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SPECTRAL ANALYSIS OF A NONLINEAR BOUNDARY–VALUE PROBLEM IN A PERFORATED DOMAIN. APPLICATIONS TO THE FRIEDRICHS INEQUALITY IN L_p

YULIA KOROLEVA

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Abstract. The paper deals with asymptotic expansion for p-Laplace boundary-value problem in a domain periodically perforated along the boundary. It is assumed that the later boundary of the domain is subject to the Neumann boundary condition while the Dirichlet condition is set on the boundary of small sets. The asymptotic expansion for the first eigenelement is constructed. This result is applied to derive the asymptotics of the best constant in the Friedrichs inequality.

1. Introduction

The homogenization of problems for differential operators in perforated domains has been addressed in fairly numerous works (see e.g., [3], [7], [19], [20], [22], [25]–[30] and the references therein). The basic goal is to determine a homogenized (limiting) problem, compute the convergence rate of solutions to the original problem and construct asymptotics of solutions (whenever possible). The direction in which the eigenvalue of the original problem shifts from that of the homogenized problem can frequently be determined from variational considerations. For example, in the case when a Dirichlet boundary condition is specified on small subsets of the boundary, the eigenvalue of the perturbed problem is obviously larger than the eigenvalue of the homogenized problem. However, for the case of Dirichlet boundary conditions on perforation inside of the domain and Neumann boundary conditions on the outer boundary such arguments are not applicable.

In this paper, we consider a singularly perturbed eigenvalue problem for the operator $-\Delta_p$ in n-dimensional domain that is periodically perforated along its boundary in the case when the diameter of the cavities, the distances between them, and the distance to the boundary are of the same order of smallness. A Dirichlet boundary condition is set on the boundaries of the cavities, while the Neumann boundary condition is specified on the remaining part of the boundary. For this problem, it can be shown that homogenized problem is one in the domain without perforation with a Dirichlet

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boundary condition on that part of the boundary near which the small perforation were situated (a detailed proof of this convergence can be found in [17]). Asymptotics of the eigenvalues to analogous spectral problems in case $p=2$ were studied in [7], [8] and [10] for different geometrical settings, in particular, for dimensions 2 and 3. It was shown that the first eigenvalue of the original problem is less than the first eigenvalue of the limit one. Let us remark that such a result can not be derived directly from variational definition of the first eigenvalues. Our goal now is to study asymptotics of the problem for the case $p>2$ in $n-$dimensional domain. Such problem requires more complicated analysis since it leads to nonlinear spectral problem. Especial interest in case $p>2$ is motivated by applications to the Friedrichs inequality for L_p space, $p>2$. For the perturbed problem, we construct a two-term asymptotic expansion of the first eigenvalue. We apply the constructed asymptotics of the considered spectral problem to estimate the best constant in the Friedrichs inequality in perforated domain. The validity of such inequality was proved in [17], while the bounds for the sharp constant were derived in [7]-[10] for $p=2$ and $n=2$.

2. Statement of the problem

Let $\Omega\subset\mathbb{R}^{n-1}\times\{x_n>0\}, n>2$ be a domain with boundary $\partial\Omega=\Gamma$. We will use the notation $\widehat{x}=(x_1,\ldots,x_{n-1})$. It is assumed that $\Gamma=\Gamma_1\cup\Gamma_{out}\cup\Gamma_0$ is piece-wise smooth and consists of the parts $\Gamma_0=[-1/2;1/2]^{n-1}\cap\{x_n=0\}$, Γ_1 is the smooth surface given by $x_n=f(\widehat{x})$. The part

$$\Gamma_{out}=\bigcup_{i=1}^{n-1}\Big[-\frac{1}{2};\frac{1}{2}\Big]^{n-2}\cap\{x_i=\pm\frac{1}{2}\}\cap\{0<x_n<f(\widehat{x})\}.$$

In the sequel $\varepsilon=\frac{1}{2\mathscr{N}+1}$ is a small parameter, $\mathscr{N}\in\mathbb{N}$, $\mathscr{N}\gg 1$. Moreover, we can represent the domain Ω as the union of parallelepipeds:

$$\Omega=\bigcup_{(i_1\ldots i_{n-1})}\Pi_\varepsilon^{i_1\ldots i_{n-1}},\ i_k=0,\ldots,\mathscr{N},\ k=\overline{1,n-1},$$

where

$$\Pi_\varepsilon^{i_1\ldots i_{n-1}}=\Big\{(\widehat{x},x_n)\in\Omega:\big(i_k-\frac{1}{2}\big)\varepsilon\leqslant x_k\leqslant\big(i_k+\frac{1}{2}\big)\varepsilon, 0\leqslant x_n\leqslant f(\widehat{x})\Big\}, k=\overline{1,n-1}.$$

Let the constants α,β satisfy the condition $0<\alpha<1,\quad 0\leqslant\beta$. Introduce small parameters $\varepsilon_1=\alpha\varepsilon,\quad \varepsilon_2=\beta\varepsilon$. Define a set of balls with radius ε_1 centered at points $\widehat{x_0}=(x_{1,0}^{i_1\ldots i_{n-1}},\ldots,x_{n-1,0}^{i_1\ldots i_{n-1}},\frac{1}{2})\in\Pi_\varepsilon^{i_1\ldots i_{n-1}}$:

$$K_\varepsilon^{i_1\ldots i_{n-1}}=\Big\{(\widehat{x},x_n)\in\Pi_\varepsilon^{i_1\ldots i_{n-1}}:\\ (x_1-x_{1,0}^{i_1\ldots i_{n-1}})^2+\ldots+(x_{n-1}-x_{n-1,0}^{i_1\ldots i_{n-1}})^2+(x_n-\frac{1}{2})^2\leqslant\varepsilon_1^2\Big\}.$$

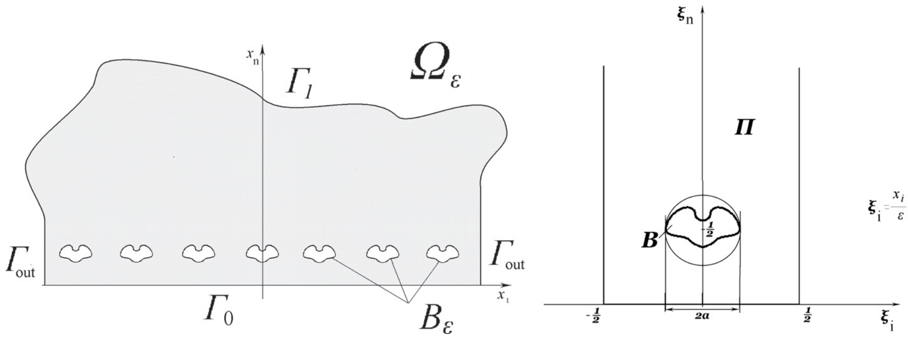

Figure 1: Structure of Ω_ε

Due to the construction, the centers of each ball are located on the distance $\frac{\varepsilon}{2}$ from the boundary Γ_0. The distances between centers of neighboring balls are ε_2 :

$$|x_{k,0}^{i_1\dots i_l \dots i_{n-1}} - x_{k,0}^{i_1\dots i_{l+1}\dots i_{n-1}}| = \varepsilon_2,\ k = \overline{1, n-1}.$$

Consider the set

$$B_\varepsilon^{i_1\dots i_{n-1}} \subset K_\varepsilon^{i_1\dots i_{n-1}} \text{ such that } \frac{meas(B_\varepsilon^{i_1\dots i_{n-1}})}{meas(K_\varepsilon^{i_1\dots i_{n-1}})} = const,$$

i.e. we choose the subset of the ball with equivalent measure. Denote

$$B_\varepsilon = \bigcup_{(i_1\dots i_{n-1})} B_\varepsilon^{i_1\dots i_{n-1}},\ \Gamma_\varepsilon = \partial B_\varepsilon.$$

According to this definition, the small set B_ε is the 1- periodic translation of the fixed $B_\varepsilon^{1\dots 1}$ with respect to vector $(i_1,\dots,i_{n-1})$ over the domain Ω. Moreover, we shall assume that the boundary of set $B_\varepsilon^{1\dots 1}$ is given by a smooth function. We denote, finally, $B = \{\left(\frac{\widehat{x}}{\varepsilon}, \frac{x_n}{\varepsilon}\right) : (\widehat{x}, x_n) \in B_\varepsilon^{1\dots 1}\}$. The figure demonstrates an example of set B in the periodic cell Π. Define the perforated domain Ω_ε as $\Omega \setminus \overline{B_\varepsilon}$. See the illustration for cut of Ω_ε on fig. 1.

In the sequel the space

$$W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon) := \{u \in W^{1,p}(\Omega_\varepsilon) :\ u|_{\Gamma_\varepsilon = 0}\}$$

will be used. We remind also the following important fact which holds for spaces $W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$ (for the proof see e.g. [6], [15], [16], [17] for different p, n and different types of perforations).

