathscr{X}(D_2)$.

REMARK 7. The total population size

$$\begin{aligned}N_\varepsilon(t) &= \int_0^\infty u_\varepsilon(t,x)dx\\ &= \int_0^t u_\varepsilon(t,x)dx + \int_t^\infty u_\varepsilon(t,x)dx\\ &= \int_0^t \beta e^{(\beta-\mu)t}e^{-\beta x}dx + \int_t^\infty \phi_\varepsilon(x-t)e^{-\mu t}dx\\ &= \beta e^{(\beta-\mu)t}\int_0^t e^{-\beta x}dx + Ke^{-\mu t} = \beta e^{(\beta-\mu)t}\left[-(1/\beta)\, e^{-\beta x}\right]_0^t + Ke^{-\mu t}\\ &= e^{(\beta-\mu)t}\left(1-e^{-\beta t}\right) + Ke^{-\mu t} = e^{(\beta-\mu)t} + (K-1)e^{-\mu t}.\end{aligned}$$

3.3.3. Case where β is a Dirac distribution δ_{x_0}

We assume that individuals give birth when they reach a certain age x_0. This corresponds to a birth distribution of the form $\beta = B\delta_{x_0}$, where B is the number of offspring and $x_0 > 1$. We consider the case where the parent survives.

The generalized problem (P_{gen}) is also well formulated as

$$(P_{gen}) \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = -\mu u, \\ u\mid_{\{t=0\}} = [\phi_\varepsilon], \end{cases}$$

where $[\phi_\varepsilon] \in \mathscr{G}([0;+\infty[)$ and (g,w) is searched in $\mathscr{G}(D_1) \times \mathscr{G}(D_2)$.

Set $\beta_\varepsilon(x) = B\varphi_\varepsilon(x-x_0)$. Assume that $x_0 > b > a$.

First, we can give a straight proof for $x_0 + x > t \geqslant x$.

THEOREM 8. *The solution (g,w) to (P_ε) is defined, if $x_0+x > t \geqslant x$, by*

$$g_\varepsilon(t,x) = cBe^{-\mu t}\left[1 - 1/2\left(h_\varepsilon(x_0 - t + x - a) - h_\varepsilon(x_0 - t + x - b)\right)\right]$$

and, if $t < x$, by

$$w_\varepsilon(t,x) = \phi_\varepsilon(x-t)e^{-\mu t}.$$

Proof. We have

$$u_\varepsilon(t,x) = \begin{cases} g_\varepsilon(t,x) = v_\varepsilon(t-x)e^{-\mu x}, & \text{if} \quad x_0 + x > t \geqslant x, \\ w_\varepsilon(t,x) = \phi_\varepsilon(x-t)e^{-\mu t}, & \text{if} \quad t < x \end{cases}$$

where $v_\varepsilon(t) = u_\varepsilon(t,0)$. We have

$$\begin{aligned} \psi_\varepsilon(t) &= \int_t^{+\infty} \phi_\varepsilon(y-t)B\varphi_\varepsilon(y-x_0)e^{-\mu t}\,dy \\ &= Be^{-\mu t}\int_0^{+\infty} \phi_\varepsilon(s)\varphi_\varepsilon(s-(x_0-t))\,ds \\ &= Be^{-\mu t}\phi_\varepsilon(x_0-t) \\ &= cBe^{-\mu t}\left[1 - (1/2)\left(h_\varepsilon(x_0-t-a) - h_\varepsilon(x_0-t-b)\right)\right]. \end{aligned}$$

The renewal equation takes the form

$$\begin{aligned} v_\varepsilon(t) &= \psi_\varepsilon(t) + B\int_0^t v_\varepsilon(t-a)\varphi_\varepsilon(a-x_0)e^{-\mu a}\,da \\ &= Be^{-\mu t}\phi_\varepsilon(x_0-t) + B\int_0^t v_\varepsilon(s)\varphi_\varepsilon(t-s-x_0)e^{-\mu(t-s)}\,ds \\ &= Be^{-\mu t}\phi_\varepsilon(x_0-t) + Be^{-\mu t}\int_0^t v_\varepsilon(s)\varphi_\varepsilon((t-x_0)-s)e^{\mu s}\,ds \end{aligned}$$

