CONTINUOUS DEPENDENCE FOR SOLUTIONS
TO 2–D BOUSSINESQ SYSTEM CHANNEL FLOW

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Abstract. This paper considers the 2-D Boussinesq system in a semi-infinite channel. By making use of the earlier work of [19] and some Soblev inequalities, the continuous dependence on the coefficient of the system is obtained. The authors also show how to bound the total energy.

1. Introduction

The Boussinesq system we consider in this paper takes the form

$$
\begin{aligned}
&\partial_t \mathbf{u} - \mu \Delta \mathbf{u} + \mathbf{u}\nabla\mathbf{u} - \gamma \mathbf{g}\theta + \nabla p = 0, \\
&div\mathbf{u} = 0, \\
&\partial_t \theta - \kappa \Delta \theta + \mathbf{u}\nabla\theta = 0.
\end{aligned}
\tag{1.1}
$$

Here $\mathbf{u} = (u_1, u_2)$ denotes the velocity of the fluid; θ is the temperature; p is the hydrostatic pressures; $\mathbf{g} = (g_1, g_2)$ is the gravity force; μ and κ are kinematic viscosity and thermal conductivity, respectively; γ is a positive constant associated to the coefficient of volume expansion. Without loss of generality, we assume that $\mu = \kappa = \gamma = 1$ in this paper.

The Boussinesq model is one of the most useful models in fluid and geophysical fluid dynamics such as atmospheric fronts and ocean circulations (see [21, 27]). In 2001 Majda [20] pointed out that the two-dimensional Boussinesq equations retain some key features of the three-dimensional Euler and Navier-Stokes equations such as the vortex stretching mechanism and the inviscid two-dimensional Boussinesq equations are identical to the Euler equations for three-dimensional axisymmetric swirling flows. For more information about Boussinesq system, we refer reders to Majda's books [21, 20]. In fact, there have been many papers devoted to the study of the Boussinesq system. The global well-posedness of the 2-D Boussinesq system was obtained by [7, 12]. [6] established the global in time existence of classical solutions to the two-dimensional anisotropic Boussinesq equations with vertical dissipation.

Different from the above papers, we consider the two-dimensional Boussinesq equations in a semi-infinite channel. In fact, a number of papers on channel flow of a incompressible viscous fluid have appeared in the literature. Horgan [11] investigated the stationary Navier-Stokes flow in a channel. Under certain assumptions, the authors proofed that the flow tended asymptotically to the fully developed flow. Lin [14] considered the entry flow problem of the transient Stokes in a semi-infinite strip when the mean value of the entry ow is zero and obtained spatial decay estimates for an energy associated with the flow. Lee and Song [13] established exponential decay bounds for a transient magnetohydrodynamic flow in a semi-infinite channel. Song [28, 29] investigated the case of nonzero entry Stokes flow and obtained improved decay results. For more papers, one can see [19, 16, 18] and papers cited therein. Obviously, most of the papers established exponential decay bounds for the solution to two-dimensional equations, but few paper in the literature paid attention to structural stability(or continuous dependence) of two-dimensional Boussinesq equations.

The purpose of the present paper is to derive the continuous dependence of external force. Structural stability questions are fundamental in that one wishes to know whether a small change in a coefficient in the equations will induce a dramatic change in the solution. In the spirit of the earlier work of [19], we derive a second order partial differential inequality which leads to the continue dependence results. Since there is a non-linear term $\mathbf{u}\nabla\mathbf{u}$ in (1.1), the argument of this paper will be more complicated and the results are interesting. The methods which are used in our paper can be extended to other similar equations.

Throughout this paper, the usual summation convention is employed with repeated Greek subscript summed from 1 to 2. The comma is used to indicate partial differentiation,i.e. $\varphi_{\alpha,\alpha} = \Sigma_{\alpha=1}^{2} \frac{\partial \varphi_{\alpha}}{\partial x_{\alpha}}$. The paper is structured as follows: In section 2 we formulate the boundary problem with provides the framework investigation. Some auxiliary inequalities are presented in section 3. Section 4 is devote to deriving a basic differential inequality that leads directly to continuous dependence for the solution. Finally, to make the estimate explicit, we derive the total energies in terms of prescribed data in section 5 and section 6, respectively.

2. Formulation

We consider the Boussinesq system with variable viscosity and thermal in a channel in $\mathbb{R}^2$. The channel is denoted by R which is defined as

$$R := \{(x_1,x_2)|x_1 \geqslant 0, 0 \leqslant x_2 \leqslant h\}, \tag{2.1}$$

where h is a fixed positive constant, and we also use the notation

$$L_z := \{(x_1,x_2)|x_1 = z, 0 \leqslant x_2 \leqslant h\}. \tag{2.2}$$

We also define

$$R_z := \{(x_1,x_2)|x_1 \geqslant z, 0 \leqslant x_2 \leqslant h\}. \tag{2.3}$$

Clearly $R_0 = R$. As in paper [19], Eq.(1.1) may be rewritten in the form

$$u_{\alpha,t} - \mu\Delta u_\alpha + u_\beta u_{\alpha,\beta} - \gamma g_\alpha \theta + p_{,\alpha} = 0, \; in \; \mathbb{R} \times [0,\infty), \tag{2.4}$$

$$u_{\alpha,\alpha} = 0, \; in \; \mathbb{R} \times [0,\infty), \tag{2.5}$$

$$\theta_{,t} - \kappa\Delta\theta + u_\alpha\theta_{,\alpha} = 0, \; in \; \mathbb{R} \times [0,\infty), \tag{2.6}$$

with the initial-boundary conditions

$$u_\alpha(x_1,x_2,0) = \theta(x_1,x_2,0) = 0 \tag{2.7}$$

$$u_\alpha(x_1,0,t) = u_\alpha(x_1,h,t) = 0, \; \theta(x_1,0,t) = \theta(x_1,h,t) = 0, \tag{2.8}$$

$$u_\alpha(0,x_2,t) = f_\alpha(x_2,t), \; \theta(0,x_2,t) = h(x_2,t), \tag{2.9}$$

for $\alpha = 1,2$. The boundary functions $f_\alpha(\alpha = 1,2)$ and h satisfy the compatibility relationships $f_\alpha(0,t) = f_\alpha(h,t) = 0, h(0,t) = h(h,t) = 0$. If data f_α and h are small enough in L_2, bounded solutions will exist in the interval [0,T]. In fact, under the condition $\int_0^h f_1 dx_2 = \int_0^h h dx_2 = 0$, the solutions of (2.4)-(2.9) will vanish as x_1 tends to ∞. However, if $\int_0^h f_1 dx_2 \neq 0$ and $\int_0^h h dx_2 \neq 0$, we suppose that (u_1,u_2) tend to $(V,0)$ for sufficiently small data, when $x_1 \to \infty$, where $V(x_2,t)$ is the solution of the problem

$$V_{,t} - V_{,22} + P(t) = 0, \quad 0 < x_2 < h, 0 < t < T, \tag{2.10}$$

$$V(x_2,0) = 0, \quad 0 \leqslant x_2 \leqslant h, \tag{2.11}$$

$$V(0,t) = V(h,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{2.12}$$

The unknown function $P(t)$ may be determined by

$$\int_0^h V(x_2,t)dx_2 = \int_0^h f_1(x_2,t)dx_2 = Q(t). \tag{2.13}$$

The problem (2.10)-(2.13) may be viewed as an inverse problem for determining $P(t)$ and hence $V(x_2,t)$. It is easy to show that this problem has a unique solution. For interesting, readers can refer to [5] in which Cannon and Zachmann investigated a similar inverse problem. Now, we let

$$w_\alpha = u_\alpha - V\delta_{\alpha 1}, \quad q_{,\alpha} = p_{,\alpha} - P(t)\delta_{\alpha 1}. \tag{2.14}$$

Then, the boundary-initial value problem (2.4)-(2.9) may be rewritten as

$$w_{\alpha t} - \mu\Delta w_\alpha + (w_\beta + V\delta_{\beta 1})w_{\alpha,\beta} + w_2 V_{,2}\delta_{\alpha 1} - g_\alpha\theta + q_{,\alpha} = 0, \tag{2.15}$$

$$w_{\alpha,\alpha} = 0, \; in \; \mathbb{R} \times [0,\infty), \tag{2.16}$$

$$\theta_t - \kappa\Delta\theta + w_\alpha\theta_{,\alpha} + \theta_{,1}V = 0, \; in \; \mathbb{R} \times [0,\infty), \tag{2.17}$$

with the initial-boundary conditions

$$w_\alpha(x_1,x_2,0) = \theta(x_1,x_2,0) = 0 \tag{2.18}$$

$$w_\alpha(x_1,0,t) = u_\alpha(x_1,h,t) = 0, \; \theta(x_1,0,t) = \theta(x_1,h,t) = 0, \tag{2.19}$$

$$w_\alpha(0,x_2,t) = f_\alpha(x_2,t) - V\delta_{\alpha 1}, \; \theta(t,0,x_2) = h(x_2,t). \tag{2.20}$$