THEOREM 1. (Friedrichs inequality) *There exists a constant $K > 0$ independent on ε such that the following Friedrichs inequality*

$$\int_{\Omega_\varepsilon} |u|^p\, dx \leqslant K \int_{\Omega_\varepsilon} |\nabla u|^p\, dx \text{ holds for } u \in W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon). \tag{2.1}$$

DEFINITION 1. For $2 \leqslant p < n$ define the operator

$$\Delta u_p(x) \equiv \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(|\nabla u(x)|^{p-2} \frac{\partial u(x)}{\partial x_j} \right).$$

We are interested to obtain the asymptotics with respect to ε of the solution to the following spectral problem:

$$\begin{cases} -\Delta_p u_\varepsilon(x) = \lambda_\varepsilon |u_\varepsilon|^{p-2} u_\varepsilon & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial_p \nu} \equiv |\nabla u|^{p-2} (\nabla u, \nu) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where ν is the unit outward normal vector to the boundary of Ω.

DEFINITION 2. One say that λ_ε is an eigenfunction to the problem (2.2) if there exists $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon) \setminus \{0\}$, satisfying the integral identity

$$\int\limits_{\Omega_\varepsilon} \sum_{j=1}^{n} |\nabla u_\varepsilon(x)|^{p-2} \frac{\partial u_\varepsilon(x)}{\partial x_j} \frac{\partial \varphi_\varepsilon(x)}{\partial x_j} \, dx = \lambda_\varepsilon \int\limits_{\Omega_\varepsilon} |u_\varepsilon|^{p-2} u_\varepsilon \varphi_\varepsilon \, dx \tag{2.3}$$

for every $\varphi_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$. The couple $(u_\varepsilon, \lambda_\varepsilon)$ is called the solution to (2.2).

One can extend the functions $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$ into B_ε by zero. For the extended function we keep the same notation. It is true that u_ε belongs to $W^{1,p}(\Omega)$, see [21]. Our goal now is to construct the asymptotics for eigenelements to problem (2.2). The main result with asymptotics of the first eigenvalue is given in Theorem 5.

3. Asymptotic expansions

To construct the asymptotics we use the method of matching of asymptotic expansions (see [14], and also [2], [5], [12], [13]). It was shown in [17] the existence of eigenelement (u_0, λ_0), $u_0 \in C^\infty(\overline{\Omega})$ such that

$$u_\varepsilon \to u_0, \quad \lambda_\varepsilon \to \lambda_0 \text{ as } \varepsilon \to 0.$$

In the sequel the eigenfunction u_0 is chosen to be normalized in $L_p(\Omega)$. Therefore it is naturally to seek the asymptotics of the solution u_ε in the form

$$u_\varepsilon(x) \sim \widehat{u}_\varepsilon(x) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2. \tag{3.1}$$

We will call (3.1) " the outer expansion" assuming that it holds outside a small neighborhood of Γ_0, mainly as $x_n > \varepsilon^\beta$, $0 < \beta < 1$. Analogously, we will seek the eigenvalue λ_ε as the formal expansion

$$\lambda_\varepsilon \sim \widehat{\lambda}_\varepsilon = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2. \tag{3.2}$$

Now we find out the boundary-value problems for u_0, u_1, u_2. Substitute expansions into the boundary-value problem (2.2). By using formulas

$$
\begin{aligned}
|u_0+\varepsilon u_1+\varepsilon^2 u_2+\ldots)|^{p-2} = & |u_0|^{p-2}+\varepsilon(p-2)|u_0|^{p-3}|u_1| \\
& +\varepsilon^2\frac{(p-2)}{2}(2u_2|u_0|^{p-3}+(p-3)|u_0|^{p-4}u_1^2)+o(\varepsilon^2),
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
\lambda_\varepsilon|u_\varepsilon|^{p-2}u_\varepsilon = & \lambda_0|u_0|^{p-2}u_0+\varepsilon((\lambda_1 u_0+\lambda_0 u_1)|u_0|^{p-2} \\
& +(p-2)\lambda_0 u_0|u_0|^{p-3}|u_1|) \\
& +\varepsilon^2[|u_0|^{p-2}(\lambda_2 u_0+\lambda_1 u_1+\lambda_0 u_2)+\frac{(p-2)}{2}\lambda_0 u_0(2u_2|u_0|^{p-3} \\
& +(p-3)|u_0|^{p-4}u_1^2)+(p-2)|u_0|^{p-3}|u_1|(\lambda_1 u_0+\lambda_0 u_1)]+o(\varepsilon^2),
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
|\nabla u_\varepsilon|^{p-2} = & (|\nabla u_0|^2+2\varepsilon(\nabla u_0,\nabla u_1)+\varepsilon^2(2(\nabla u_2,\nabla u_0)+|\nabla u_1|^2)+\ldots)^{\frac{p-2}{2}} \\
= & |\nabla u_0|^{p-2}+\varepsilon(p-2)|\nabla u_0|^{p-4}(\nabla u_0,\nabla u_1) \\
& +\varepsilon^2\frac{p-2}{2}|\nabla u_0|^{p-6}(2(\nabla u_2,\nabla u_0)|\nabla u_0|^2 \\
& +|\nabla u_0|^2|\nabla u_1|^2+(p-4)(\nabla u_0,\nabla u_1)^2)+o(\varepsilon^2),
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
& \frac{\partial}{\partial x_j}\left(|\nabla u_\varepsilon|^{p-2}\left(\frac{\partial u_0}{\partial x_j}+\varepsilon\frac{\partial u_1}{\partial x_j}+\varepsilon^2\frac{\partial u_2}{\partial x_j}+o(\varepsilon^2)\right)\right) \\
& = \frac{\partial}{\partial x_j}\left(|\nabla u_0|^{p-2}\left(\frac{\partial u_0}{\partial x_j}\right)\right) \\
& \quad +\varepsilon\frac{\partial}{\partial x_j}\left((p-2)|\nabla u_0|^{p-4}(\nabla u_0,\nabla u_1)\frac{\partial u_0}{\partial x_j}+|\nabla u_0|^{p-2}\frac{\partial u_1}{\partial x_j}\right) \\
& \quad +\varepsilon^2\left[\frac{\partial}{\partial x_j}\left(\left(\frac{p-2}{2}|\nabla u_0|^{p-6}(2(\nabla u_2,\nabla u_0)|\nabla u_0|^2\right.\right.\right. \\
& \quad \left.\left.+|\nabla u_0|^2|\nabla u_1|^2+(p-4)(\nabla u_0,\nabla u_1)^2)\right)\frac{\partial u_0}{\partial x_j}\right) \\
& \quad \left.+\frac{\partial}{\partial x_j}\left(|u_0|^{p-2}\frac{\partial u_2}{\partial x_j}\right)+\frac{\partial}{\partial x_j}\left(2(\nabla u_0,\nabla u_1)\frac{\partial u_1}{\partial x_j}\right)\right]+o(\varepsilon^2),
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
|\nabla u_\varepsilon|^{p-2}\frac{\partial u_\varepsilon}{\partial \nu} = & |\nabla u_0|^{p-2}\frac{\partial u_0}{\partial \nu} \\
& +\varepsilon\left((p-2)|\nabla u_0|^{p-4}(\nabla u_0,\nabla u_1)\frac{\partial u_0}{\partial \nu}+|\nabla u_0|^{p-2}\frac{\partial u_1}{\partial \nu}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon^2\Bigg[\bigg(\frac{p-2}{2}|\nabla u_0|^{p-6}(2(\nabla u_2,\nabla u_0)|\nabla u_0|^2\\
&+|\nabla u_0|^2|\nabla u_1|^2+(p-4)(\nabla u_0,\nabla u_1)^2)\bigg)\frac{\partial u_0}{\partial \nu}\\
&+|\nabla u_0|^{p-2}\frac{\partial u_2}{\partial \nu}+2(\nabla u_0,\nabla u_1)\frac{\partial u_1}{\partial \nu}\Bigg]+o(\varepsilon^2),
\end{aligned}
\tag{3.7}
$$

and collecting the terms with ε^0 we obtain the following problem for (u_0,λ_0) :

$$
\begin{cases}
-\Delta_p u_0(x)=\lambda_0|u_0|^{p-2}u_0 & \text{in } \Omega,\\
u_0=0 & \text{on } \Gamma_0,\\
\frac{\partial u_0}{\partial_p \nu}\equiv|\nabla u_0|^{p-2}\frac{\partial u_0}{\partial \nu}=0 & \text{on } \partial\Omega\setminus\overline{\Gamma}_0.
\end{cases}
\tag{3.8}
$$

As usually, we understand the solution to this boundary-value problem in the weak sense i.e. iff $u_0\in W^{1,p}(\Omega,\Gamma_0)\setminus\{0\}$, satisfies

$$
\int\limits_\Omega\sum_{j=1}^n|\nabla u_0(x)|^{p-2}\frac{\partial u_0(x)}{\partial x_j}\frac{\partial \varphi(x)}{\partial x_j}\,dx=\lambda_0\int\limits_\Omega|u_0|^{p-2}u_0\varphi\,dx
\tag{3.9}
$$

for every $\varphi\in W^{1,p}(\Omega,\Gamma_0)$. The existence and uniqueness of the smooth $(C^\infty(\overline{\Omega}))$ normalized solution follows from the general theory of monotone nonlinear elliptic operators. The detailed proof is given in [17]. In addition, the following fact was proved in [17].