and

$$v_\varepsilon(0) = Be^{-\mu t}\phi_\varepsilon(x_0) = cBe^{-\mu t}.$$

Differentiation gives

$$
\begin{aligned}
(v_\varepsilon)'(t) &= -\mu Be^{-\mu t}\phi_\varepsilon(x_0-t) - Be^{-\mu t}\phi_\varepsilon'(x_0-t)\\
&\quad + Be^{-\mu t}v_\varepsilon(t)\varphi_\varepsilon(-x_0)e^{\mu t} - \mu Be^{-\mu t}\int_0^t v_\varepsilon(s)\varphi_\varepsilon((t-x_0)-s)e^{\mu s}ds\\
&= -\mu(Be^{-\mu t}\phi_\varepsilon(x_0-t) + Be^{-\mu t}\int_0^t v_\varepsilon(s)\varphi_\varepsilon((t-x_0)-s)e^{\mu s}ds)\\
&\quad - Be^{-\mu t}\phi_\varepsilon'(x_0-t)\\
&= -\mu v_\varepsilon(t) - Be^{-\mu t}\phi_\varepsilon'(x_0-t).
\end{aligned}
$$

Consequently we have to solve the equation

$$(v_\varepsilon)'(t) = -\mu v_\varepsilon(t) - Be^{-\mu t}\phi_\varepsilon'(x_0-t). \tag{E2}$$

Consider the homogenous equation $(v_\varepsilon)'(t) = -\mu v_\varepsilon(t)$. The solution is given by $v_\varepsilon(t) = Ke^{-\mu t}$. Then take $v_\varepsilon(t) = K(t)e^{-\mu t}$. Thus v_ε is solution to (E2) if and only if

$$-Be^{-\mu t}\phi_\varepsilon'(x_0-t) = K'(t)e^{-\mu t},$$

then $K'(t) = -B\phi_\varepsilon'(x_0-t)$ and $K(t) = B\phi_\varepsilon(x_0-t)+k$. As

$$v_\varepsilon(0) = Be^{-\mu t}\phi_\varepsilon(x_0) = cBe^{-\mu t},$$

thus $v_\varepsilon(t) = B\phi_\varepsilon(x_0-t)e^{-\mu t}$. We have

$$v_\varepsilon(t-x)e^{-\mu x} = B\phi_\varepsilon(x_0-t+x)e^{-\mu(t-x)}e^{-\mu x} = B\phi_\varepsilon(x_0+x-t)e^{-\mu t}.$$

Then

$$v_\varepsilon(t-x)e^{-\mu x} = cBe^{-\mu t}[1-(1/2)(h_\varepsilon((x_0-t)+x-a) - h_\varepsilon((x_0-t)+x-b))].$$

This gives the result.

THEOREM 9. *The solution $(g_\varepsilon, w_\varepsilon)$ to (P_ε) is defined, if $t \geqslant x$, by*

$$g_\varepsilon(t,x) = cB^m e^{-\mu t}\left[1-\frac{1}{2}(h_\varepsilon(mx_0+x-t-a) - h_\varepsilon(mx_0+x-t-b))\right]$$

and, if $t < x$, by

$$w_\varepsilon(t,x) = \phi_\varepsilon(x-t)e^{-\mu t}.$$

Proof. According to Theorem 1, we have

$$u_\varepsilon(t,x) = \begin{cases} g_\varepsilon(t,x) = B^m\phi_\varepsilon(mx_0+x-t)e^{-\mu t}, t \geqslant x,\\ w_\varepsilon(t,x) = \phi_\varepsilon(x-t)e^{-\mu t}, \text{ if } t < x \end{cases}$$

where $m = Ent((t-x)/x_0+1)$. We have

$$\phi_\varepsilon(x) = c - \frac{1}{2}c(h_\varepsilon(x-a) - h_\varepsilon(x-b))$$

then

$$\phi_\varepsilon(mx_0+x-t)=\left[c-\frac{1}{2}c(h_\varepsilon(mx_0+x-t-a)-h_\varepsilon(mx_0+x-t-b))\right].$$

Then

$$\begin{aligned} g_\varepsilon(t,x) &= B^m\left[c-\frac{1}{2}c(h_\varepsilon(mx_0+x-t-a)-h_\varepsilon(mx_0+x-t-b))\right]e^{-\mu t} \\ &= cB^m e^{-\mu t}\left[1-\frac{1}{2}(h_\varepsilon(mx_0+x-t-a)-h_\varepsilon(mx_0+x-t-b))\right]. \end{aligned}$$