Under the condition (2.13), it follows that at each instant of time t

$$\int_{L_0} w_1 dx_2 = 0.$$

Since (w_1,w_2) is divergence-free, we have

$$\int_{L_z} w_1 dx_2 = \int_{L_0} w_1 dx_2 + \int_0^z \int_{L_\xi} w_{1,1} dx_2 d\xi$$
$$= -\int_0^z \int_{L_\xi} w_{2,2} dx_2 d\xi = 0.$$

Now, we list some results which have been derived in [19]. These results will be used in the next sections of this paper. In paper [19], the authors defined energy expressions as

$$E(z,t) = \int_0^t \int_{R_z} w_{\alpha,\beta} w_{\alpha,\beta} dA d\eta + \rho_1 \int_0^t \int_{R_z} w_{\alpha,\eta} w_{\alpha,\eta} dA d\eta$$
$$+ \int_0^t \int_{R_z} \theta_{,\alpha} \theta_{,\alpha} dA d\eta + \rho_2 \int_0^t \int_{R_z} \theta_{,\eta}^2 dA d\eta$$
$$= E_1(z,t) + \rho_1 E_2(z,t) + E_3(z,t) + \rho_2 E_4(z,t), \tag{2.21}$$

and

$$E_5(z,t) = \int_0^t \int_{R_z} w_{\alpha,\beta\eta} w_{\alpha,\beta\eta} dA d\eta, \; E_6(z,t) = \int_0^t \int_{R_z} \theta_{,\alpha\eta} \theta_{,\alpha\eta} dA d\eta. \tag{2.22}$$

In section 4 of [19], the authors proofed

$$E(z,t) \leqslant m_1 E(0,t) e^{-m_2 z}, \tag{2.23}$$

for some computable positive constants m_1 and m_2. They named this result as "Spatial decay estimates". To make the estimates explicit, the bounds for $E_i(0,t), (i = 1,2,3,4,5,6)$ were also derived in section 5 of that paper. We write the bounds for $E_i(0,t), (i = 1,2,3,4,5,6)$ as

$$E_i(0,t) \leqslant \widetilde{Q}_i, \tag{2.24}$$

where $\widetilde{Q}_i(i = 1,2,3,4,5,6)$ are positive constants which only depend on the known data.

In this paper, we continue their work. We want to establish the continuous dependence on g_α. Let (w_α, θ, q) and $(w_\alpha^*, \theta^*, q^*)$ be solutions of (2.15)-(2.20), but with different coefficients g_α and g_α^*, respectively. We set

$$v_\alpha = w_\alpha - w_\alpha^*, \ T = \theta - \theta^*, \ \pi = q - q^*, \ \tau_\alpha = g_\alpha - g_\alpha^*. \tag{2.25}$$

Then, it is easy to find that (v_α, T, π) satisfy

$$v_{\alpha t} - \Delta v_\alpha + v_\beta w_{\alpha,\beta} + w_\beta^* v_{\alpha,\beta} + V v_{\alpha,1} + v_2 V_{,2} \delta_{\alpha 1} - \tau_\alpha \theta - g_\alpha^* T + \pi_{,\alpha} = 0, \quad (2.26)$$

$$v_{\alpha,\alpha} = 0, \ in \ \mathbb{R} \times [0,\infty), \tag{2.27}$$

$$T_t - \Delta T + v_\alpha \theta_{,\alpha} + w_\alpha^* T_{,\alpha} + T_{,1} V = 0, \ in \ \mathbb{R} \times [0,\infty), \tag{2.28}$$

with the initial-boundary conditions

$$v_\alpha(x_1, x_2, 0) = T(x_1, x_2, 0) = 0 \tag{2.29}$$

$$v_\alpha(x_1, 0, t) = v_\alpha(x_1, h, t) = 0, \ T(x_1, 0, t) = T(x_1, h, t) = 0, \tag{2.30}$$

$$v_\alpha(0, x_2, t) = 0, \ T(0, x_2, t) = 0. \tag{2.31}$$

Our main result of this paper may be written as:

THEOREM 1. *Supposing the known data h, f_α and κ, μ, γ small enough. Then the solution (w_α, θ, p) of the initial-boundary value problem (2.15)-(2.20) depends continuously on change in the coefficient g_α, as shown explicitly in inequality (4.38) which derives a relation of the form*

$$\Psi(z,t) \leqslant \tau^2 L, \tag{2.32}$$

where L is an a priori constant and $\Psi(z,t)$ will be defined in (3.33). In (2.32), we note that $\tau^2 = \max\{\tau_\alpha \tau_\alpha\}$.

3. The definition for $\Psi(z,t)$

To proof theorem 1, we first seek the definition for $\Psi(z,t)$. To do this, we introduce a stream function $\varphi(x_1, x_2, t)$ such that

$$v_1 = \varphi_{,2}, \quad v_2 = -\varphi_{,1}. \tag{3.1}$$

We can eliminate the troublesome pressure term $\pi_{,\alpha}$ in (2.26). The equation (2.26)-(2.28) may be transformed into the following form

$$(\Delta\varphi)_t - \Delta^2\varphi - \varphi_{,\beta}\Delta w_\beta + w_\beta^*(\Delta\varphi)_{,\beta} - \varphi_{,1}V_{,22} + V(\Delta\varphi)_{,1} + (\tau_2\theta)_{,1} - (\tau_1\theta)_{,2}$$
$$+ (g_2^*T)_{,1} - (g_1^*T)_{,2} = 0, \ in \ \mathbb{R} \times [0,\infty), \quad (3.2)$$

$$T_t - \Delta T + \varphi_{,2}\theta_{,1} - \varphi_{,1}\theta_{,2} + w_\alpha^* T_{,\alpha} + T_{,1} V = 0, \ in \ \mathbb{R} \times [0,\infty), \tag{3.3}$$

with the initial-boundary conditions

$$\varphi(x_1,x_2,0) = T(x_1,x_2,0) = 0, \tag{3.4}$$

$$\begin{cases} \varphi(x_1,0,t) = \varphi(x_1,h,t) = 0, \\ \varphi_{,2}(x_1,0,t) = \varphi_{,2}(x_1,h,t) = 0, \\ T(x_1,0,t) = T(x_1,h,t) = 0, \end{cases} \tag{3.5}$$

$$\varphi(0,x_2,t) = \varphi_{,1}(0,x_2,t) = 0, \ T(0,x_2,t) = 0. \tag{3.6}$$

To get the result we want, we must seek definition of $\Psi(z,t)$. We get our goal in three steps.

Step 1: Definition for $\Phi_1(z,t)$.

Let φ be a solution of (3.2)-(3.6), we start with the integral

$$\int_0^t \int_{L_z} \varphi_{,1\eta} \varphi \, dx_2 d\eta. \tag{3.7}$$

Making use of the divergence theorem, Eq.(3.2) and the initial-boundary conditions and integrating by parts, we have

$$\int_0^t \int_{L_z} \varphi_{,1\eta}\varphi \, dx_2 d\eta = -\int_0^t \int_z^\infty \int_{L_\xi} (\Delta\varphi)_{,\eta}\varphi \, dAd\eta - \frac{1}{2}\int_z^\infty \int_{L_\xi} \varphi_{,\alpha}\varphi_{,\alpha}dA|_{\eta=t}$$

$$= \int_0^t \int_{R_z} [-\Delta^2\varphi - \varphi_{,\beta}\Delta w_\beta + w_\beta^*(\Delta\varphi)_{,\beta} - \varphi_{,1}V_{,22} + V(\Delta\varphi)_{,1} + (\tau_2\theta)_{,1} - (\tau_1\theta)_{,2}$$

$$+ (g_2^*T)_{,1} - (g_1^*T)_{,2}]\varphi \, dAd\eta - \frac{1}{2}\int_{R_z} \varphi_{,\alpha}\varphi_{,\alpha}dA|_{\eta=t}, \tag{3.8}$$

from which it follows that

$$\int_0^t \int_{R_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\eta + \frac{1}{2}\int_{R_z} \varphi_{,\alpha}\varphi_{,\alpha}dA|_{\eta=t}$$

$$= -\int_0^t \int_{L_z} \varphi_{,1\eta}\varphi \, dx_2 d\eta + \int_0^t \int_{L_z} \Delta\varphi_{,1}\varphi \, dx_2 d\eta$$

$$- \int_0^t \int_{L_z} \varphi_{,1\alpha}\varphi_{,\alpha}dx_2 d\eta + \int_0^t \int_{R_z} [w_\beta^*(\Delta\varphi)_{,\beta} - \varphi_{,\beta}\Delta w_\beta]\varphi \, dAd\eta$$

$$+ \int_0^t \int_{R_z} [V(\Delta\varphi)_{,1} - \varphi_{,1}V_{,22}]\varphi \, dAd\eta$$

$$+ \int_0^t \int_{R_z} [(\tau_2\theta)_{,1} - (\tau_1\theta)_{,2}]\varphi \, dAd\eta + \int_0^t \int_{R_z} [(g_2^*T)_{,1} - (g_1^*T)_{,2}]\varphi \, dAd\eta. \tag{3.9}$$