THEOREM 2. *The spectrum of problems (2.2) and (3.8) is nonempty closed set. Let $\lambda_\varepsilon^1,\lambda_0^1$ be the first eigenvalues of problems (2.2) and (3.8) respectively. Then*

$$
\lambda_\varepsilon^1\to\lambda_0^1\quad \text{as } \varepsilon\to 0.
$$

Moreover, if u_ε, u_0 are corresponding eigenfunctions, normalized in L_p, then up to a subsequence,

$$
\|u_\varepsilon-u_0\|_{W^{1,p}}\to 0.
$$

The main goal in present paper is to derive the asymptotics of λ_ε^1 and estimate the rate of the convergence λ_ε^1 to λ_0^1. This result can be applied to estimate the sharp constant in the Friedrichs inequality (2.1) (see Section 7).

Analogously, collecting the terms in front of ε, and ε^2 one obtain the following boundary-value problems for unknown u_1 and u_2 functions:

$$
\begin{cases}
\sum\limits_j\frac{\partial}{\partial x_j}\left((p-2)|\nabla u_0|^{p-4}(\nabla u_0,\nabla u_1)\frac{\partial u_0}{\partial x_j}+|\nabla u_0|^{p-2}\frac{\partial u_1}{\partial x_j}\right) & \\
\qquad\qquad=(\lambda_1u_0+\lambda_0u_1)|u_0|^{p-2}+(p-2)\lambda_0u_0|u_0|^{p-3}u_1, & \text{in } \Omega,\\
u_1=\alpha_1^0(\widehat{x}) & \text{on } \Gamma_0,\\
\frac{\partial u_1}{\partial \nu}=0 & \text{on } \partial\Omega\setminus\overline{\Gamma}_0.
\end{cases}
\tag{3.10}
$$

$$\begin{cases} -\sum\limits_j \Big[\frac{\partial}{\partial x_j}\Big(\Big(\frac{p-2}{2}|\nabla u_0|^{p-6}(2(\nabla u_2,\nabla u_0)|\nabla u_0|^2+|\nabla u_0|^2|\nabla u_1|^2 \\ \qquad +(p-4)(\nabla u_0,\nabla u_1)^2)\Big)\frac{\partial u_0}{\partial x_j}\Big) \\ \qquad +\frac{\partial}{\partial x_j}\Big(|u_0|^{p-2}\frac{\partial u_2}{\partial x_j}\Big)+\frac{\partial}{\partial x_j}\Big(2(\nabla u_0,\nabla u_1)\frac{\partial u_1}{\partial x_j}\Big)\Big] \\ = |u_0|^{p-2}(\lambda_2 u_0+\lambda_1 u_1+\lambda_0 u_2)+\frac{(p-2)}{2}\lambda_0 u_0(2u_2|u_0|^{p-3}+|u_0|^{p-4}u_1^2) \\ \quad +(p-2)|u_0|^{p-3}|u_1|(\lambda_1 u_0+\lambda_0 u_1) & \text{in } \Omega, \\ u_2=\alpha_2^0(\widehat{x}) & \text{on } \Gamma_0, \\ \frac{\partial u_2}{\partial \nu}=0 & \text{on } \partial\Omega\setminus\overline{\Gamma}_0. \end{cases} \tag{3.11}$$

The existence of the solution for (3.10) and (3.11) follows from the general theory of linear elliptic operators with smooth coefficients. It is given in Lemma 1.

4. The internal expansion

However, the approximation $\widehat{u}_\varepsilon(x)\equiv u_0+\varepsilon u_1+\varepsilon^2 u_2$ does not satisfy to the condition on Γ_ε. This forces us to introduce an additional term in the asymptotic expansion for u_ε to satisfy the appropriate boundary condition. In a small neighborhood of Γ_0, mainly on the set $\{(\widehat{x},x_n):\left[-\frac{1}{2};\frac{1}{2}\right]^{n-1},\ 0\leqslant x_n<2\varepsilon^\beta\},\ 0<\beta<1,$ we construct an "internal expansion". The solution $u_0(x)\in C^\infty(\overline{\Omega})$ satisfies to the problem (3.8), is even with respect to x_i. With this property, one has where

$$u_0(\widehat{x},0)=0,\qquad \frac{\partial u_0}{\partial x_i}\bigg|_{x_i=\pm\frac{1}{2}}=0\ \text{ on } \Gamma_{out}.$$

Due to the regularity, the equation

$$\Delta_p u_0=\lambda_0|u_0|^{p-2}u_0$$

holds up to the boundary, hence, $\Delta_p u_0=0$ and $\frac{\partial u_0}{\partial x_i}=0$ on $\Gamma_0,\ i=\overline{1,n-1}$. Now we rewrite the p-Laplace operator as follows:

$$\Delta_p u_0=|\nabla u_0|^{p-4}\Big(|\nabla u_0|^{p-4}\Delta u_0+(p-2)\sum_{i,j=1}^n\frac{\partial u_0}{\partial x_i}\frac{\partial u_0}{\partial x_j}\frac{\partial^2 u_0}{\partial x_i\partial x_j}\Big).$$

On Γ_0 it reduces to

$$0=\Delta_p u_0=\Big(\frac{\partial u_0}{\partial x_n}\Big)^{p-4}\Big(\Big(\frac{\partial u_0}{\partial x_n}\Big)^2\frac{\partial^2 u_0}{\partial x_n^2}+(p-2)\Big(\frac{\partial u_0}{\partial x_n}\Big)^2\frac{\partial^2 u_0}{\partial x_n^2}\Big)$$
$$=(p-1)\Big(\frac{\partial u_0}{\partial x_n}\Big)^{p-2}\frac{\partial^2 u_0}{\partial x_n^2}.$$

From this we conclude that

$$\frac{\partial^2 u_0}{\partial x_n^2} = 0 \text{ on } \Gamma_0. \tag{4.1}$$

Thus, one can write the following expansion in the Taylor series with respect to x_n :

$$u_0(x) = \alpha_0^1(\widehat{x})x_n + O(x_n^3) \quad \text{as } x_n \to 0, \tag{4.2}$$

where

$$\alpha_0^1 = \left.\frac{\partial u_0}{\partial x_n}\right|_{x_n=0} \in C^\infty\left([-1/2,1/2]^{n-1}\right) \tag{4.3}$$

satisfies

$$\left.\frac{\partial \alpha_0^1}{\partial x_i}\right|_{x_i=\pm\frac{1}{2}} = 0, \qquad i = \overline{1,n-1}, \text{ on } \Gamma_{out}. \tag{4.4}$$

Due to the smoothness (see Lemma 1), the functions u_1, u_2 can also be decomposed in the Taylor series

$$\begin{aligned} u_1(x) &= \alpha_1^0(\widehat{x}) + \alpha_1^1(\widehat{x})x_n + O(x_n^2), \\ u_2(x) &= \alpha_2^0(\widehat{x}) + O(x_n) \end{aligned} \tag{4.5}$$

as $x_n \to 0$. Moreover, $\alpha_1^1, \alpha_2^0 \in C^\infty\left([-1/2,1/2]^{n-1}\right)$ and due to (3.10)

$$\left.\frac{\partial \alpha_1^1}{\partial x_i}\right|_{x_i=\pm\frac{1}{2}} = 0, \; \left.\frac{\partial \alpha_2^0}{\partial x_i}\right|_{x_i=\pm\frac{1}{2}} = 0, \; i = \overline{1,n-1} \text{ on } \Gamma_{out}. \tag{4.6}$$

LEMMA 1. *Let $\alpha_1^0, \alpha_2^0 \in C^\infty\left([-1/2,1/2]^{n-1}\right)$ and satisfies (4.6). Then there exist constants λ_1, λ_2 and functions $u_1(x), u_2(x) \in C^\infty(\overline{\Omega})$, solving the problems (3.10), (3.11) and satisfying $\int_\Omega |u_0|^{p-2}u_0 u_1\,dx = 0$. Moreover, λ_1 can be find as*

$$\lambda_1 = -(p-1)\int_{\Gamma_0} |\nabla u_0|^{p-2}\alpha_0^1\alpha_1^0\,d\widehat{x}. \tag{4.7}$$