COROLLARY 1. *The solution* (u_ε) *to* (P_ε) *in* $\mathrm{C}^\infty(D)$ *is defined by*

$$u_\varepsilon(t,x)=\sum_{m=0}^{+\infty}B^m k\Phi_\varepsilon(mx_0+x-t)e^{-\mu t}.$$

REMARK 8. Set

$$T_m=\{(t,x):x+(m-1)x_0\leqslant t<x+mx_0;t\geqslant 0,x\geqslant 0\}.$$

For $(t,x)\in T_m$ we have $u_\varepsilon(t,x)=B^m\phi_\varepsilon(mx_0+x-t)e^{-\mu t}$ and

$$\lim_{\varepsilon\to 0}u_\varepsilon(t,x)=cB^m ke^{-\mu t}\left[1-\frac{1}{2}(H_a(mx_0+x-t)-H_b(mx_0+x-t))\right].$$

At time $t=(m-1)x_0$ a new cohort is completely created imitating the previous cohort for $0\leqslant x\leqslant x_0$.

THEOREM 10. *Taking* $u=[u_\varepsilon]_{\mathscr{G}(D)}$ *then* u *is a solution to* (P_{gen}) *in* $\mathscr{G}(D)$.

For $(t,x)\in D$, we have $u_\varepsilon(t,x)=B^m\phi_\varepsilon(mx_0+x-t)e^{-\mu t}$ where $m=Ent((t-x)/x_0+1)$ if $t\geqslant x$, $m=0$ if $t<x$. Technical estimates prove that $\left(P_{K,l}(u_\varepsilon)\right)_\varepsilon\in|A|$ thus $(u_\varepsilon)_\varepsilon\in\mathscr{X}(D)$.

4. Conclusion

Our research relates to the persistent character of shocks in demography. Our results indicate the propagation mechanism for some demographic shocks and show the persistent effect of shocks in population dynamics whose amplitude diminishes with time. This study could be used also to disease transmission in age-structured models. We watch appearance of traveling waves which have been observed frequently in the spread of epidemics. Our investigation in the framework of Colombeau algebra can be extended to analyze a wide spectrum of situations (significant immigration, black-out...) in various types of shocks. Economic policies need to take into account variations of population in reply to shocks.

REFERENCES

[1] J. ARAGONA, *Colombeau generalized functions on quasi-regular sets*, Publicationes Mathematicae Debrecen, **68**, 3-4 (2006), 371–399.

[2] J. ARAGONA, A. R. G. GARCIA AND S. O. JURIAANS, *Generalized solutions of a nonlinear parabolic equation with generalized functions as initial data*, Nonlinear Analysis, **71** (2009), 5187–5207.

[3] BANQUE MONDIALE, *L'IDA en action. Gérer les risques naturels, réduire les risques liés au développement,* Washington D. C., 2008.

[4] J. R. BARNETT AND M. WEBBER, *Accomodating Migration to Promote Adaptation to Climate Change,* World Bank Policy Research, Working Paper **5270**, 2010.

[5] H. A. BIAGIONI, *A nonlinear theory of Generalized Functions*, Lecture Notes in Mathematics **1421**, Springer-Verlag, 1990.

[6] E. CANALES, *1808-1814: Démographie et guerre en Espagne*, Annales historiques de la Révolution française, **336** (2004), 37–52.

[7] S. J. CHAPMAN, M. J. PLANK, A. JAMES AND A. B. BASSE, *A nonlinear model of age size-structured populations with applications to cell cycles*, The ANZIAM Journal **49**, 2 (2007), 151–169.

[8] P. COLLIER, *On the economic consequences of Civil War*, Oxford Economic Papers, **51** (1999), 168-183.

[9] J. F. COLOMBEAU, *New Generalized Functions and Multiplication of Distributions*, North Holland, Amsterdam, Oxford, New-York, 1984.

[10] J. F. COLOMBEAU, *Elementary introduction to new generalized functions*, North Holland Mathematics Studies **113**, North-Holland, Amsterdam, 1984.