Integrating by parts, we have

$$\int_0^t \int_{L_z} \Delta\varphi_{,1}\varphi dx_2 d\eta - \int_0^t \int_{L_z} \varphi_{,1\alpha}\varphi_{,\alpha} dx_2 d\eta = \frac{\partial}{\partial z} \int_0^t \int_{L_z} [\varphi_{,11}\varphi - \varphi_{,\alpha}\varphi_{,\alpha}] dx_2 d\eta,$$

(3.10)

$$\int_0^t \int_{R_z} [w_\beta^*(\Delta\varphi)_{,\beta} - \varphi_{,\beta}\Delta w_\beta]\varphi dAd\eta$$

$$= -\int_0^t \int_{L_z} w_1^* \Delta\varphi\varphi dx_2 d\eta - \int_0^t \int_{R_z} w_\beta^* \Delta\varphi\varphi_{,\beta} dAd\eta$$

$$+ \frac{1}{2} \int_0^t \int_{L_z} \varphi^2 w_{1,11} dx_2 d\eta + \frac{1}{2} \int_0^t \int_{L_z} \varphi^2 w_{1,22} dx_2 d\eta$$

$$= -\int_0^t \int_{L_z} w_1^* \Delta\varphi\varphi dx_2 d\eta - \int_0^t \int_{R_z} w_\beta^* \Delta\varphi\varphi_{,\beta} dAd\eta$$

$$- \frac{1}{2} \int_0^t \int_{L_z} \varphi^2 w_{2,12} dx_2 d\eta + \frac{1}{2} \int_0^t \int_{L_z} \varphi^2 w_{1,22} dx_2 d\eta$$

$$= -\int_0^t \int_{L_z} w_1^* \Delta\varphi\varphi dx_2 d\eta - \int_0^t \int_{R_z} w_\beta^* \Delta\varphi\varphi_{,\beta} dAd\eta$$

$$+ \int_0^t \int_{L_z} \varphi\varphi_{,2} w_{2,1} dx_2 d\eta - \int_0^t \int_{L_z} \varphi\varphi_{,2} w_{1,2} dx_2 d\eta,$$

(3.11)

where we have used the fact $w_{\alpha,\alpha} = 0$.

$$\int_0^t \int_{R_z} [V(\Delta\varphi)_{,1} - \varphi_{,1}V_{,22}]\varphi dAd\eta = \frac{1}{2}\int_0^t \int_{L_z} V\varphi_{,\alpha}\varphi_{,\alpha} dx_2 d\eta$$

$$- \int_0^t \int_{L_z} V\varphi_{,11}\varphi dx_2 d\eta + \int_0^t \int_{R_z} V_{,2}\varphi_{,1}\varphi_{,2} dAd\eta$$

$$= \frac{1}{2}\int_0^t \int_{L_z} V\varphi_{,\alpha}\varphi_{,\alpha} dx_2 d\eta - \frac{\partial}{\partial z}\int_0^t \int_{L_z} V\varphi_{,1}\varphi dx_2 d\eta$$

$$+ \int_0^t \int_{L_z} V\varphi_{,1}^2 dx_2 d\eta + \int_0^t \int_{R_z} V_{,2}\varphi_{,1}\varphi_{,2} dAd\eta,$$

(3.12)

$$\int_0^t \int_{R_z} [(\tau_2\theta)_{,1} - (\tau_1\theta)_{,2}]\varphi dAd\eta$$

$$= -\int_0^t \int_{L_z} \tau_2\theta\varphi dx_2 d\eta - \int_0^t \int_{R_z} \tau_2\theta\varphi_{,1} dAd\eta + \int_0^t \int_{R_z} \tau_1\theta\varphi_{,2} dAd\eta. \quad (3.13)$$

$$\int_0^t \int_{R_z} [(g_2^*T)_{,1} - (g_1^*T)_{,2}]\varphi dAd\eta$$

$$= -\int_0^t \int_{L_z} g_2^*T\varphi dx_2 d\eta - \int_0^t \int_{R_z} g_2^*T\varphi_{,1} dAd\eta + \int_0^t \int_{R_z} g_1^*T\varphi_{,2} dAd\eta. \quad (3.14)$$

So, we define

$$\Phi_1(z,t) = \int_0^t \int_{R_z} \varphi_{,\alpha\beta} \varphi_{,\alpha\beta} dAd\eta. \tag{3.15}$$

Combining (3.9)-(3.14), we have

$$\Phi_1(z,t) + \frac{1}{2}\int_{R_z} \varphi_{,\alpha}\varphi_{,\alpha}dA|_{\eta=t} = y_{11}(z,t) + y_{12}(z,t) + y_{13}(z,t), \tag{3.16}$$

where

$$y_{11}(z,t) = \frac{\partial}{\partial z}\int_0^t\int_{L_z}[\varphi_{,11}\varphi - \varphi_{,\alpha}\varphi_{,\alpha} - V\varphi_{,1}\varphi]dx_2d\eta,$$

$$y_{12}(z,t) = \int_0^t\int_{L_z}[-\varphi_{,1\eta}\varphi - w_1^*\Delta\varphi\varphi + \varphi\varphi_{,2}w_{2,1} - \varphi\varphi_{,2}w_{1,2} + \frac{1}{2}V\varphi_{,\alpha}\varphi_{,\alpha}$$
$$+ V\varphi_{,1}^2 - V\varphi_{,12}\varphi - \tau_2\theta\varphi - g_2^*T\varphi]dx_2d\eta,$$

$$y_{13}(z,t) = \int_0^t\int_{R_z}\big[-w_\beta^*\Delta\varphi\varphi_{,\beta} + V_{,2}\varphi_{,1}\varphi_{,2} + \tau_1\theta\varphi_{,2} - \tau_2\theta\varphi_{,1}$$
$$- g_1^*T\varphi_{,2} - g_2^*T\varphi_{,1}\big]dAd\eta. \tag{3.17}$$

Step 2: Definition of $\Phi_2(z,t)$. We consider the integral

$$\int_0^t\int_{R_z}\varphi_{,\alpha\eta}\varphi_{,\alpha\eta}dAd\eta. \tag{3.18}$$

By integrating by parts, using the divergence theorem and the initial-boundary conditions, we have

$$\int_0^t\int_{R_z}\varphi_{,\alpha\eta}\varphi_{,\alpha\eta}dAd\eta$$

$$= -\int_0^t\int_{L_z}\varphi_{,1\eta}\varphi_{,\eta}dx_2d\eta - \int_0^t\int_{R_z}\varphi_{,\eta}(\Delta\varphi)_{,\eta}dAd\eta$$

$$= -\frac{1}{2}\frac{\partial}{\partial z}\int_0^t\int_{L_z}\varphi_{,\eta}^2dx_2d\eta - \int_0^t\int_{R_z}\Delta^2\varphi\varphi_{,\eta}dAd\eta$$

$$- \int_0^t\int_{R_z}[\varphi_{,\beta}\Delta w_\beta - w_\beta^*(\Delta\varphi)_{,\beta}]\varphi_{,\eta}dAd\eta$$

$$+ \int_0^t\int_{R_z}[\varphi_{,1}V_{,22} - V(\Delta\varphi)_{,1}]\varphi_{,\eta}dAd\eta$$

$$+ \int_0^t\int_{R_z}[(\tau_2\theta)_{,1} - (\tau_1\theta)_{,2}]\varphi_{,\eta}dAd\eta$$

$$+ \int_0^t\int_{R_z}[(g_2^*T)_{,1} - (g_1^*T)_{,2}]\varphi_{,\eta}dAd\eta. \tag{3.19}$$