Proof. The existence of the smooth solutions u_1, u_2 follows from the classical results on regular solutions of linear elliptic operators with smooth coefficients (see e.g., [1]). In order to get u_1 as the unique solution one can add the condition

$$\int_\Omega |u_0|^{p-2}u_0 u_1\,dx = 0. \tag{4.8}$$

By multiplying (3.10) with u_0 and twice integrating by parts over Ω the obtained equation, we find that

$$\lambda_1\int_\Omega |u_0|^p\,dx + \lambda_0(p-1)\int_\Omega u_1 u_0|u_0|^{p-2}\,dx$$

$$
\begin{aligned}
&= -\int\limits_{\Omega} \sum_j \frac{\partial}{\partial x_j} \left((p-2)|\nabla u_0|^{p-4} (\nabla u_0, \nabla u_1) \frac{\partial u_0}{\partial x_j} + |\nabla u_0|^{p-2} \frac{\partial u_1}{\partial x_j} \right) u_0 \, dx \\
&= (p-1) \int\limits_{\Omega} |\nabla u_0|^{p-2} (\nabla u_0, \nabla u_1) \, dx = -(p-1) \int\limits_{\Omega} \sum_j \frac{\partial}{\partial x_j} \left(|\nabla u_0|^{p-2} \frac{\partial u_0}{\partial x_j} \right) u_1 \\
&\quad + (p-1) \int\limits_{\partial\Omega} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial \nu} u_1 \, d\Omega = \lambda_0 (p-1) \int\limits_{\Omega} |u_0|^{p-2} u_0 u_1 \, dx \\
&\quad - (p-1) \int\limits_{\Gamma_0} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial x_n} u_1 \, d\widehat{x}.
\end{aligned} \tag{4.9}
$$

Taking into account the fact that u_0 is the normalized in $L_p(\Omega)$ solution of (3.8) and since u_1 satisfies (4.8), we can deduce that

$$
\lambda_1 = -(p-1) \int\limits_{\Gamma_0} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial x_n} u_1 \, d\widehat{x} = -(p-1) \int\limits_{\Gamma_0} |\nabla u_0|^{p-2} \alpha_0^1 \alpha_1^0 \, d\widehat{x}. \tag{4.10}
$$

The formula (4.10) proves (4.7) and the proof is complete. $\square$

Insert (4.2) and (4.5) into (3.1) and make the substitution $\xi_n = \frac{x_n}{\varepsilon}$. It yields

$$
u_\varepsilon(x) = \varepsilon V_1(\xi_n; \widehat{x}) + \varepsilon^2 V_2(\xi_n; \widehat{x}) + O(x_n^3 + \varepsilon^2 + \varepsilon^2 x_2) \tag{4.11}
$$

as $\varepsilon \xi_n = x_n \to 0$, where

$$
V_1(\xi_n; \widehat{x}) = \alpha_0^1(\widehat{x}) \xi_n + \alpha_1^0(\widehat{x}), \quad V_2(\xi_2; \widehat{x}) = \alpha_1^1(\widehat{x}) \xi_n + \alpha_2^0(\widehat{x}). \tag{4.12}
$$

Following the method of matched asymptotic expansions, we conclude that the inner expansion must have the structure

$$
u_\varepsilon(x) = \varepsilon v_1(\xi; \widehat{x}) + \varepsilon^2 v_2(\xi; \widehat{x}) + O(\varepsilon^3), \tag{4.13}
$$

where $\xi = (\xi_1, \dots \xi_n) = \frac{x}{\varepsilon} = \left(\frac{x_1}{\varepsilon}, \dots \frac{x_n}{\varepsilon} \right)$. Here x_i is a "slow" variable while ξ_i is a "fast" one. Denote by $\widehat{v}_\varepsilon(x) = \varepsilon v_1 + \varepsilon^2 v_2$. Definition (4.12) implies the asymptotics

$$
v_1(\xi; \widehat{x}) \sim V_1(\xi_n; \widehat{x}), \; v_2(\xi; \widehat{x}) \sim V_2(\xi_n; \widehat{x}) \qquad \text{as} \quad \xi_n \to +\infty. \tag{4.14}
$$

REMARK 1. In the sequel we construct v_1 and v_2 as 1-periodic functions with respect to ξ_i (depending on a fixed "slow" parameter x_i).

With new scaling, the domain Ω can be divided in the union of periodic cells $\Pi = \Omega \cap \{\xi : -\frac{1}{2} < \xi_i < \frac{1}{2}, \, \xi_n > 0\}$.
Let $\gamma = \{\xi : -\frac{1}{2} < \xi_i < \frac{1}{2}, \, \xi_n = 0\}$. Rewriting Δ_p operator in $(\xi, \widehat{x})$ variables and applying it to the function $\widehat{v}_\varepsilon$, we get

$$
\Delta_p \widehat{v}_\varepsilon = \sum_{i=1}^{n-1} \left[\frac{\partial}{\partial x_i} \left(\left| \frac{\partial \widehat{v}_\varepsilon}{\partial x_i} \right|^{p-2} \frac{\partial \widehat{v}_\varepsilon}{\partial x_i} \right) \right.
$$

$$
\begin{aligned}
&+\frac{1}{\varepsilon}\left(\frac{\partial}{\partial \xi_i}\left(\left|\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}+\frac{1}{\varepsilon}\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-2}\left(\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}+\frac{1}{\varepsilon}\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right)\right)\right)\Bigg]\\
&+\frac{1}{\varepsilon^p}\frac{\partial}{\partial \xi_n}\left(\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_n}\right|^{p-2}\left(\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_n}\right)\right)\\
=&\sum_{i=1}^{n-1}\Bigg[\frac{\partial}{\partial x_i}\left(\left|\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right|^{p-2}\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right)\\
&+\frac{1}{\varepsilon^p}\left(\frac{\partial}{\partial \xi_i}\left(\left|\varepsilon\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}+\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-2}\left(\varepsilon\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}+\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right)\right)\right)\Bigg]\\
&+\frac{1}{\varepsilon^p}\frac{\partial}{\partial \xi_n}\left(\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_n}\right|^{p-2}\left(\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_n}\right)\right).
\end{aligned}
$$

The Taylor decomposition implies that

$$
\begin{aligned}
&\left|\varepsilon\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}+\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-2}\left(\varepsilon\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}+\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right)\\
&\quad=\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-2}\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}+\varepsilon\left((p-2)\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-3}\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}+\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-2}\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right)\\
&\qquad+\varepsilon^2\Bigg((p-3)(p-2)\left|\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right|^{2}\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-4}\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\\
&\qquad\qquad+(p-2)\left(\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right)^2\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-3}\Bigg)+O(\varepsilon^3).
\end{aligned}
$$

Substitute this inside of previous formula, we get the following:

$$
\begin{aligned}
\Delta_p\widehat{v}_\varepsilon=&\sum_{i=1}^{n-1}\Bigg[\frac{\partial}{\partial x_i}\left(\left|\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right|^{p-2}\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right)\\
&+\frac{1}{\varepsilon^p}\frac{\partial}{\partial \xi_i}\Bigg(\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-2}\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}+\varepsilon\left((p-2)\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-3}\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}+\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-2}\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right)\\
&+\varepsilon^2\left((p-3)(p-2)\left|\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right|^{2}\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-4}\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}+(p-2)\left(\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right)^2\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-3}\right)\Bigg)\\
&+\frac{1}{\varepsilon^p}\frac{\partial}{\partial \xi_n}\left(\left|\frac{\partial \overline{u}_\varepsilon}{\partial \xi_n}\right|^{p-2}\left(\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_n}\right)\right)\Bigg]+O(\varepsilon^{p-3}).
\end{aligned}
$$

Using the expansions for Δ_p (see the previous step) and collecting the terms in front of the same power of ε in the equation $-\Delta_p\widehat{v}_\varepsilon=\lambda_\varepsilon|\widehat{v}_\varepsilon|^{p-2}\widehat{v}_\varepsilon,$ we deduce that

$$
-\Delta_p^{\xi}v_1=0,\ \text{where}\ \Delta_p^{\xi}v_1=\sum_{i=1}^{n}\frac{\partial}{\partial \xi_i}\left(\left|\frac{\partial v_1}{\partial \xi_i}\right|^{p-2}\frac{\partial v_1}{\partial \xi_i}\right).
$$

The boundary conditions

$$\widehat{v}_\varepsilon = 0 \text{ on } \Gamma_\varepsilon$$

implies $v_1 = v_2 = 0$ on ∂B. On the part Γ_0 one has

$$0 = \frac{\partial \widehat{v}_\varepsilon}{\partial_p x_n} = \left|\frac{\partial \widehat{v}_\varepsilon}{\partial x_n}\right|^{p-2} \frac{\partial \widehat{v}_\varepsilon}{\partial x_n} = \frac{1}{\varepsilon^{p-1}} \left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_n}\right|^{p-2} \frac{\partial \widehat{v}_\varepsilon}{\partial \xi_n}.$$

Hence, $\frac{\partial v_1}{\partial \xi_n} = \frac{\partial v_2}{\partial \xi_n} = 0$ on γ. On the boundary Γ_{out} we have

$$\begin{aligned} 0 &= \frac{\partial \widehat{v}_\varepsilon}{\partial_p \nu} = \left|\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right|^{p-2} \frac{\partial \widehat{v}_\varepsilon}{\partial x_i} \\ &= \pm \left|\frac{\partial \widehat{v}_\varepsilon}{\varepsilon \partial \xi_i} + \frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right|^{p-2} \left(\frac{\partial \widehat{v}_\varepsilon}{\varepsilon \partial \xi_1} + \frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right) \\ &= \pm \frac{1}{\varepsilon^{p-2}} \left(\left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-2} + \varepsilon(p-2) \left|\frac{\partial \widehat{v}_\varepsilon}{\partial \xi_i}\right|^{p-3} \left|\frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right| + o(\varepsilon) \right) \left(\frac{\partial \widehat{v}_\varepsilon}{\varepsilon \partial \xi_i} + \frac{\partial \widehat{v}_\varepsilon}{\partial x_i}\right). \end{aligned}$$

