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ON THE OSCILLATION CERTAIN FOURTH ORDER NONLINEAR DYNAMIC EQUATIONS WITH A NONLINEAR MIDDLE TERM

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Abstract. New oscillation criteria for certain fourth order nonlinear dynamic equations with nonlinear middle term are established.

1. Introduction

This article deals with the oscillatory behavior of certain fourth order nonlinear dynamic equations with nonlinear middle term

$$\Big(a(t)\phi_\alpha(x^{\Delta\Delta\Delta}(t))\Big)^{\Delta}+p(t)\phi_\beta(x^{\Delta\Delta}(h(t)))+q(t)\phi_\beta(x(g(t)))=0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}\subseteq\mathbb{R}$ with $\sup\mathbb{T}=\infty$. We assume that

(i) $\phi_\lambda(u):=|u|^{\lambda}\operatorname{sgn}u$ for $\lambda>0$;

(ii) $a,p,q:\mathbb{T}\to(0,\infty)$ are real valued, rd-continuous functions such that

$$\int_0^{\infty}a^{-1/\alpha}(s)\Delta s=\infty; \tag{1.2}$$

(iii) $g,h:\mathbb{T}\to\mathbb{T}$ are real valued, rd-continuous functions such that $g^{\Delta}(t)\geqslant 0$, $h^{\Delta}(t)\geqslant 0$, $g(t)\leqslant t$, $h(t)\leqslant t$ for $t\geqslant t_0\in\mathbb{T}$ and $\lim_{t\to\infty}g(t)=\lim_{t\to\infty}h(t)=\infty$;

We recall that a solution of equation (1.1) is said to nonoscillatory if there exists $t_0\in\mathbb{T}$ such that $x(t)x(\sigma(t))>0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$, where the forward jump operator $\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}$, otherwise it is said to be oscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D Thesis in 1988 in order to unify continuous and discrete analysis, see [12]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [1]). This new theory of

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these so-called "dynamic equations" not only unifies the corresponding theories for the differential equations and difference equations cases, but it also extends these classical cases to cases "in between". That is, we are able to treat the so-called $q-$difference equations when $\mathbb{T}=q^{\mathbb{N}_0}:=\{q^n: n\in\mathbb{N}_0 \text{ for } q>1\}$ (which has important applications in quantum theory and can be applied to different types of time scales like $\mathbb{T}=h\mathbb{N}$, $\mathbb{T}=\mathbb{N}^2$ and $\mathbb{T}=\mathbb{T}_n$ the set of the harmonic numbers. The books on the subject of time scales by Bohner and Peterson [1] summarize and organize much of time scale calculus.

Recently, there has been an increasing interest in studying the oscillatory behavior of all order dynamic equations on time scales, see, for example [2, 1, 5, 6, 7, 8, 10, 3, 4, 9, 11] and the references contained therein.

The study content on the oscillatory and asymptotic behavior of second order dynamic equations on time scales is very rich.

In contrast, the study of oscillation criteria of fourth order dynamic equations is relatively less. To the best of our knowledge, the oscillatory behavior of fourth order nonlinear dynamic equations with nonlinear middle term has not been studied till now.

Our aim here is to initiate such a study by establishing some new criteria for the oscillation of equation (1.1) and some related equation. Our approach is to reduce the problem in such a way that specific oscillation results for second order equations can be adapted for fourth order case.

We may also extend the results obtained to fourth order nonlinear dynamic equations with a nonlinear middle term and with distributed deviating arguments on time scale of the form

$$\left(a(t)\phi_\alpha(x^{\Delta\Delta\Delta}(t))\right)^\Delta+p(t)\phi_\beta(x^{\Delta\Delta}(h(t)))+\int_b^c q_1(t,\tau)\phi_\beta(x(g_1(t,\tau)))\Delta\tau=0, \quad (1.3)$$

where (i)–(iii) hold, $0<b<c$,

(iv) $q_1:\mathbb{T}\times[b,c]\to[0,\infty)$ is real valued, rd-continuous function;

(v) $g_1:\mathbb{T}\times[b,c]\to\mathbb{T}$ is decreasing with respect to second variable, $g_1(t,\tau)\leqslant t$ and $\lim_{t\to\infty}g_1(t,\tau)=\infty$, $t\geqslant t_0\in\mathbb{T}$ and $\tau\in[b,c]$.