By the divergence theorem, we have

$$-\int_0^t\int_{R_z}\Delta^2\varphi\varphi_{,\eta}dAd\eta$$

$$= \int_0^t \int_{L_z} \Delta\varphi_{,1}\varphi_{,\eta}dx_2d\eta + \int_0^t \int_{R_z} \Delta\varphi_{,\alpha}\varphi_{,\alpha\eta}dx_2d\eta$$

$$= \frac{\partial}{\partial z}\int_0^t \int_{L_z} \varphi_{,11}\varphi_{,\eta}dx_2d\eta - \int_0^t \int_{L_z} \varphi_{,11}\varphi_{,1\eta}dx_2d\eta$$

$$+ \int_0^t \int_{L_z} \varphi_{,122}\varphi_{,\eta}dx_2d\eta - \int_0^t \int_{L_z} \varphi_{,1\alpha}\varphi_{,\alpha\eta}dx_2d\eta$$

$$- \frac{1}{2}\int_{R_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dA|_{\eta=t}$$

$$= \frac{\partial}{\partial z}\int_0^t \int_{L_z} \varphi_{,11}\varphi_{,\eta}dx_2d\eta - 2\int_0^t \int_{L_z} \varphi_{,1\alpha}\varphi_{,\alpha\eta}dx_2d\eta$$

$$- \frac{1}{2}\int_{R_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dA|_{\eta=t}, \tag{3.20}$$

and

$$- \int_0^t \int_{R_z} [\varphi_{,\beta}\Delta w_\beta - w_\beta^*(\Delta\varphi)_{,\beta}]\varphi_{,\eta}dAd\eta$$

$$= \int_0^t \int_{L_z} \varphi_{,\beta}w_{\beta,1}\varphi_{,\eta}dx_2d\eta + \int_0^t \int_{R_z} \varphi_{,\alpha\beta}w_{\beta,\alpha}\varphi_{,\eta}dAd\eta$$

$$+ \int_0^t \int_{R_z} \varphi_{,\beta}w_{\beta,\alpha}\varphi_{,\alpha\eta}dAd\eta - \int_0^t \int_{L_z} w_\beta^*\varphi_{,1\beta}\varphi_{,\eta}dx_2d\eta$$

$$- \int_0^t \int_{R_z} w_{\beta,\alpha}^*\varphi_{\alpha\beta}\varphi_{,\eta}dAd\eta - \int_0^t \int_{R_z} w_\beta^*\varphi_{,\alpha\beta}\varphi_{,\alpha\eta}dAd\eta$$

$$= \int_0^t \int_{L_z} \varphi_{,\beta}w_{\beta,1}\varphi_{,\eta}dx_2d\eta + \int_0^t \int_{R_z} \varphi_{,\alpha\beta}v_{\beta,\alpha}\varphi_{,\eta}dAd\eta$$

$$+ \int_{R_z} \varphi_{,\beta}w_{\beta,\alpha}\varphi_{,\alpha}dA|_{\eta=t} - \int_0^t \int_{R_z} \varphi_{,\beta\eta}w_{\beta,\alpha}\varphi_{,\alpha}dAd\eta$$

$$- \int_0^t \int_{R_z} \varphi_{,\beta}w_{\beta,\alpha\eta}\varphi_{,\alpha}dAd\eta - \int_0^t \int_{L_z} w_\beta^*\varphi_{,1\beta}\varphi_{,\eta}dx_2d\eta$$

$$- \int_0^t \int_{R_z} w_\beta^*\varphi_{,\alpha\beta}\varphi_{,\alpha\eta}dAd\eta$$

$$= \int_0^t \int_{L_z} \varphi_{,\beta}w_{\beta,1}\varphi_{,\eta}dx_2d\eta + \int_{R_z} \varphi_{,\beta}w_{\beta,\alpha}\varphi_{,\alpha}dA|_{\eta=t}$$

$$+ \int_0^t \int_{L_z} \varphi_{,\beta\eta}w_\beta\varphi_{,1}dAd\eta + \int_0^t \int_{R_z} \varphi_{,\alpha\beta\eta}w_\beta\varphi_{,\alpha}dAd\eta$$

$$+ \int_0^t \int_{R_z} \varphi_{,\beta\eta}w_\beta\Delta\varphi dAd\eta + \int_0^t \int_{L_z} \varphi_{,\beta}w_{\beta,\eta}\varphi_{,1}dx_2d\eta$$

$$+ \int_0^t \int_{R_z} \varphi_{,\alpha\beta}w_{\beta,\eta}\varphi_{,\alpha}dAd\eta + \int_0^t \int_{R_z} \varphi_{,\beta}w_{\beta,\eta}\Delta\varphi dAd\eta$$

$$- \int_0^t \int_{L_z} w_\beta^*\varphi_{,1\beta}\varphi_{,\eta}dx_2d\eta - \int_0^t \int_{R_z} w_\beta^*\varphi_{,\alpha\beta}\varphi_{,\alpha\eta}dAd\eta$$

$$= \int_0^t \int_{L_z} \varphi_{,\beta} w_{\beta,1} \varphi_{,\eta} dx_2 d\eta + \int_{R_z} \varphi_{,\beta} w_{\beta,\alpha} \varphi_{,\alpha} dA|_{\eta=t}$$

$$+ \int_0^t \int_{L_z} \varphi_{,\beta\eta} w_\beta \varphi_{,1} dA d\eta - \int_0^t \int_{L_z} \varphi_{,\alpha\eta} w_1 \varphi_{,\alpha} dx_2 d\eta$$

$$- \int_0^t \int_{R_z} \varphi_{,\alpha\eta} w_\beta \varphi_{,\alpha\beta} dA d\eta + \int_0^t \int_{R_z} \varphi_{,\beta\eta} w_\beta \Delta \varphi dA d\eta$$

$$+ \int_0^t \int_{L_z} \varphi_{,\beta} w_{\beta,\eta} \varphi_{,1} dx_2 d\eta + \frac{1}{2} \int_0^t \int_{L_z} \varphi_{,\alpha} w_{1,\eta} \varphi_{,\alpha} dx_2 d\eta$$

$$+ \int_0^t \int_{R_z} \varphi_{,\beta} w_{\beta,\eta} \Delta \varphi dA d\eta - \int_0^t \int_{L_z} w_\beta^* \varphi_{,1\beta} \varphi_{,\eta} dx_2 d\eta$$

$$- \int_0^t \int_{R_z} w_\beta^* \varphi_{,\alpha\beta} \varphi_{,\alpha\eta} dA d\eta, \tag{3.21}$$

where we have used that $\int_0^t \int_{R_z} \varphi_{,\alpha\beta} v_{\beta,\alpha} \varphi_{,\eta} dA d\eta = 0$ in view of (3.1).

$$\int_0^t \int_{R_z} [\varphi_{,1} V_{,22} - V(\Delta \varphi)_{,1}] \varphi_{,\eta} dA d\eta$$

$$= - \int_0^t \int_{L_z} \varphi_{,12} \varphi_{,\eta} V dx_2 d\eta - \int_0^t \int_{R_z} \varphi_{,1} \varphi_{,2\eta} V_{,2} dA d\eta + \int_0^t \int_{R_z} \varphi_{,1\alpha} \varphi_{,\alpha\eta} V dA d\eta, \tag{3.22}$$

$$\int_0^t \int_{R_z} [(\tau_2 \theta)_{,1} - (\tau_1 \theta)_{,2}] \varphi_{,\eta} dA d\eta$$

$$= - \int_0^t \int_{L_z} \tau_2 \theta \varphi_{,\eta} dx_2 d\eta + \int_0^t \int_{R_z} \tau_1 \theta \varphi_{,2\eta} dA d\eta - \int_0^t \int_{R_z} \tau_2 \theta \varphi_{,1\eta} dA d\eta, \tag{3.23}$$

$$\int_0^t \int_{R_z} [(g_2^* T)_{,1} - (g_1^* T)_{,2}] \varphi_{,\eta} dA d\eta$$

$$= - \int_0^t \int_{L_z} g_2^* T \varphi_{,\eta} dx_2 d\eta + \int_0^t \int_{R_z} g_1^* T \varphi_{,2\eta} dA d\eta - \int_0^t \int_{R_z} g_2^* T \varphi_{,1\eta} dA d\eta. \tag{3.24}$$

If we define

$$\Phi_2(z,t) = \int_0^t \int_{R_z} \varphi_{,\alpha\eta} \varphi_{,\alpha\eta} dA d\eta, \tag{3.25}$$

and combine(3.19)-(3.24), we conclude that

$$\Phi_2(z,t) + \frac{1}{2} \int_{R_z} \varphi_{,\alpha\beta} \varphi_{,\alpha\beta} dA|_{\eta=t} = y_{21}(z,t) + y_{22}(z,t) + y_{23}(z,t), \tag{3.26}$$

where

$$y_{21}(z,t) = \frac{\partial}{\partial z} \int_0^t \int_{L_z} [\varphi_{,11} \varphi_{,\eta} - \frac{1}{2} \varphi_{,\eta}^2] dx_2 d\eta,$$

$$y_{22}(z,t) = \int_0^t \int_{L_z} [-2\varphi_{,1\alpha}\varphi_{,\alpha\eta} + \varphi_{,\beta}w_{\beta,1}\varphi_{,\eta} + \varphi_{,\beta\eta}w_\beta\varphi_{,1} - \varphi_{,\alpha\eta}w_1\varphi_{,\alpha} + \varphi_{,\beta}w_{\beta,\eta}\varphi_{,1}$$
$$- \frac{1}{2}\varphi_{,\alpha}w_{1,\eta}\varphi_{,\alpha} - w_\beta^*\varphi_{,1\beta}\varphi_{,\eta} - \varphi_{,12}\varphi_{,\eta}V - \tau_2\theta\varphi_{,\eta} - g_2^*T\varphi_{,\eta}]dx_2 d\eta,$$

$$y_{23}(z,t) = \int_{R_z} \varphi_{,\beta}w_{\beta,\alpha}\varphi_{,\alpha}dA|_{\eta=t}$$
$$+ \int_0^t \int_{R_z} [-\varphi_{,\alpha\eta}w_\beta\varphi_{,\alpha\beta} + \varphi_{,\beta\eta}w_\beta\Delta\varphi + \varphi_{,\beta}w_{\beta,\eta}\Delta\varphi - w_\beta^*\varphi_{,\alpha\beta}\varphi_{,\alpha\eta}$$
$$- \varphi_{,1}\varphi_{,2\eta}V_{,2} + \varphi_{,1\alpha}\varphi_{,\alpha\eta}V + \tau_1\theta\varphi_{,2\eta} - \tau_2\theta\varphi_{,1\eta} - g_2^*T\varphi_{,1\eta}]dAd\eta.$$
$$(3.27)$$