Analogously, collecting the terms in front of the same power of ε, we conclude that

$$\sum_i \left|\frac{\partial v_1}{\partial \xi_i}\right|^{p-2} \frac{\partial v_1}{\partial \xi_i} = 0, \; i = 1, \ldots, n$$

on corresponding parts of the boundary Γ_{out}. Thus, the functions v_1 is the solution to the following problem in periodic cell Π:

$$\begin{cases} \Delta_\xi v_1 = 0 \quad \text{in } \Pi \backslash \overline{B}, \\ v_1 = 0 \quad \text{on } \partial B, \\ \left|\frac{\partial v_1}{\partial \xi_n}\right|^{p-2} \frac{\partial v_1}{\partial \xi_n} = 0 \quad \text{on } \gamma, \\ \left|\frac{\partial v_1}{\partial \xi_i}\right|^{p-2} \frac{\partial v_1}{\partial \xi_i}(\xi; x_i = \pm\frac{1}{2}) = 0 \quad \text{as } \xi_i = \pm\frac{1}{2}, \\ v_1 \sim V_1 \quad \text{as } \xi_n \to +\infty. \end{cases} \tag{4.15}$$

An analogous technique one can use to obtain the boundary-value problem for the function v_2 with asymptotics

$$v_2 \sim V_2 \quad \text{as } \xi_n \to +\infty.$$

Here we omit the details due to heavy technical formulas. For an example of explicit boundary-value problem for v_2 and its solution in case $p = n = 2$ we refer to [7].

5. Solvability of problem (4.15).

We analyze problem (4.15) and simultaneously determine the function $\alpha_1^0(\widehat{x})$. The following Lemma is useful for our analysis. The proof of an analogous statement can be find in [23].

LEMMA 2. *Let* $e^{\delta_0\xi_n}F \in L_q(\Pi \setminus B)$ $e^{\delta_0\xi_n}H \in L_p(\partial\Pi)$, $\delta_0 > 0$. *Then there exists the unique weak solution of*

$$\begin{cases} -\Delta_p Z = F & \text{in } \Pi \setminus \overline{B} \\ Z = K & \text{on } \partial B, \\ \dfrac{\partial Z}{\partial \nu} = H & \text{on } \partial\Pi. \end{cases}$$

This solution can be represented as

$$Z(\xi) = C_1 + Z'(\xi),$$

where C_1 *is a constant and* Z' *satisfies* $e^{\delta\xi_n}Z' \in W^{1,p}(\Pi\setminus\overline{B})$, *where* δ *such that* $\delta \leqslant \delta_0$. *Moreover, the asymptotics*

$$|\nabla Z(\xi)|^{p-2} \sim C_2 + O(e^{-\delta\xi_n})$$

holds as $\xi_n \to \infty$.

Consider the problem

$$\begin{cases} \Delta_p Y = 0 & \text{in } \Pi\setminus\overline{B}, \\ Y = -\xi_n & \text{on } \partial B, \\ \frac{\partial Y}{\partial \xi_n} = -1 & \text{on } \gamma, \\ \frac{\partial Y}{\partial \xi_i} = 0 & \text{as } \xi_i = \pm\frac{1}{2}. \end{cases} \tag{5.1}$$

By Lemma 2 there exists the weak solution to boundary-value problem (5.1) such that

$$Y(\xi) = C_1(B) + O(e^{-\delta\xi_n}),\ |\nabla Y(\xi)|^{p-2} \sim C_2(B) + O(e^{-\delta\xi_n}). \tag{5.2}$$

Due to the symmetry of B with respect to axis $\xi_i = 0$ the function Y is even with respect to ξ_i. Denote by $\Pi^R = \Pi \cap \{\xi_n > R\}$, $\gamma_R = \{\xi \in \Pi, \xi_n = R\}$, $y_R = Y\big|_{\gamma_R}$. The function Y is evidently the unique bounded solution of boundary-value problem

$$\begin{cases} \Delta_p Y = 0 & \text{in } \Pi^R, \\ \frac{\partial Y}{\partial \xi_n} = y_R & \text{on } \gamma_R, \\ \frac{\partial Y}{\partial \xi_i} = 0 & \text{on } \partial\Pi_R \setminus \overline{\gamma_R}, \end{cases}$$

when R is large. Thus, it has the asymptotics (5.2) as $\xi_n \to \infty$.

Let us derive the formula for $C(B) = C_1(B)C_2(B)$:

$$C(B) = \int\limits_{\Pi\setminus\overline{B}} |\nabla Y|^p d\xi + \int\limits_B \sum_{i=1}^n \frac{\partial}{\partial \xi_i}(|\nabla Y|^{p-2}\xi_n)\, d\xi. \tag{5.3}$$

Denote $\Pi_R = \Pi \cap \{\xi_n < R\}$. Multiply the equation in (5.1) with $Y + \xi_n$ and integrate the result over $\Pi_R \setminus \overline{B}$.

$$
\begin{aligned}
0 &= \int\limits_{\Pi_R \setminus B} \Delta_p Y (Y+\xi_n)\, d\xi = \int\limits_{\Pi_R \setminus B} \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}\left(|\nabla Y|^{p-2}\frac{\partial Y}{\partial \xi_i}\right)(Y+\xi_n)\, d\xi \\
&= -\int\limits_{\Pi_R \setminus B} \sum_{i=1}^{n} |\nabla Y|^{p-2}\frac{\partial Y}{\partial \xi_i}\frac{\partial (Y+\xi_n)}{\partial \xi_i}\, d\xi - \int\limits_{\gamma} \frac{\partial Y}{\partial \xi_n}|\nabla Y|^{p-2}(Y+0)\, d\widehat{\xi} \\
&\quad + \int\limits_{\Gamma_R} \frac{\partial Y}{\partial \xi_n}|\nabla Y|^{p-2}(Y+\xi_n)\, d\widehat{\xi} + \int\limits_{\partial B} \sum_{i=1}^{n} \frac{\partial Y}{\partial \nu_B}|\nabla Y|^{p-2}(Y+\xi_n)\, dB \\
&\quad + \int\limits_{\partial \Pi} \sum_{i=1}^{n} \frac{\partial Y}{\partial \xi_i}|\nabla Y|^{p-2}(Y+\xi_n)\, d\Pi = -\int\limits_{\Pi_R \setminus B} \sum_{i=1}^{n} |\nabla Y|^{p-2}\frac{\partial Y}{\partial \xi_i}\frac{\partial (Y+\xi_n)}{\partial \xi_i}\, d\xi \\
&\quad + \int\limits_{\gamma} |\nabla Y|^{p-2} Y\, d\widehat{\xi} + \int\limits_{\Gamma_R} O(e^{-\delta R})|\nabla Y|^{p-2}(C+R+O(e^{-\delta R}))\, d\widehat{\xi}.
\end{aligned}
$$

Integrating by parts again we have

$$
\begin{aligned}
0 &= \int\limits_{\Pi_R \setminus B} \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}\left(|\nabla Y|^{p-2}\frac{\partial}{\partial \xi_i}(Y+\xi_n)\right) Y\, d\xi + \int\limits_{\gamma} |\nabla Y|^{p-2} Y\, d\widehat{\xi} \\
&\quad + \int\limits_{\gamma} \frac{\partial}{\partial \xi_n}(Y+\xi_n)|\nabla Y|^{p-2} Y\, d\widehat{\xi} - \int\limits_{\partial B} Y|\nabla Y|^{p-2}\frac{\partial}{\partial \nu_B}(Y+\xi_n)\, ds_\xi \\
&\quad - \int\limits_{\Gamma_R} Y|\nabla Y|^{p-2}\frac{\partial}{\partial \xi_n}(Y+\xi_n)\, d\widehat{\xi} \\
&= \int\limits_{\Pi_R \setminus B} \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}\left(|\nabla Y|^{p-2}\frac{\partial Y}{\partial \xi_i}\right) Y\, d\xi + \int\limits_{\Pi_R \setminus B} \frac{\partial}{\partial \xi_n}\left(|\nabla Y|^{p-2}\frac{\partial \xi_n}{\partial \xi_n}\right) Y\, d\xi \\
&\quad + \int\limits_{\gamma} Y|\nabla Y|^{p-2}\, d\widehat{\xi} + \int\limits_{\partial B} Y|\nabla Y|^{p-2}\frac{\partial Y}{\partial \nu_B}\, ds_\xi - \int\limits_{\partial B} \xi_n|\nabla Y|^{p-2}\frac{\partial \xi_n}{\partial \nu_B}\, ds_\xi \\
&\quad - \int\limits_{\Gamma_R} (C_1(B)+O(e^{-\delta R}))(C_2(B)+O(e^{-\delta R}))(1+O(e^{-\delta R}))\, d\widehat{\xi} \\
&= \int\limits_{\Pi \setminus B} (C_1(B)+O(e^{-\delta R}))O(e^{-\delta R})\, d\xi + \int\limits_{\gamma} Y|\nabla Y|^{p-2}\, d\widehat{\xi} + \int\limits_{\partial B} Y|\nabla Y|^{p-2}\frac{\partial Y}{\partial \nu_B}\, ds_\xi \\
&\quad - \int\limits_{\partial B} \xi_n|\nabla Y|^{p-2}\frac{\partial \xi_n}{\partial \nu_B}\, ds_\xi \\
&\quad - \int\limits_{\Gamma_R} (C_1(B)+O(e^{-\delta R}))(C_2(B)+O(e^{-\delta R}))(1+O(e^{-\delta R}))\, d\widehat{\xi} \to
\end{aligned}
$$

$$\int\limits_{\gamma} Y|\nabla Y|^{p-2}\, d\widehat{\xi} + \int\limits_{\partial B} Y|\nabla Y|^{p-2}\frac{\partial Y}{\partial \nu_B}\, ds_\xi - \int\limits_{\partial B} \xi_n|\nabla Y|^{p-2}\frac{\partial \xi_n}{\partial \nu_B}\, ds_\xi - C(B).$$