[11] J. M. CUSHING AND J. LI, *Juvenile versus adult competition*, Journal of Mathematical Biology **29** (1991), 457–473.

[12] A. DELCROIX, *Remarks on the embedding of spaces of distributions into spaces of Colombeau generalized functions*, Novi Sad Journal of Mathematics, **35**, 2 (2005), 27–40.

[13] A. DELCROIX, V. DÉVOUÉ AND J.-A. MARTI, *Generalized solutions of singular differential problems. Relationship with classical solutions*, Journal of Mathematical Analysis and Applications **353**, 1 (2009), 386–402.

[14] A. DELCROIX, V. DÉVOUÉ AND J.-A. MARTI, *Well posed problems in algebras of generalized function,* Applicable Analysis, **90**, 11 (2011), 1747–1761.

[15] R. DILAO AND A. LAKMECHE, *On the weak solutions of the McKendrick equation: Existence of demography cycles*, Mathematical Modeling of Natural Phenomena, **1**, 1 (2006), 1–32.

[16] M IANNELLI, *Mathematical Theory of Age-Structured Population Dynamics*, Giardini Editori e stampatori in Pisa 1995.

[17] M. D. JOHNSTON, C. M. EDWARDS, W. F. BODMER, P. K. MAINI AND S. J. CHAPMAN, *Mathematical modeling of cell population dynamics in the colonic crypt and in colorectal cancer*, Proceedings of the National Academy of Sciences of the United States of America, **104**, 10 (2007), 4008–4013.

[18] I. KMIT, *A distributional solution to a hyperbolic problem arising in population dynamics*, Electronic Journal of Differential Equations, **132 (**2007), 1–23.

[19] J. A. LOTKA, *Théorie analytique des associations biologiques*, Hermann, Paris, 1934 and 1939.

[20] J.-A. MARTI, *Fundamental structures and asymptotic microlocalization in sheaves of generalized functions*, Integral Transforms and Special Functions **6**, 1-4 (1998), 223–228.

[21] J.-A. MARTI, $(\mathscr{C},\mathscr{E},\mathscr{P})$ *-Sheaf structure and applications,* Nonlinear theory of generalized functions (eds M. Grosser and al.), Research Notes in Mathematics, Chapman & Hall/CRC (1999), 175–186.

[22] J.-A. MARTI, Non linear Algebraic analysis of delta shock wave to Burgers' equation, Pacific Journal of Mathematics, **210**, 1 (2003), 165–187.

[23] A. G. MCKENDRICK, *Applications of Mathematics to Medical Problems*, Proceedings of the Edinburgh Mathematical Society, **44** (1926), 98–130.

[24] NORHAYATI AND G. C. WAKE, *The solution and stability of a nonlinear age-structured population model*, The ANZIAM Journal, **45** (2003), 153–165.

[25] ORGANISATION DE L'UNITÉ AFRICAINE, *Rapport sur le génocide du Rwanda. Le génocide qu'on aurait pu stopper*, Nations Unies 2000.

[26] B. PERTHAME, S. MISCHLER, J. CLAIRAMBAULT AND B. LAROCHE, *A mathematical model of the cell cycle and its control,* Rapport de recherche **4892**, Institut National de Recherche en Informatique et en Automatique, 2007.

[27] L. SCHWARTZ, *Théorie des Distributions*, Hermann, Paris, 1966.

[28] D. SCHWEISGUTH, *Japon: Séisme et tsunami, quel impact sur la croissance?*, Publications de l'OCFCE, Centre de recherche en économie de Sciences Politiques 2011.

[29] M. VERPOORTEN AND L. BERLAGE, *Economic Mobility in Rural Rwanda: A Study of the Effects of War and Genocide at the Household Level*, Journal of African Economies, **3** (2007), 349–392.

(Received November 17, 2015)
(Revised April 19, 2016)

Elina Dévoué
Université des Antilles
Faculté de Droit et d'Economie de Martinique, CEREGMIA
Campus de Schoelcher BP 7209
97275 Schoelcher cedex. Martinique. F.W.I.
e-mail: elina.devoue@martinique.univ-ag.fr

Victor Dévoué
Université des Antilles. CEREGMIA.
Campus de Schoelcher BP 7209.
97275 Schoelcher cedex. Martinique. F.W.I.
e-mail: devoue-vi@orange.fr