Step 3: Definition of $\Phi_3(z,t)$

We define

$$\Phi_3(z,t) = \int_0^t \int_{R_z} T_{,\alpha}T_{,\alpha}dAd\eta. \tag{3.28}$$

Integrating by parts and using the equation (3.3) and the initial-boundary conditions, we have

$$\Phi_3(z,t) = -\int_0^t \int_{L_z} TT_{,1}dx_2 d\eta - \int_0^t \int_{R_z} \Delta TT dAd\eta$$
$$= -\int_0^t \int_{L_z} TT_{,1}dx_2 d\eta - \int_0^t \int_{R_z} [T_{,\eta} + \varphi_{,2}T_{,1} - \varphi_{,1}T_{,2} - w_\alpha^*T_{,\alpha} + VT_{,1}]TdAd\eta$$
$$\leqslant -\frac{\partial}{\partial z}\int_0^t \int_{L_z} T^2 dx_2 d\eta - \int_0^t \int_{R_z} \varphi_{,2}T_{,1}TdAd\eta + \int_0^t \int_{R_z} \varphi_{,1}T_{,2}TdAd\eta$$
$$- \frac{1}{2}\int_0^t \int_{L_z} w_1^*T^2 dx_2 d\eta + \frac{1}{2}\int_0^t \int_{L_z} VT^2 dx_2 d\eta. \tag{3.29}$$

We conclude that

$$\Phi_3(z,t) \leqslant y_{31}(z,t) + y_{32}(z,t) + y_{33}(z,t), \tag{3.30}$$

where

$$y_{31}(z,t) = -\frac{\partial}{\partial z}\int_0^t \int_{L_z} T^2 dx_2 d\eta,$$
$$y_{32}(z,t) = \frac{1}{2}\int_0^t \int_{L_z} [-w_1^*T^2 + VT^2]dx_2 d\eta, \tag{3.31}$$
$$y_{33}(z,t) = \int_0^t \int_{R_z} [-\varphi_{,2}T_{,1}T + \varphi_{,1}T_{,2}T]dAd\eta.$$

Now, we first define

$$\Phi(z,t) = \int_0^t \int_{R_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\eta$$

$$+\delta\int_0^t \int_{R_z} \varphi_{,\alpha\eta}\varphi_{,\alpha\eta}dAd\eta + \int_0^t \int_{R_z} T_{,\alpha}T_{,\alpha}dAd\eta, \quad (3.32)$$

where $\delta < 1$ is a positive constant to be determined later. Then, we let

$$\Psi(z,t) = \int_z^\infty \Phi(\xi,t)d\xi. \quad (3.33)$$

Clearly, we find

$$-\frac{\partial}{\partial z}\Psi(z,t) = \Phi(z,t), \quad (3.34)$$

and

$$\frac{\partial^2}{\partial z^2}\Psi(z,t) = \int_0^t \int_{L_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dx_2d\eta$$

$$+\delta\int_0^t \int_{L_z} \varphi_{,\alpha\eta}\varphi_{,\alpha\eta}dx_2d\eta + \int_0^t \int_{L_z} \theta_{,\alpha}\theta_{,\alpha}dx_2d\eta. \quad (3.35)$$

In next section, we derive our main results.

4. Continuous dependence on the coefficients g_α

In this section, to derive continuous dependence results we will use the basic inequalities which have been derived in section 3. In addition, we will also frequently use the following well-known inequalities:
(1).If $\omega(x_2) \in C^1(0,h)$ and $\omega(0) = \omega(h) = 0$, then

$$\int_{L_z} \omega^2 dx_2 \leqslant \frac{h^2}{\pi^2}\int_{L_z} \omega_{,2}^2 dx_2. \quad (4.1)$$

(2).If $\omega(x_2) \in C^2(0,h)$ and $\omega(0) = \omega_{,2}(0) = \omega(h) = \omega_{,2}(h) = 0$, then

$$\int_{L_z} \omega_{,2}^2 dx_2 \leqslant \frac{h^2}{4\pi^2}\int_{L_z} \omega_{,22}^2 dx_2. \quad (4.2)$$

(3).If $\omega(x_2) \in C^2(0,h)$ and $\omega(0) = \omega_{,2}(0) = \omega(h) = \omega_{,2}(h) = 0$, then

$$\int_{L_z} \omega^2 dx_2 \leqslant (\frac{2}{3})^4\frac{h^4}{\pi^4}\int_{L_z} \omega_{,22}^2 dx_2. \quad (4.3)$$

For proofs of these inequalities see Refs [10, 22]. In addition to (4.1)-(4.3), we also use some Sobolev inequality in $R_z \times [0,T]$ which have been widely used in the study of

Navier-Stokes equations or Boussinesq equations (see e.g. Refs.[19, 23, 24, 25, 26]). Let ω satisfying the condition in (2) and $\lim_{x_1 \to \infty} \omega(x_1, x_2, t) = 0$, i.e.,

$$\int_{R_z} \omega^4 dA \leqslant \int_{R_z} \omega^2 dA \int_{R_z} \omega_{,\alpha} \omega_{,\alpha} dA, \tag{4.4}$$

$$\max_{(0,t)} \int_{R_z} \omega^4 dA \leqslant 36 \int_0^t \int_{R_z} \omega_{,\alpha} \omega_{,\alpha} dA d\eta \int_0^t \int_{R_z} \omega_{,\eta}^2 dA d\eta. \tag{4.5}$$

In (4.4) and (4.5), if the function ω satisfy the additional homogeneous boundary condition, then the following inequalities hold

$$\int_R \omega^4 dA \leqslant \frac{1}{2} \int_R \omega^2 dA \int_R \omega_{,\alpha} \omega_{,\alpha} dA, \tag{4.6}$$

$$\max_{(0,t)} \int_R \omega^4 dA \leqslant 18 \int_0^t \int_R \omega_{,\alpha} \omega_{,\alpha} dA d\eta \int_0^t \int_R \omega_{,\eta}^2 dA d\eta. \tag{4.7}$$

The proofs for inequalities (4.4)-(4.7) may be found in paper [15]. In addition to inequalities (4.1)-(4.7), we derive the following inequalities which will be used in this paper. Let ω satisfy the additional homogeneous boundary condition and

$$\lim_{x_1 \to \infty} \omega(x_1, x_2, t) = 0,$$

we observe that

$$\omega^3 = 3 \int_0^{x_2} \omega^2 w_{,s} ds = -3 \int_{x_2}^h \omega^2 w_{,s} ds,$$

$$\omega^3 = 3 \int_z^\infty \omega^2 w_{,\xi} d\xi.$$

So,

$$|\omega|^3 \leqslant \frac{3}{2} \int_0^h \omega^2 |w_{,s}| ds,$$

and

$$|\omega|^3 \leqslant 3 \int_z^\infty \omega^2 |w_{,\xi}| d\xi.$$

Thus, we have

$$\int_{R_z} \omega^6 dA \leqslant \frac{9}{4} \int_{R_z} \omega^4 dA \int_{R_z} \omega_{,\alpha} \omega_{,\alpha} dA \leqslant \frac{9h^2}{4\pi^2} \left(\int_{R_z} \omega_{,\alpha} \omega_{,\alpha} dA \right)^3,$$

where we have used (4.1) and (4.4). We also note that

$$\int_0^h \omega^4 dA = -4 \int_{R_z} \omega^3 \omega_{,1} dA$$

$$\leqslant 4\Big(\int_{R_z}\omega_{,1}^2 dA\Big)^{\frac{1}{2}}\Big(\int_{R_z}\omega^6 dA\Big)^{\frac{1}{2}}$$

$$\leqslant \frac{9h^2}{\pi^2}\Big(\int_{R_z}\omega_{,\alpha}\omega_{,\alpha}dA\Big)^3. \tag{4.8}$$