Multiply the equation of the problem (5.1) with Y, integrate by parts the obtained equality over $\Pi_R \setminus \overline{B}$ and pass to the limit as $R \to +\infty$. Then we get

$$0 = -\int\limits_{\Pi\setminus\overline{B}} |\nabla Y|^p\, d\xi + \int\limits_{\partial B} \frac{\partial Y}{\partial \nu_B}|\nabla Y|^{p-2} Y\, ds_\xi + \int\limits_{\gamma} Y|\nabla Y|^{p-2}\, d\widehat{\xi}. \tag{5.4}$$

Hence,

$$C(B) = \int\limits_{\Pi\setminus\overline{B}} |\nabla Y|^p\, d\xi - \int\limits_{\partial B} |\nabla Y|^{p-2}\xi_n\frac{\partial \xi_n}{\partial \nu}\, ds_\xi. \tag{5.5}$$

Integrating by parts the left-hand side of

$$0 = \int\limits_{B} |\nabla Y|^{p-2}\xi_n \Delta_p \xi_n d\xi,$$

we get that

$$\int\limits_{\partial B} |\nabla Y|^{p-2}\xi_n\frac{\partial \xi_n}{\partial \nu}\, ds_\xi = \int\limits_{B} \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}(|\nabla Y|^{p-2}\xi_n)\, d\xi, \tag{5.6}$$

where ν is outer normal vector to B. Hence, combining (5.5) and (5.6), we get the formula

$$C(B) = \int\limits_{\Pi\setminus\overline{B}} |\nabla Y|^p\, d\xi + \int\limits_{B} \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}(|\nabla Y|^{p-2}\xi_n)\, d\xi.$$

Define

$$X(\xi) = Y(\xi) + \xi_n.$$

The definition of Y implies that X can be extended as a 1-periodic function of ξ_i. Keeping the same notation for the extended function, we set

$$v_1(\xi;\widehat{x}) = \alpha_0^1(\widehat{x})X(\xi). \tag{5.7}$$

Then, by virtue of (5) and (4.12), v_1 is a 1-periodic function of ξ_i that solves problem (4.15) and

$$v_1(\xi;\widehat{x}) = V_1(\xi_n;\widehat{x}) + O(e^{-\delta\xi_n}), \quad \text{as } \xi_n \to +\infty, \tag{5.8}$$

$$\text{for} \qquad \alpha_1^0(\widehat{x}) = \alpha_0^1(\widehat{x})C(B). \tag{5.9}$$

In particular, the properties of α_0^1 give that

$$\frac{\partial v_1}{\partial x_i}(\xi;\widehat{x}) = 0 \text{ at } x_i = \pm\frac{1}{2},\ i = \overline{1, n-1}, \tag{5.10}$$

what is required in Lemma 1. The formula (5.9) together with (4.7) imply that

$$\lambda_1 = -C(B)(p-1)\int\limits_{\Gamma_0} |\nabla u_0|^{p-2}(\alpha_0^1)^2\, d\widehat{x}. \tag{5.11}$$

Analogously one can construct the function v_2.

6. Verification of the asymptotics.

In the sequel we assume that $p-2>\varepsilon$. Let $\chi(t)$ be an infinitely differentiable cutoff function that vanishes identically for $t<1$ and equals unity for $t>2$, and let $\chi_\beta(x_n)=\chi\left(\frac{x_n}{\varepsilon^\beta}\right)$. Define

$$\widehat{U}_\varepsilon(x)=\chi_\beta(x_n)\,\widehat{u}_\varepsilon(x)+\left(1-\chi_\beta(x_n)\right)\widehat{v}_\varepsilon(x).$$

Obviously,

$$\lim_{\varepsilon\to0}\|\widehat{U}_\varepsilon\|_{L_p(\Omega_\varepsilon)}\geqslant 1. \tag{6.1}$$

Now we show that the function U_ε approximate the solution to problem (2.2) in domain Ω_ε and verify the constructed asymptotic expansions.

6.1. Homogenization theorems

We use the results from [17]. The following facts were proved

THEOREM 3. *Assume that $F\in L_q(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$, $2\leqslant p<\infty$, $q>1$ and K is an arbitrary compact set belonging to the complex plane $\mathbb{C}$, K does not contain the eigenvalues of the problem (3.8). Then the following statements hold:*

1. There exists a number $\varepsilon_0>0$, such that the unique solution to the problem

$$\begin{cases} -\Delta_p U_\varepsilon(x)=\lambda|U_\varepsilon|^{p-2}U_\varepsilon+F & \text{in } \Omega_\varepsilon,\\ U_\varepsilon=0 & \text{on } \Gamma_\varepsilon,\\ \frac{\partial U_\varepsilon}{\partial_p\nu}\equiv|\nabla U_\varepsilon|^{p-2}(\nabla U_\varepsilon,\nu)=0 & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

does exist for all $\varepsilon<\varepsilon_0$ and for all $\lambda\in K$. Moreover, the uniform (in ε and λ) estimate

$$\|U_\varepsilon\|_{W^{1,p}}\leqslant C\|F\|_{L_q} \tag{6.3}$$

is valid, where C does not depend on U_ε and F;

2. It yields that

$$\|U_\varepsilon-U_0\|_{W^{1,p}}\to 0 \text{ when } \varepsilon\to 0, \tag{6.4}$$

where U_0 is the unique solution of the problem

$$\begin{cases} -\Delta_p U_0(x)=\lambda|U_0|^{p-2}U_0+F & \text{in } \Omega,\\ U_0=0 & \text{on } \Gamma_0,\\ \frac{\partial U_0}{\partial_p\nu}\equiv|\nabla U_0|^{p-2}\frac{\partial U_0}{\partial\nu}=0 & \text{on } \partial\Omega\setminus\overline{\Gamma}_0. \end{cases} \tag{6.5}$$

Here the solutions to problems (6.2) and (6.5) are understood in the weak sense i.e. iff $U_\delta,\delta=\begin{cases}\varepsilon,\\ 0\end{cases}$ satisfies to the integral identity:

$$\int_\Omega\sum_{j=1}^n|\nabla U_\delta(x)|^{p-2}\frac{\partial U_\delta(x)}{\partial x_j}\frac{\partial\varphi(x)}{\partial x_j}\,dx=\lambda\int_\Omega|U_\delta|^{p-2}U_\delta\varphi\,dx+\int_\Omega F\varphi\,dx \tag{6.6}$$

for every $\varphi \in W^{1,p}(\Omega,\Gamma_0)$.

In addition, an estimate for the solution to problem (6.2) in a neighborhood of λ_0^1 was obtained.