2. Main Results

We shall employ the following lemmas. Consider the inequality

$$\left(a(t)\phi_\alpha\left(x^\Delta(t)\right)\right)^\Delta+p(t)\phi_\beta\left(x(h(t))\right)\leqslant 0, \quad (2.1)$$

where the numbers α and β are positive real numbers and the functions a, h and p are as in equation (1.1).

LEMMA 1. *If the inequality* (2.1) *has an eventually positive solution, then the equation*

$$\left(a(t)\phi_\alpha\left(x^\Delta(t)\right)\right)^\Delta+p(t)\phi_\beta\left(x(h(t))\right)=0, \quad (2.2)$$

also has an eventually positive solution.

Proof. Let $x(t)$ be an eventually positive solution of inequality (2.1). It is easy to see that $x^{\Delta} > 0$ eventually. Let t_0 be sufficiently large so that $x(t) > 0$, $x(h(t)) > 0$ and $y(t) := a(t)\phi_\alpha\left(x^{\Delta}(t)\right)$ for $t \in [t_0,\infty)_{\mathbb{T}}$. Then in view of

$$x(t) = x(t_0) + \int_{t_0}^{t} \phi_\alpha^{-1}\left(\frac{y(s)}{a(s)}\right)\Delta s.$$

There is a $t_1 \geqslant t_0$ such that $h(t) \geqslant t_0$, for $t \geqslant t_1$. Inequality (2.1) becomes

$$y^{\Delta}(t) + p(t)\phi_\beta\left(x(t_0) + \int_{t_0}^{h(t)} \phi_\alpha^{-1}\left(\frac{y(s)}{a(s)}\right)\Delta s\right) \leqslant 0. \tag{2.3}$$

Integrating (2.3) from t to $v \geqslant t \geqslant t_1$ and letting $v \to \infty$, we have

$$y(t) \geqslant G(t,y(t)), \qquad \text{for } t \in [t_1,\infty)_{\mathbb{T}},$$

where

$$G(t,y(t)) := \int_t^{\infty} p(v)\phi_\beta\left(x(t_0) + \int_{t_0}^{h(v)} \phi_\alpha^{-1}\left(\frac{y(s)}{a(s)}\right)\Delta s\right)\Delta v.$$

Now, we define a sequence of successive approximations $\left\{w_j(t)\right\}$ as follows:

$$\begin{aligned} w_0(t) &: = y(t) \\ w_{j+1}(t) &: = G(t,w_j(t)),\ j = 0,1,2,\ldots \end{aligned}$$

It is easy to show that

$$0 < w_j(t) \leqslant y(t) \qquad \text{and} \qquad w_{j+1}(t) \leqslant w_j(t),\ j = 0,1,2,\ldots$$

Then, the sequence $\left\{w_j(t)\right\}$ is nonincreasing and bounded for each $t \geqslant t_1$. This means,we may define $w(t) := \lim_{j\to\infty} w_j(t) \geqslant 0$. Since

$$0 \leqslant w(t) \leqslant w_j(t) \leqslant y(t), \qquad \text{for all } j \geqslant 0,$$

we find that

$$\int_{t_1}^{t} w_j(s)\Delta s \leqslant \int_{t_1}^{t} y(s)\Delta s.$$

By the Lebesgue's dominated convergence theorem on time scale, one can easily find

$$w(t) = G(t,w(t)).$$

Therefore

$$w^{\Delta}(t) = -p(t)\phi_\beta\left(x(t_0) + \int_{t_0}^{h(t)} \phi_\alpha^{-1}\left(\frac{w(s)}{a(s)}\right)\Delta s\right) = -p(t)\phi_\beta\left(m(h(t))\right), \tag{2.4}$$

where

$$m(t) := x(t_0) + \int_{t_0}^{t} \phi_\alpha^{-1}\left(\frac{w(s)}{a(s)}\right)\Delta s.$$

Then

$$m(t) > 0 \qquad \text{and} \qquad a(t)\phi_\alpha\left(m^\Delta(t)\right) = w(t), \qquad \text{for } t \geqslant t_1.$$

Equation (2.4) then gives

$$\left(a(t)\phi_\alpha\left(m^\Delta(t)\right)\right)^\Delta + p(t)\phi_\beta\left(m(h(t))\right) = 0.$$

Hence equation (2) has a positive solution $m(t)$. This completes the proof.