A combination of (3.16), (3.26) and (3.30) leads to

$$\Psi(z,t)+\frac{1}{2}\delta\int_z^\infty\int_{R_\xi}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\xi+\frac{1}{2}\int_z^\infty\int_{R_\xi}\varphi_{,\alpha}\varphi_{,\alpha}dAd\xi\leqslant J_1+J_2+J_3, \tag{4.9}$$

where

$$
\begin{aligned}
J_1 &= \int_0^t\int_{L_z}[\varphi_{,\alpha}\varphi_{,\alpha}+\frac{1}{2}\delta\varphi_{,\eta}^2+T^2-\varphi_{,11}\varphi-\delta\varphi_{,11}\varphi_{,\eta}]dx_2d\eta,\\
J_2 &= \int_z^\infty[y_{12}(\xi,t)+\delta y_{22}(\xi,t)+y_{32}(\xi,t)]d\xi,\\
J_3 &= \int_z^\infty[y_{13}(\xi,t)+\delta y_{23}(\xi,t)+y_{33}(\xi,t)]d\xi.
\end{aligned}
\tag{4.10}
$$

Making use of the Schwarz's inequality, (4.1), (4.2), (4.3) and the Arithmetic-Geometric mean inequality, we have

$$
\begin{aligned}
J_1 &\leqslant \frac{h^2}{\pi^2}\int_0^t\int_{L_z}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dx_2d\eta+\frac{h^2}{2\pi^2}\delta\int_0^t\int_{L_z}\varphi_{,\alpha\eta}\varphi_{,\alpha\eta}dx_2d\eta\\
&\quad+\frac{h^2}{\pi^2}\int_0^t\int_{L_z}T_{,\alpha}T_{,\alpha}dx_2d\eta+\frac{4h^2}{9\pi^2}\Big(\int_0^t\int_{L_z}\varphi_{,11}^2dx_2d\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{L_z}\varphi_{,22}^2dx_2d\eta\Big)^{\frac{1}{2}}\\
&\quad+\frac{h}{\pi}\delta\Big(\int_0^t\int_{L_z}\varphi_{,11}^2dx_2d\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{L_z}\varphi_{,2\eta}^2dx_2d\eta\Big)^{\frac{1}{2}}\\
&\leqslant k_1\frac{\partial^2}{\partial z^2}\Psi(z,t),
\end{aligned}
\tag{4.11}
$$

where

$$k_1=\frac{h^2}{\pi^2}+\frac{h^2}{2\pi^2}\delta+\frac{2h^2}{9\pi^2}+\frac{h}{2\pi}\delta. \tag{4.12}$$

Next, we seek a bound for J_2. Making use of the Schwarz's inequality, (3.1)-(3.5), (2.24), and the Arithmetic-Geometric mean inequality again, we obtain

$$
\begin{aligned}
&\int_z^\infty y_{12}(\xi,t)d\xi\\
&=\int_0^t\int_{R_z}\Big[-\varphi_{,1\eta}\varphi-w_1^*\Delta\varphi\varphi+\varphi\varphi_{,2}w_{2,1}-\varphi\varphi_{,2}w_{1,2}+\frac{1}{2}V\varphi_{,\alpha}\varphi_{,\alpha}\\
&\qquad\qquad\qquad+V\varphi_{,1}^2-V\varphi_{,12}\varphi-\tau_2\theta\varphi-g_2^*T\varphi\Big]dAd\eta\\
&\leqslant\frac{4h^2}{9\pi^2}\Big(\int_0^t\int_{R_z}\varphi_{,1\eta}^2dAd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}\varphi_{,22}^2dAd\eta\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$+\max_t\left(\int_{R_z}(w_1^*)^4dA\right)^{\frac{1}{4}}\left(\int_0^t\int_{R_z}\varphi^4dAd\eta\right)^{\frac{1}{4}}\left(\int_0^t\int_{R_z}(\Delta\varphi)^2dAd\eta\right)^{\frac{1}{2}}$$

$$+\max_t\left(\int_{R_z}\varphi^4dA\right)^{\frac{1}{4}}\left(\int_0^t\int_{R_z}\varphi_{,2}^4dAd\eta\right)^{\frac{1}{4}}\left(\int_0^t\int_{R_z}w_{2,1}^2dAd\eta\right)^{\frac{1}{2}}$$

$$+\max_t\left(\int_{R_z}\varphi^4dA\right)^{\frac{1}{4}}\left(\int_0^t\int_{R_z}\varphi_{,2}^4dAd\eta\right)^{\frac{1}{4}}\left(\int_0^t\int_{R_z}w_{1,2}^2dAd\eta\right)^{\frac{1}{2}}$$

$$+\frac{h^2}{2\pi^2}|V|_{\max}\int_0^t\int_{R_z}\varphi_{,2\alpha}\varphi_{,2\alpha}dAd\eta+\frac{h^2}{\pi^2}|V|_{\max}\int_0^t\int_{R_z}\varphi_{,12}^2dAd\eta$$

$$+\frac{4h^3\tau}{9\pi^3}\left(\int_0^t\int_{R_z}\theta_{,2}^2dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{R_z}\varphi_{,22}^2dAd\eta\right)^{\frac{1}{2}}$$

$$+\frac{4h^3g^*}{9\pi^3}\left(\int_0^t\int_{R_z}T_{,2}^2dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{R_z}\varphi_{,22}^2dAd\eta\right)^{\frac{1}{2}}$$

$$\leqslant\frac{2h^2}{9\pi^2}\int_0^t\int_{R_z}\varphi_{,1\eta}^2dAd\eta+[\frac{2h^2}{9\pi^2}+\frac{2(1+g^*)h^3}{9\pi^3}]\int_0^t\int_{R_z}\varphi_{,22}^2dAd\eta$$

$$+\frac{2}{3}\sqrt{\frac{6h^3}{\pi^3}}\sqrt[4]{\widetilde{Q}_1}\sqrt[4]{\widetilde{Q}_2}\int_0^t\int_{R_z}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\eta$$

$$+2\sqrt{\frac{6h^3Q_1}{\pi^3}}\left(\int_0^t\int_{R_z}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\eta\right)^{\frac{3}{4}}\left(\int_0^t\int_{R_z}\varphi_{,\alpha\eta}\varphi_{,\alpha\eta}dAd\eta\right)^{\frac{1}{4}}$$

$$+\frac{h^2}{2\pi^2}|V|_{\max}\int_0^t\int_{R_z}\varphi_{,2\alpha}\varphi_{,2\alpha}dAd\eta+\frac{h^2}{\pi^2}|V|_{\max}\int_0^t\int_{R_z}\varphi_{,12}^2dAd\eta$$

$$+\frac{2h^3\tau^2}{9\pi^3}\int_0^t\int_{R_z}\theta_{,2}^2dAd\eta+\frac{2h^3g^*}{9\pi^3}\int_0^t\int_{R_z}T_{,2}^2dAd\eta,\tag{4.13}$$

where

$$\tau^2=\max\{\tau_\alpha\tau_\alpha\},\ (g^*)^2=\max\{g_\alpha^*g_\alpha^*\}.$$

Using the Young's inequality for $a>0$, and $b>0$,i.e.,

$$ab\leqslant\frac{a^p}{p}+\frac{a^q}{q},\quad\frac{1}{p}+\frac{1}{q}=1,p,q>0.\tag{4.14}$$

in the fourth term on the right of (4.13), we have

$$\int_z^\infty y_{12}(\xi,t)d\xi\leqslant k_2(-\frac{\partial}{\partial z}\Psi(z,t))+\frac{2h^3\tau^2}{9\pi^3}\int_0^t\int_{R_z}\theta_{,2}^2dAd\eta\tag{4.15}$$

where

$$k_2=\frac{2h^2}{9\pi^2}+\frac{3h^2}{2\pi^2}|V|_{\max}+\frac{2(1+g^*)h^3}{9\pi^3}+\frac{2}{3}\sqrt{\frac{6h^3}{\pi^3}}\sqrt[4]{\widetilde{Q}_1}\sqrt[4]{\widetilde{Q}_2}+\frac{3}{2\delta}\sqrt{\frac{6h^3\widetilde{Q}_1}{\pi^3}}.\tag{4.16}$$

Similar to the derivations of (4.15), we may have

$$
\begin{aligned}
&\int_z^\infty y_{22}(\xi,t)d\xi \leqslant k_3(-\frac{\partial}{\partial z}\Psi(z,t)) + \frac{h^2\tau^2}{2\pi^2}\int_0^t\int_{R_z}\theta_{,2}^2 dAd\eta, \\
&\int_z^\infty y_{32}(\xi,t)d\xi \leqslant k_4(-\frac{\partial}{\partial z}\Psi(z,t)),
\end{aligned}
\tag{4.17}
$$

where

$$
\begin{aligned}
k_3 &= [1 + \sqrt{6\widetilde{Q}_1}\sqrt{\frac{h}{\pi}} + \sqrt{\frac{3h}{2\pi}}\sqrt[4]{\widetilde{Q}_1\widetilde{Q}_2} + \frac{h}{2\pi} + \frac{h^2(1+g^*)}{2\pi^2}]\frac{1}{\delta}, \\
k_4 &= \frac{h^2}{2\pi^2}\sqrt{\widetilde{Q}_1} + \frac{h^2}{2\pi^2}|V|_{\max}.
\end{aligned}
\tag{4.18}
$$