LEMMA 3. *Let λ is close to λ_0^1 and λ_ε^1 converges to λ_0^1. Then the following estimate holds:*

$$\|U_\varepsilon\|^p_{W^{1,p}} \leqslant C\frac{\|F\|^2_{L_q}}{|\lambda^1_\varepsilon - \lambda|}. \tag{6.7}$$

We need to obtain the estimates for the constructed functions. According to our notations,

$$\widehat{u}_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2, \quad \widehat{\lambda}_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2.$$

LEMMA 4. *The function $\widehat{u}_\varepsilon$ solves the problem*

$$\begin{cases} -\Delta_p \widehat{u}_\varepsilon = \widehat{\lambda}^\varepsilon |\widehat{u}_\varepsilon|^{p-2}\widehat{u}_\varepsilon + \widehat{f}^u_\varepsilon & \text{in } \Omega_\varepsilon, \\ \frac{\partial \widehat{u}_\varepsilon}{\partial_p \nu} = 0 & \text{on } \partial\Omega\setminus\overline{\Gamma}_0, \end{cases} \tag{6.8}$$

where

$$\|\widehat{f}^u_\varepsilon\|_{L_q(\Omega_\varepsilon)} = O(\varepsilon^3). \tag{6.9}$$

Proof. We define $\widehat{f}^u_\varepsilon$ through

$$-\widehat{f}^u_\varepsilon = \Delta_p \widehat{u}_\varepsilon + \widehat{\lambda}^\varepsilon |\widehat{u}_\varepsilon|^{p-2}\widehat{u}_\varepsilon.$$

By using the expansions (3.3)–(3.7), equating terms with the same power of ε and taking into account boundary-value problems for $u_0, u_1, u_2,$ we derive that

$$\begin{aligned} -\widehat{f}^u_\varepsilon &= \Delta_p \widehat{u}_\varepsilon + \widehat{\lambda}^\varepsilon |\widehat{u}_\varepsilon|^{p-2}\widehat{u}_\varepsilon = \Delta_p u_0 + \lambda_0 |u_0|^{p-2} u_0 \\ &\quad + \varepsilon\Bigg(\sum_{j=1}^n \frac{\partial}{\partial x_j}\Bigg((p-2)|\nabla u_0|^{p-4}(\nabla u_0, \nabla u_1)\frac{\partial u_0}{\partial x_j} + |\nabla u_0|^{p-2}\frac{\partial u_1}{\partial x_j}\Bigg) \\ &\qquad + (\lambda_1 u_0 + \lambda_0 u_1)|u_0|^{p-2} + (p-2)\lambda_0 u_0 |u_0|^{p-3}|u_1|\Bigg) + \ldots + O(\varepsilon^3). \end{aligned}$$

Due to the boundary-value problems for u_0 and u_1, u_2 the terms in front of ε^0, ε and ε^2 becomes 0, thus

$$-\widehat{f}^u_\varepsilon = O(\varepsilon^3), \text{ and } \|\widehat{f}^u_\varepsilon\|_{L_q} = O(\varepsilon^3).$$

The boundary conditions for $\widehat{u}_\varepsilon$ is fulfilled by the boundary conditions for u_0, u_1 and u_2. □

LEMMA 5. *The function $\widehat{v}_\varepsilon$ satisfies*

$$|\Delta_p \widehat{v}_\varepsilon + \widehat{\lambda}^\varepsilon |\widehat{v}_\varepsilon|^{p-2}\widehat{v}_\varepsilon| = O(\varepsilon^{\beta(p-1)}) \text{ as } x_n < 2\varepsilon^\beta \tag{6.10}$$

and

$$\widehat{v}_\varepsilon = 0 \text{ on } \partial\Gamma_\varepsilon, \ \ |\nabla \widehat{v}_\varepsilon|^{p-2}\frac{\partial \widehat{v}_\varepsilon}{\partial \nu} = 0 \text{ on } (\partial\Omega \setminus \overline{\Gamma}_0) \cap \{x_n < 2\varepsilon^\beta\}. \tag{6.11}$$

Proof. The proof is analogous to the previous theorem. We shall use the definition of $\widehat{v}_\varepsilon$ and asymptotics $v_1 \sim \alpha_0^1 \xi_n + \alpha_1^0$, $v_2 \sim \alpha_1^1 \xi_n + \alpha_2^0$ as $x_n \to 0$ to deduce that $\widehat{v}_\varepsilon = O(\varepsilon^\beta)$. The next step consists of rewriting operator Δ_p in $(x,\xi)-$ coordinates and estimating the function $f_\varepsilon^v = -\Delta_p \widehat{v}_\varepsilon - \widehat{\lambda}^\varepsilon |\widehat{v}_\varepsilon|^{p-2}\widehat{v}_\varepsilon$. We omit the details.

The boundary conditions (6.11) follows directly from the definition of $\widehat{v}_\varepsilon$. $\square$

LEMMA 6. *If $\varepsilon^\beta < x_n < 2\varepsilon^\beta$ then*

$$\widehat{u}_\varepsilon - \widehat{v}_\varepsilon = O(\varepsilon^{3\beta}), \frac{\partial}{\partial x_n}(\widehat{u}_\varepsilon - \widehat{v}_\varepsilon) = O(\varepsilon^{2\beta})$$

and

$$\sum_{i=1}^n \left|\frac{\partial \widehat{U}_\varepsilon}{\partial x_i}\right|^p - \hat{\lambda}_\varepsilon |\widehat{U}_\varepsilon|^p = O(\varepsilon^{\beta p}).$$

Proof. Now we estimate the function

$$\widehat{U}_\varepsilon = \chi_\beta \widehat{u}_\varepsilon + (1-\chi_\beta)\widehat{v}_\varepsilon$$

on the set $\varepsilon^\beta < x < 2\,\varepsilon^\beta$. By the definition,

$$\begin{aligned}\widehat{u}_\varepsilon - \widehat{v}_\varepsilon = \alpha_0^1 x_n + \varepsilon\alpha_1^0 + \varepsilon\alpha_1^1 x_n + \varepsilon^2\alpha_2^0 + O(x_n^3 + \varepsilon x_n^2 + \varepsilon^2 x_n)\\ - \varepsilon\alpha_0^1 \xi_n - \varepsilon\alpha_1^0 - \varepsilon^2(\alpha_1^1\xi_n + \alpha_2^0) = O(\varepsilon^{3\beta}).\end{aligned}$$

Thus, using the asymptotics for $\widehat{v}_\varepsilon$, we derive that

$$U_\varepsilon = \chi_\beta(\widehat{u}_\varepsilon - \widehat{v}_\varepsilon) + \widehat{v}_\varepsilon = O(\varepsilon^{3\beta}) + O(\varepsilon^\beta) = O(\varepsilon^\beta),$$

therefore the term $\widehat{\lambda}^\varepsilon |U_\varepsilon|^p = O(\varepsilon^{\beta p})$. Next we estimate the derivatives of U_ε. Thinking analogously, one has for $i = 1,\ldots,n-1$:

$$\frac{\partial \widehat{U}_\varepsilon}{\partial x_i} = O\left(\varepsilon^\beta\right) \Rightarrow \left|\frac{\partial \widehat{U}_\varepsilon}{\partial x_i}\right|^p = O(\varepsilon^{\beta p}).$$

For $i = n$ one has

$$\frac{\partial}{\partial x_n}(\widehat{u}_\varepsilon - \widehat{v}_\varepsilon) = \alpha_0^1 + \varepsilon\alpha_1^1 + O(x_n^2 + \varepsilon x_n + \varepsilon^2) - \alpha_0^1 - \varepsilon\alpha_1^1 = O(\varepsilon^{2\beta}) \text{ as } x_n < 2\varepsilon^\beta,$$

hence

$$\left|\frac{\partial \widehat{U}_\varepsilon}{\partial x_n}\right|^p = O(\varepsilon^{2\beta p}).$$

Summing up, we get that

$$\sum_{i=1}^{n}\left|\frac{\partial \widehat{U}_\varepsilon}{\partial x_i}\right|^p - \hat{\lambda}_\varepsilon |\widehat{U}_\varepsilon|^p = O(\varepsilon^{\beta p}). \quad \square$$

THEOREM 4. *The function $\widehat{U}_\varepsilon$ satisfies the boundary-value problem*

$$\begin{cases} -\Delta_p U_\varepsilon = \widehat{\lambda}^\varepsilon |U_\varepsilon|^{p-2} U_\varepsilon + F_\varepsilon & \text{in } \Omega_\varepsilon, \\ U_\varepsilon = 0 & \text{on } \Gamma_\varepsilon, \\ \frac{\partial U_\varepsilon}{\partial_p \nu} = 0 & \text{on } \Gamma, \end{cases} \tag{6.12}$$

where

$$\|\widehat{F}_\varepsilon\|_{L_q(\Omega_\varepsilon)} = O(\varepsilon^{\frac{\beta(p+1)}{q}}). \tag{6.13}$$

Proof. First we observe that the boundary conditions are fulfilled due to the definition of $\widehat{U}_\varepsilon$ and boundary conditions for $u_0, u_1, u_2, \alpha_0^1, \alpha_1^0, \alpha_1^1, \alpha_2^0$. Consider now the equation for $\widehat{U}_\varepsilon$. Denote

$$-\widehat{F}_\varepsilon = \Delta_p \widehat{U}_\varepsilon + \widehat{\lambda}^\varepsilon |\widehat{U}_\varepsilon|^{p-2} \widehat{U}_\varepsilon.$$