In the following two lemmas, we consider the second order dynamic equation

$$\left(a(t)\phi_\alpha\left(x^\Delta(t)\right)\right)^\Delta = Q(t)\phi_\beta\left(x(h(t))\right), \tag{2.5}$$

where the numbers α and β are the ratio of positive odd integers and the functions a and h are as in equation (1.1) and $Q : \mathbb{T} \to [0,\infty)$ is real valued, rd-continuous function.

LEMMA 2. *If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} Q(s)\left(\int_{h(s)}^{h(t)} \frac{\Delta\tau}{a^{1/\alpha}(\tau)}\right)^\beta \Delta s > \begin{cases} 1 & \text{if} \quad \beta = \alpha \\ 0 & \text{if} \quad \beta < \alpha, \end{cases} \tag{2.6}$$

then all bounded solutions of equation (2.5) *are oscillatory.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of equation (2.5), say $x(t) > 0$ and $x(h(t)) > 0$ for $t \geqslant t_1$ for some $t_1 \in [t_0,\infty)_\mathbb{T}$. Then there exists a $t_2 \geqslant t_1$ such that

$$x(t) > 0,\ x^\Delta(t) < 0 \text{ and } \left(a(t)\phi_\alpha\left(x^\Delta(t)\right)\right)^\Delta \geqslant 0 \quad \text{for } t \geqslant t_2. \tag{2.7}$$

Now, for $v \geqslant u \geqslant t_2$ we have

$$\begin{aligned} x(u) \geqslant x(u) - x(v) &= -\int_u^v x^\Delta(\tau)\Delta\tau = -\int_u^v a^{-1/\alpha}(\tau)\phi_\alpha^{-1}\left[a(\tau)\phi_\alpha\left(x^\Delta(\tau)\right)\right]\Delta\tau \\ &\geqslant \phi_\alpha^{-1}\left[a(v)\phi_\alpha\left(-x^\Delta(v)\right)\right]\int_u^v a^{-1/\alpha}(\tau)\Delta\tau. \end{aligned} \tag{2.8}$$

For $t \geqslant s \geqslant t_2$, setting $u = h(s)$ and $v = h(t)$ in (2.8), we get

$$x(h(s)) \geqslant \phi_\alpha^{-1}\left[a(h(t))\phi_\alpha\left(-x^\Delta(h(t))\right)\right]\int_{h(s)}^{h(t)} a^{-1/\alpha}(\tau)\Delta\tau. \tag{2.9}$$

Integrating (2.5) from $h(t) \geqslant t_2$ to t, we have

$$\begin{aligned} -a(h(t))\phi_\alpha\left(x^\Delta(h(t))\right) &\geqslant a(t)\phi_\alpha\left(x^\Delta(t)\right) - a(h(t))\phi_\alpha\left(x^\Delta(h(t))\right) \\ &= \int_{h(t)}^{t} Q(s)\phi_\beta\left(x(h(s))\right)\Delta s. \end{aligned} \tag{2.10}$$

Using (2.9) in (2.10), we have

$$a(h(t))\phi_\alpha\left(-x^\Delta(h(t))\right)$$
$$\geqslant \phi_{\beta/\alpha}\left[a(h(t))\phi_\alpha\left(-x^\Delta(h(t))\right)\right]\int_{h(t)}^{t} Q(s)\left(\int_{h(s)}^{h(t)} a^{-1/\alpha}(\tau)\Delta\tau\right)^{\beta}\Delta s,$$

or

$$\left[a(h(t))\phi_\alpha\left(-x^\Delta(h(t))\right)\right]^{1-\beta/\alpha} \geqslant \int_{h(t)}^{t} Q(s)\left(\int_{h(s)}^{h(t)} a^{-1/\alpha}(\tau)\Delta\tau\right)^{\beta}\Delta s. \tag{2.11}$$

We take the $\limsup$ as $t\to\infty$ of both sides of the above inequality. If $\alpha=\beta$, the contradiction is obvious. If $\beta<\alpha$ the left hand side of (2.11) is positive and must decrease to zero (to present a contradiction to positively of $x(t)$). Thus contradicts (2.6) and completes the proof of the lemma.