Combining (4.15)-(4.17), we have

$$
J_2 \leqslant (k_2 + k_3\delta + k_4)(-\frac{\partial}{\partial z}\Psi(z,t)) + [\frac{2h^3}{9\pi^3} + \frac{h^2\delta}{2\pi^2}]\tau^2\int_0^t\int_{R_z}\theta_{,2}^2 dAd\eta.
\tag{4.19}
$$

Now, we derive bound for J_3. We first seek bounds for $y_{i3}(z,t), (i=1,2,3)$. Making use of the Schwarz's inequality, (3.1)-(3.5) and the Arithmetic-Geometric mean inequality again, we obtain

$$
\begin{aligned}
y_{13}(z,t) \leqslant{}& \max_{(0,t)}\left(\int_{R_z}(w_\beta^* w_\beta^*)^2 dA\right)^{\frac{1}{4}}\left(\int_0^t\int_{R_z}(\varphi_{,\beta}\varphi_{,\beta})^2 dAd\eta\right)^{\frac{1}{4}}\left(\int_0^t\int_{R_z}(\Delta\varphi)^2 dAd\eta\right)^{\frac{1}{2}} \\
&+ |V_{,2}|_{\max}\frac{h^2}{\sqrt{2}\pi^2}\left(\int_0^t\int_{R_z}\varphi_{,12}^2 dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{R_z}\varphi_{,22}^2 dAd\eta\right)^{\frac{1}{2}} \\
&+ \frac{h^2\tau}{\sqrt{2}\pi^2}\left(\int_0^t\int_{R_z}\theta_{,2}^2 dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{R_z}\varphi_{,22}^2 dAd\eta\right)^{\frac{1}{2}} \\
&+ \frac{h^2\tau}{\pi^2}\left(\int_0^t\int_{R_z}\theta_{,2}^2 dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{R_z}\varphi_{,12}^2 dAd\eta\right)^{\frac{1}{2}} \\
&+ \frac{h^2 g^*}{\sqrt{2}\pi^2}\left(\int_0^t\int_{R_z}T_{,2}^2 dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{R_z}\varphi_{,22}^2 dAd\eta\right)^{\frac{1}{2}} \\
&+ \frac{h^2 g^*}{\pi^2}\left(\int_0^t\int_{R_z}T_{,2}^2 dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{R_z}\varphi_{,12}^2 dAd\eta\right)^{\frac{1}{2}} \\
\leqslant{}& \sqrt{\frac{6h}{\pi}}\sqrt[4]{\widetilde{Q}_1\widetilde{Q}_2}\int_0^t\int_{R_z}\varphi_{,\alpha\beta}\varphi_{,\alpha\beta} dAd\eta \\
&+ [|V_{,2}|_{\max}\frac{h^2}{2\sqrt{2}\pi^2} + \frac{h^2}{2\sqrt{2}\pi^2} + \frac{2\sqrt{2}h^2 g^*}{2\pi^2}]\int_0^t\int_{R_z}\varphi_{,22}^2 dAd\eta \\
&+ [|V_{,2}|_{\max}\frac{h^2}{2\sqrt{2}\pi^2} + \frac{h^2}{2\pi^2} + \frac{h^2 g^*}{2\pi^2}]\int_0^t\int_{R_z}\varphi_{,12}^2 dAd\eta
\end{aligned}
$$

$$+ \frac{h^2 \tau^2}{\pi^2} \int_0^t \int_{R_z} \theta_{,2}^2 dA d\eta + \frac{h^2 g^*}{\pi^2} \int_0^t \int_{R_z} T_{,2}^2 dA d\eta$$

$$\leqslant k_5 \Phi(z,t) + \frac{h^2 \tau^2}{\pi^2} \int_0^t \int_{R_z} \theta_{,2}^2 dA d\eta, \tag{4.20}$$

where

$$k_5 = \sqrt{\frac{6h}{\pi}} \sqrt[4]{\widetilde{Q}_1 \widetilde{Q}_2} + |V_{,2}|_{\max} \frac{h^2}{2\pi^2} + \frac{h^2}{2\pi^2} + \frac{h^2 g^*}{2\pi^2}. \tag{4.21}$$

Since w_β vanish at $x_2 = 0, h$, we have

$$w_\beta w_\beta = 2 \int_0^{x_2} w_\beta w_{\beta,2} dx_2 = -2 \int_{x_2}^h w_\beta w_{\beta,2} dx_2.$$

which leads to

$$w_\beta w_\beta \leqslant \int_0^h |w_\beta w_{\beta,2}| dx_2 \leqslant \left(\int_0^h w_\beta w_\beta dx_2 \right)^{\frac{1}{2}} \left(\int_0^h w_{\beta,2} w_{\beta,2} dx_2 \right)^{\frac{1}{2}}$$
$$\leqslant \frac{h}{\pi} \int_0^h w_{\beta,2} w_{\beta,2} dx_2. \tag{4.22}$$

We now use this result to bound $y_{23}(z,t)$. For the first term of (3.27), using Schwarz's inequality and (4.8) we have

$$- \int_0^t \int_{R_z} \varphi_{,\alpha\eta} w_\beta \varphi_{,\alpha\beta} dA d\eta$$

$$\leqslant \int_0^t \left(\int_z^\infty \int_{L_\xi} \varphi_{,\alpha\eta} \varphi_{,\alpha\eta} w_\beta w_\beta dA \right)^{\frac{1}{2}} \left(\int_{R_z} \varphi_{,\alpha\beta} \varphi_{,\alpha\beta} dA \right)^{\frac{1}{2}} d\eta$$

$$\leqslant \sqrt{\frac{h}{\pi}} \int_0^t \left(\int_z^\infty \int_{L_\xi} w_{\beta,2} w_{\beta,2} dx_2 \int_{L_\xi} \varphi_{,\alpha\eta} \varphi_{,\alpha\eta} dx_2 d\xi \right)^{\frac{1}{2}} \left(\int_{R_z} \varphi_{,\alpha\beta} \varphi_{,\alpha\beta} dA \right)^{\frac{1}{2}} d\eta$$

$$\leqslant \sqrt{\frac{h\sqrt{h}}{\pi}} \int_0^t \left(\int_z^\infty \left(\int_{L_\xi} (w_{\beta,2} w_{\beta,2})^2 dx_2 \right)^{\frac{1}{2}} \int_{L_\xi} \varphi_{,\alpha\eta} \varphi_{,\alpha\eta} dx_2 d\xi \right)^{\frac{1}{2}}$$
$$\cdot \left(\int_{R_z} \varphi_{,\alpha\beta} \varphi_{,\alpha\beta} dA \right)^{\frac{1}{2}} d\eta$$

$$\leqslant \sqrt[8]{\frac{72 h^7}{\pi^5}} \int_0^t \left(\int_{R_z} w_{\beta,\alpha2} w_{\beta,\alpha2} dA \right)^{\frac{1}{2}} \left(\int_{R_z} \varphi_{,\alpha\eta} \varphi_{,\alpha\eta} dA \right)^{\frac{1}{2}} \left(\int_{R_z} \varphi_{,\alpha\beta} \varphi_{,\alpha\beta} dA \right)^{\frac{1}{2}} d\eta. \tag{4.23}$$

Thus, as the derivation of (4.23) and using (4.1) and (4.4) we have

$$y_{23}(z,t) \leqslant \left(\int_{R_z} (\varphi_{,\beta} \varphi_{,\beta})^2 dA \right)^{\frac{1}{2}} \left(\int_{R_z} w_{\beta,\alpha} w_{\beta,\alpha} dA \right)^{\frac{1}{2}} \Big|_{\eta=t}$$

$$+ \sqrt[8]{\frac{72 h^7}{\pi^5}} \int_0^t \left(\int_{R_z} w_{\beta,\alpha2} w_{\beta,\alpha2} dA \right)^{\frac{1}{2}} \left(\int_{R_z} \varphi_{,\alpha\eta} \varphi_{,\alpha\eta} dA \right)^{\frac{1}{2}}$$

$$\cdot \left(\int_{R_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dA \right)^{\frac{1}{2}} d\eta$$

$$+ \sqrt{\frac{h}{\pi}} \widetilde{Q}_2^{\frac{1}{4}} \widetilde{Q}_5^{\frac{1}{4}} \int_0^t \int_{R_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\eta$$

$$+ \frac{h}{\pi}|V_{,2}|_{\max} \left(\int_0^t \int_{R_z} \varphi_{,12}^2 dAd\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_z} \varphi_{,2\eta}^2 dAd\eta \right)^{\frac{1}{2}}$$

$$+ |V|_{\max} \left(\int_0^t \int_{R_z} \varphi_{,1\alpha}\varphi_{,1\alpha}dAd\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_z} \varphi_{,\alpha\eta}\varphi_{,\alpha\eta}dAd\eta \right)^{\frac{1}{2}}$$