Then the integral identity

$$\int_\Omega \sum_{i=1}^{n}\left|\frac{\partial \widehat{U}_\varepsilon}{\partial x_i}\right|^p dx = \hat{\lambda}_\varepsilon \int_\Omega |\widehat{U}_\varepsilon|^p \, dx + \int_\Omega \widehat{F}_\varepsilon \widehat{U}_\varepsilon \, dx$$

holds. Let us estimate now

$$\left|\int_\Omega \widehat{F}_\varepsilon \widehat{U}_\varepsilon \, dx\right| = \left|\int_\Omega \sum_{i=1}^{n}\left|\frac{\partial \widehat{U}_\varepsilon}{\partial x_i}\right|^p dx - \hat{\lambda}_\varepsilon \int_\Omega |\widehat{U}_\varepsilon|^p \, dx\right| = I_1 + I_2 + I_3,$$

where

$$I_1 = \left|\int_{\Omega \cap \{x > 2\varepsilon^\beta\}} \left(\sum_{i=1}^{n}\left|\frac{\partial \widehat{u}_\varepsilon}{\partial x_i}\right|^p - \hat{\lambda}_\varepsilon |\widehat{u}_\varepsilon|^p\right) dx\right|,$$

$$I_2 = \left|\int_{\Omega \cap \{\varepsilon^\beta < x < 2\varepsilon^\beta\}} \left(\sum_{i=1}^{n}\left|\frac{\partial \widehat{U}_\varepsilon}{\partial x_i}\right|^p - \hat{\lambda}_\varepsilon |\widehat{U}_\varepsilon|^p\right) dx\right|,$$

$$I_3 = \left| \int\limits_{\Omega\cap\{x<2\varepsilon^\beta\}} \left(\sum_{i=1}^n \left| \frac{\partial \widehat{v}_\varepsilon}{\partial x_i} \right|^p - \hat{\lambda}_\varepsilon |\widehat{v}_\varepsilon|^p \right) dx \right|.$$

Let us estimate each integral $I_k, k = 1,2,3.$ Taking into account the properties of χ_β and Lemma 4 one derive

$$I_1 = \int\limits_{\Omega\cap\{x>2\varepsilon^\beta\}} |\Delta_p \widehat{u}_\varepsilon + \widehat{\lambda}^\varepsilon |\widehat{u}_\varepsilon|^{p-2}\widehat{u}_\varepsilon|^q \, dx = O(\varepsilon^3),$$

therefore

$$\left| \int\limits_{\Omega\cap\{x>2\varepsilon^\beta\}} \widehat{F}_\varepsilon \widehat{U}_\varepsilon \, dx \right| = \left| \int\limits_{\Omega\cap\{x>2\varepsilon^\beta\}} \widehat{F}_\varepsilon \widehat{u}_\varepsilon \, dx \right| = O(\varepsilon^3) \Rightarrow \widehat{F}_\varepsilon = O(\varepsilon^3) \text{ as } x > 2\varepsilon^\beta.$$

In view of Lemma 6 we can estimate

$$I_2 = \left| \int\limits_{\Omega\cap\{\varepsilon^\beta<x<2\varepsilon^\beta\}} \left(\sum_{i=1}^n \left| \frac{\partial \widehat{U}_\varepsilon}{\partial x_i} \right|^p - \hat{\lambda}_\varepsilon |\widehat{U}_\varepsilon|^p \right) dx \right| = O(\varepsilon^{\beta p})\varepsilon^\beta = O(\varepsilon^{\beta(p+1)})$$

which yields

$$\left| \int\limits_{\Omega\cap\{\varepsilon^\beta<x<2\varepsilon^\beta\}} \widehat{F}_\varepsilon \widehat{U}_\varepsilon \, dx \right| = \left| \int\limits_{\Omega\cap\{\varepsilon^\beta<x<2\varepsilon^\beta\}} \widehat{F}_\varepsilon O(\varepsilon^\beta) \, dx \right| = O(\varepsilon^{\beta(p+1)}) \Rightarrow$$

$$\widehat{F}_\varepsilon = O(\varepsilon^{\beta(p-1)}) \text{ as } \varepsilon^\beta < x < 2\varepsilon^\beta.$$

Finally, due to Lemma 5

$$I_3 = \left| \int\limits_{\Omega\cap\{x<2\varepsilon^\beta\}} \left(\sum_{i=1}^n \left| \frac{\partial \widehat{v}_\varepsilon}{\partial x_i} \right|^p dx - \hat{\lambda}_\varepsilon |\widehat{v}_\varepsilon|^p \, dx \right) \right| = O(\varepsilon^{\beta p})\varepsilon^\beta = O(\varepsilon^{\beta(p+1)})$$

and analogously, that implies the asymptotics

$$\widehat{F}_\varepsilon = O(\varepsilon^{\beta(p-1)}) \text{ as } x < \varepsilon^\beta.$$

Consequently,

$$\|\widehat{F}_\varepsilon\|_{L_q(\Omega_\varepsilon)} = \left(\int\limits_{\Omega\cap\{x>2\varepsilon^\beta\}} \widehat{F}_\varepsilon^q \, dx + \int\limits_{\Omega\cap\{\varepsilon^\beta<x<2\varepsilon^\beta\}} \widehat{F}_\varepsilon^q \, dx + \int\limits_{\Omega\cap\{x<\varepsilon^\beta\}} \widehat{F}_\varepsilon^q \, dx \right)^{\frac{1}{q}}$$

$$= \left(O(\varepsilon^{3q}) + 2O(\varepsilon^{q\beta(p-1)+\beta}) \right)^{\frac{1}{q}} = |(p-1)q = p| = O(\varepsilon^{\beta\frac{(p+1)}{q}}). \quad \square$$

THEOREM 5. *The asymptotics*

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + O(\varepsilon^{\frac{2\beta(p+1)}{q}}),\ 0<\beta<1, \tag{6.14}$$

holds for the first eigenvalue to spectral problem (2.2), where λ_1 is given by (5.11).

Proof. Due to the estimate (6.7) the solution to (6.12) satisfies

$$\|\widehat{U}_\varepsilon\|^p_{L_p(\Omega_\varepsilon)} \leqslant C\frac{\|\widehat{F}\|^2_{L_q(\Omega_\varepsilon)}}{\left|\lambda_1^\varepsilon - \widehat{\lambda}^\varepsilon\right|}.$$

This together with (6.1) implies

$$\left|\lambda_\varepsilon - \widehat{\lambda}^\varepsilon\right| \leqslant C\|\widehat{F}\|^2_{L_q(\Omega_\varepsilon)} = O(\varepsilon^{\frac{2\beta(p+1)}{q}}),$$

hence the asymptotic (6.14) holds. □

7. Asymptotics of the sharp constant in the Friedrichs inequality.

The obtained rate of the convergence between λ^1_ε and λ^1_0 gives the following statement:

THEOREM 6. *Let K_ε be the sharp constant for the Friedrichs inequality*

$$\int\limits_{\Omega_\varepsilon} u_\varepsilon^p\,dx \leqslant K_\varepsilon \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p\,dx,\ u_\varepsilon \in W^{1,p}(\Omega_\varepsilon,\Gamma_\varepsilon).$$

Then it converges to K the sharp constant in the inequality

$$\int\limits_{\Omega} u^p\,dx \leqslant K \int\limits_{\Omega_\varepsilon} |\nabla u|^p\,dx,\ u \in W^{1,p}(\Omega,\Gamma_0)$$

and the rate of convergence can be estimated as

$$|K_\varepsilon - K| \leqslant C(\varepsilon + \varepsilon^{\frac{2\beta(p+1)}{q}}),\ \ 0<\beta<1. \tag{7.1}$$

Proof. We remark that if one choose φ_ε as u_ε in the integral identity (2.3) and φ as u_0 in (3.8) then it reads as

$$\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^p\,dx = \lambda_\varepsilon \int\limits_{\Omega_\varepsilon} |u_\varepsilon|^p\,dx \text{ and } \int\limits_{\Omega} |\nabla u_0(x)|^p\,dx = \lambda_0 \int\limits_{\Omega} |u_0|^p\,dx.$$

From the variational definition it holds that

$$\lambda^1_\varepsilon = \inf_{u_\varepsilon \in W^{1,p}(\Omega_\varepsilon,\Gamma_\varepsilon)\setminus\{0\}} \frac{\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p\,dx}{\int\limits_{\Omega_\varepsilon} |u_\varepsilon|^p\,dx}, \text{ and } \lambda^1_0 = \inf_{u_0 \in W^{1,p}(\Omega,\Gamma_0)\setminus\{0\}} \frac{\int\limits_{\Omega} |\nabla u_0|^p\,dx}{\int\limits_{\Omega} |u_0|^p\,dx},$$

thus, the sharp constants in corresponding Friedrichs inequalities are $K_\varepsilon = (\lambda_\varepsilon^1)^{-1}$ and $K = (\lambda_0^1)^{-1}$. The estimates (7.1) follows evidently from asymptotics (6.14):

$$|K_\varepsilon - K| = \frac{|\lambda_\varepsilon^1 - \lambda_0^1|}{|\lambda_0^1||\lambda_\varepsilon^1|} \leqslant \frac{|\lambda_\varepsilon^1 - \lambda_0^1|}{|\lambda_0^1|^2} = \varepsilon C_1 + \varepsilon^{\frac{2\beta(p+1)}{q}} C_2 \leqslant C(\varepsilon + \varepsilon^{\frac{2\beta(p+1)}{q}}). \quad \square$$

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Yulia Koroleva
Department of Higher Mathematics
Gubkin State University of Oil and Gas (National Research University)
Leninskij prospect 65-1
Moscow, 119991, Russia
e-mail: koroleva.y@gubkin.ru