LEMMA 3. *If*

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\frac{1}{a^{1/\alpha}(s)}\left(\int_s^t Q(\tau)\Delta\tau\right)^{1/\alpha}\Delta s > \begin{cases} 1 & \text{if} \quad \beta=\alpha \\ 0 & \text{if} \quad \beta<\alpha, \end{cases} \tag{2.12}$$

then all bounded solutions of equation (2.5) *are oscillatory.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of equation (2.5), say $x(t)>0$ and $x(h(t))>0$ for $t\geqslant t_1$ for some $t_1\in[t_0,\infty)_{\mathbb{T}}$. As in the Lemma 2, we obtain (2.7). Integrating (2.5) from s to $t\geqslant s\geqslant t_2$, we have

$$-a(s)\phi_\alpha\left(x^\Delta(s)\right)\geqslant a(t)\phi_\alpha\left(x^\Delta(t)\right)-a(s)\phi_\alpha\left(x^\Delta(s)\right)=\int_s^t Q(\tau)\phi_\beta\left(x(h(\tau))\right)\Delta\tau.$$

Thus

$$-a(s)\phi_\alpha\left(x^\Delta(s)\right)\geqslant\int_s^t Q(\tau)\phi_\beta\left(x(h(\tau))\right)\Delta\tau$$

or

$$-x^\Delta(s)\geqslant\frac{1}{a^{1/\alpha}(s)}\left(\int_s^t Q(\tau)\phi_\beta\left(x(h(\tau))\right)\Delta\tau\right)^{1/\alpha}$$
$$\geqslant\phi_{\beta/\alpha}\left(x(h(t))\right)\frac{1}{a^{1/\alpha}(s)}\left(\int_s^t Q(\tau)\Delta\tau\right)^{1/\alpha}.$$

Integrating this inequality from $h(t)$ to t, we get

$$x(h(t))\geqslant\phi_{\beta/\alpha}\left(x(h(t))\right)\int_{h(t)}^{t}\frac{1}{a^{1/\alpha}(s)}\left(\int_s^t Q(\tau)\Delta\tau\right)^{1/\alpha}\Delta s,$$

or

$$x^{1-\beta/\alpha}(h(t)) \geqslant \int_{h(t)}^{t} \frac{1}{a^{1/\alpha}(s)} \left(\int_{s}^{t} Q(\tau)\Delta\tau \right)^{1/\alpha} \Delta s.$$

The rest of the proof is similar to that of Lemma 2 and hence is omitted. This completes the proof.

Now we are ready to establish the main results of this article.

THEOREM 1. *Let conditions (i)–(iii) and $g(t) \leqslant h(t)$ for $t \geqslant t_0 \in \mathbb{T}$ hold. If equation (2) is oscillatory and condition (2.6) or (2.12) holds with*

$$Q(t) := cg^{\beta}(t)q(t)(h(t)-g(t))^{\beta} - p(t) \geqslant 0 \textit{ for } t \geqslant t_0 \textit{ and for some } c \in (0,1),$$

then every solution x of equation (1.1), either $x(t)$ or $x^{\Delta\Delta}(t)$ is oscillatory.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(h(t)) > 0$ and $x(g(t)) > 0$ for $t \geqslant t_1$ for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$. We consider the two cases:

(I) $x^{\Delta\Delta}(t) > 0$ for $t \geqslant t_1$, (II) $x^{\Delta\Delta}(t) < 0$ for $t \geqslant t_1$.

If case (I) holds, then (1.1) becomes

$$\left(a(t)\phi_{\alpha}\left(y^{\Delta}(t)\right)\right)^{\Delta} + p(t)\phi_{\beta}(y(h(t))) \leqslant 0 \quad \text{for } t \geqslant t_2 \geqslant t_1, \tag{2.13}$$

where $y(t) := x^{\Delta\Delta}(t) > 0$. By Lemma 1, equations (2.2) has a positive solution, a contradiction. Finally, if case (II) holds, $x^{\Delta}(t)$ must be positive, otherwise, we obtain a contradiction to the fact $x(t) > 0$. Now there exists a $t_3 \geqslant t_2$ and a constant $c_1 \in (0,1)$ such that

$$x(g(t)) \geqslant c_1\, g(t)x^{\Delta}(g(t)) > 0 \quad \text{for } t \geqslant t_3.$$

Using this inequality in equation (1.1), we get

$$\left(a(t)\phi_{\alpha}\left(y^{\Delta\Delta}(t)\right)\right)^{\Delta} + p(t)\phi_{\beta}\left(y^{\Delta}(h(t))\right) + c_1^{\beta}q(t)g^{\beta}(t)\phi_{\beta}(y(g(t))) \leqslant 0 \quad \text{for } t \geqslant t_3, \tag{2.14}$$

where $y(t) := x^{\Delta}(t) > 0$ for $t \geqslant t_3$. Now, we see that $y(t) > 0$ and $y^{\Delta}(t) < 0$ and we must have $y^{\Delta\Delta}(t) > 0$, for otherwise, we obtain a contradiction to $y(t) > 0$. For $v \geqslant u \geqslant t_3$ we have

$$y(u) \geqslant y(u) - y(v) = -\int_{u}^{v} y^{\Delta}(\tau)\Delta\tau = \left(-y^{\Delta}(v)\right)(v-u).$$

Setting $u = g(t)$ and $v = h(t)$, we get

$$y(g(t)) \geqslant \left(-y^{\Delta}(h(t))\right)(h(t)-g(t)) \quad \text{for } t \geqslant t_3. \tag{2.15}$$

Using (2.15) in inequality (2.14), we get

$$\left(a(t)\phi_{\alpha}\left(w^{\Delta}(t)\right)\right)^{\Delta} \geqslant \left(c_1^{\beta}q(t)g^{\beta}(t)(h(t)-g(t))^{\beta} - p(t)\right)\phi_{\beta}(w(h(t))) \quad \text{for } t \geqslant t_3,$$

where $w(t) = -y^{\Delta}(t) > 0$ for $t \geqslant t_3$. Proceeding as in the proofs of Lemma 2 and Lemma 3, we arrive at the desired conclusion completing the proof of the theorem.

Next, we consider equation (1.3). Set

$$\bar{q}(t) := \int_b^c q_1(t,\tau)\Delta\tau, \quad g_1(t,c) := \bar{g}(t) \quad \text{and} \quad \bar{g}(t) \leqslant h(t),$$

and

$$\bar{Q}(t) := c\bar{q}(t)\left(\bar{g}(t)\right)^{\beta}\left(h(t)-\bar{g}(t)\right)^{\beta} - p(t), \quad \text{for } t \geqslant t_0 \in \mathbb{T}.$$

Now, we have the following oscillation result for equation (1.3).

THEOREM 2. *Let the hypotheses of Theorem 2.1, (iv) and (v) hold, with Q be replaced by $\bar{Q}$. Then the conclusion of Theorem 2.1 holds.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.3), say $x(t) > 0$, and $x(g_1(t,\tau)) > 0$ for $t \geqslant t_0 \in \mathbb{T}$ and $\tau \in [b,c]$. Proceeding as in the proof of Theorem 2.1, we consider the two Cases (I) and (II). The proof of Case (I) is similar to that of Theorem 2.1 and is omitted. Next, if Case (II) holds, one can easily find

$$\left(a(t)\phi_{\alpha}\left(x^{\Delta\Delta\Delta}(t)\right)\right)^{\Delta} + p(t)\phi_{\beta}\left(x^{\Delta\Delta}(h(t))\right) + \left(\int_b^c q_1(t,\tau)\Delta\tau\right)\phi_{\beta}\left(x\left(g_1(t,c)\right)\right) \leqslant 0,$$

or

$$\left(a(t)\phi_{\alpha}\left(x^{\Delta\Delta\Delta}(t)\right)\right)^{\Delta} + p(t)\phi_{\beta}\left(x^{\Delta\Delta}(h(t))\right) + \bar{q}(t)\phi_{\beta}\left(x\left(\bar{g}(t)\right)\right) \leqslant 0.$$

The rest of the proof is similar the proof of Theorem 2.1-Case (II) and is omitted. This completes the proof of the theorem.

General Remarks

1. The results of this paper are presented in a form that is essentially new and of a high degree of generality.

2. The results here are valid for various types of time scales e.g., $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ and $\mathbb{T} = \mathbb{N}_0^2$, etc. (see [1]).

3. Finally, we note that our oscillation results are applicable to equations (1.1) and (1.3) if $h(t) < t$. Thus as is well known it is the delay in equation (1.1) or (1.3) that can generate the oscillation.

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