$$+ \frac{h}{\pi}\tau \left(\int_0^t \int_{R_z} \theta_{,2}^2 dAd\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_z} \varphi_{,2\eta}^2 dAd\eta \right)^{\frac{1}{2}}$$

$$+ \frac{h}{\pi}\tau \left(\int_0^t \int_{R_z} \theta_{,2}^2 dAd\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_z} \varphi_{,1\eta}^2 dAd\eta \right)^{\frac{1}{2}}$$

$$+ \frac{h}{\pi}g^* \left(\int_0^t \int_{R_z} T_{,2}^2 dAd\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_z} \varphi_{,2\eta}^2 dAd\eta \right)^{\frac{1}{2}}$$

$$+ \frac{h}{\pi}g^* \left(\int_0^t \int_{R_z} T_{,2}^2 dAd\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_z} \varphi_{,1\eta}^2 dAd\eta \right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{2} \int_{R_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dA|_{\eta=t} + \frac{1}{2}\widehat{Q}_1 \int_{R_z} \varphi_{,\alpha}\varphi_{,\alpha}dA|_{\eta=t}$$

$$+ \frac{h\tau^2}{\pi} \int_0^t \int_{R_z} \theta_{,2}^2 dAd\eta + \frac{hg^*}{\pi} \int_0^t \int_{R_z} T_{,2}^2 dAd\eta$$

$$+ [\frac{1}{2}\sqrt[8]{\frac{72h^7}{\pi^5}} E_7^{\frac{1}{2}}(0,t) + \sqrt{\frac{h}{\pi}}Q_2^{\frac{1}{4}}Q_5^{\frac{1}{4}} + \frac{h}{2\pi}|V_{,2}|_{\max}$$

$$+ \frac{1}{2}|V|_{\max}] \int_0^t \int_{R_z} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\eta$$

$$+ [\frac{1}{2}\sqrt[8]{\frac{72h^7}{\pi^5}} E_7^{\frac{1}{2}}(0,t) + \frac{h}{2\pi}|V_{,2}|_{\max} + \frac{1}{2}|V|_{\max}$$

$$+ \frac{h}{2\pi}(1+g^*)] \int_0^t \int_{R_z} \varphi_{,\alpha\eta}\varphi_{,\alpha\eta}dAd\eta. \qquad (4.24)$$

Making use of the method of [19](see section 5 of that paper), an upper bound for $\int_{R_z} w_{\beta,\alpha}w_{\beta,\alpha}dA$ in terms of known data can be obtained. In (4.24), we have noted that

$$\int_{R_z} w_{\beta,\alpha}w_{\beta,\alpha}dA|_{\eta=t} \leqslant \widehat{Q}_1,$$

and we use the notation

$$E_7(0,t) = \int_0^t \int_{R_z} w_{\beta,\alpha2}w_{\beta,\alpha2}dAd\eta. \qquad (4.25)$$

The upper bound for $E_7(0,t)$ will be derived in section 6. Using the Young inequality (4.14) in (4.22), we have

$$y_{23}(z,t) \leqslant k_6 \Phi(z,t) + \frac{h\tau^2}{\pi} \int_0^t \int_{R_z} \theta_{,2}^2 dAd\eta + \frac{1}{2}\widehat{Q}_1 \int_{R_z} \varphi_{,\alpha}\varphi_{,\alpha}dA|_{\eta=t}, \tag{4.26}$$

where

$$k_6 = \frac{1}{2}\sqrt[8]{\frac{72h^7}{\pi^5}}E_7^{\frac{1}{2}}(0,t) + \sqrt{\frac{h}{\pi}}Q_2^{\frac{1}{4}}Q_5^{\frac{1}{4}} + \frac{h}{2\pi}|V_{,2}|_{\max} + \frac{1}{2}|V|_{\max} + \frac{h}{2\pi}(1+g^*). \tag{4.27}$$

Using (4.1) and (4.4), we have

$$y_{33} \leqslant k_7 \Phi(z,t), \tag{4.28}$$

where

$$k_7 = 2\sqrt{\frac{6h}{\pi}}\Phi^{\frac{1}{2}}(0,t).$$

Combining $(4.10)_3$, (4.20), (4.26) and (4.28), we conclude that

$$J_3 \leqslant (k_5 + \delta k_6 + k_7)\Psi(z,t) + (\frac{h^2}{\pi^2} + \frac{h}{\pi})\tau^2 \int_0^t \int_z^\infty \int_{R_\xi} \theta_{,2}^2 dAd\xi d\eta$$
$$+ \frac{1}{2}\delta\widehat{Q}_1 \int_z^\infty \int_{R_\xi} \varphi_{,\alpha}\varphi_{,\alpha}dAd\xi|_{\eta=t}. \tag{4.29}$$

From (2.21) and (2.23), we derive

$$\int_0^t \int_{R_z} \theta_{,2}^2 dAd\eta \leqslant m_1 E(0,t)e^{-m_2 z}, \tag{4.30}$$

Inserting (4.11), (4.19), (4.29) and (4.30) into (4.9) and choosing $\delta \leqslant \min\{1, \widehat{Q}_1^{-1}\}$, we have

$$\Psi(z,t) \leqslant \frac{\partial^2}{\partial z^2}\Psi(z,t) - \widetilde{k}_1\frac{\partial}{\partial z}\Psi(z,t)) + \widehat{k}_2\Psi(z,t) + \tau^2\widetilde{k}_3 e^{-m_2 z}, \tag{4.31}$$

where

$$\widetilde{k}_1 = \frac{1}{k_1}(k_2 + k_3\delta + k_4), \quad \widehat{k}_2 = \frac{1}{k_1}(k_5 + \delta k_6 + k_7),$$
$$\widetilde{k}_3 = \frac{1}{k_1}[\frac{2h^3}{9\pi^3} + \frac{h^2\delta}{2\pi^2} + \frac{h^2}{\pi^2 m_2} + \frac{h}{m_2\pi}]m_1 E(0,t). \tag{4.32}$$

The definitions of $\widetilde{k}_i, (i=1,3)$ and $\widehat{k}_2$ involve $|V|_{\max}$, $|V_{,2}|_{\max}$, $\Phi(0,t)$ and $E_7(0,t)$. We note that the bounds for $|V|_{\max}$ and $|V_{,2}|_{\max}$ are well studied in Section 5 of [15]. While, the bounds for $\Phi(0,t)$ and $E_7(0,t)$ will be proved in section 5 and section 6

of this paper, respectively. We require h, δ or/and the data small enough such that $1 - \widehat{k}_2 > 0$. This requirement may be viewed as a Reynolds number type of restriction on our problem. From (4.31), we obtain an inequality of the form

$$\frac{\partial^2}{\partial z^2} \Psi(z,t) - \widetilde{k}_1 \frac{\partial}{\partial z} \Psi(z,t)) - \widetilde{k}_2 \Psi(z,t) \geqslant -\tau^2 \widetilde{k}_3 e^{-m_2 z}, \qquad (4.33)$$

where $\widetilde{k}_2 = 1 - \widehat{k}_2 > 0$. From (4.33) it follows that

$$(\frac{\partial}{\partial z} - a)(\frac{\partial}{\partial z} \Psi(z,t) + b\Psi(z,t)) \geqslant -\tau^2 \widetilde{k}_3 e^{-m_2 z},$$

or

$$\frac{\partial}{\partial z} \{e^{-az}[\frac{\partial}{\partial z} \Psi(z,t) + b\Psi(z,t)]\} \geqslant -\tau^2 \widetilde{k}_3 e^{-(a+m_2)z}, \qquad (4.34)$$

where

$$a = \frac{\widetilde{k}_1 + \sqrt{\widetilde{k}_1^2 + 4\widetilde{k}_2}}{2}, \quad b = \frac{-\widetilde{k}_1 + \sqrt{\widetilde{k}_1^2 + 4\widetilde{k}_2}}{2}.$$

An integration of (4.34) from z to ∞ leads to

$$\frac{\partial}{\partial z} \Psi(z,t) + b\Psi(z,t) \leqslant \tau^2 \frac{1}{a+m_2} \widetilde{k}_3 e^{-m_2 z}, \qquad (4.35)$$

or

$$\frac{\partial}{\partial z}[\Psi(z,t)e^{bz}] \leqslant \tau^2 \frac{1}{m_2} \widetilde{k}_3 e^{(b-m_2)z}. \qquad (4.36)$$

Integrating (4.36) from 0 to z, we have

$$\Psi(z,t) \leqslant \Psi(0,t)e^{-bz} + \tau^2 \frac{1}{(a+m_2)(b-m_2)} \widetilde{k}_3 (e^{-m_2 z} - e^{-bz}). \qquad (4.37)$$

In order to make the estimate (4.37) explicit, one requires the upper bound for $\Psi(0,t)$. Such a bound will be derived in section 5 of this paper. Using (5.20), (4.37) can be rewritten as

$$\Psi(z,t) \leqslant \tau^2 C e^{-bz} + \tau^2 \frac{1}{(a+m_2)(b-m_2)} \widetilde{k}_3 (e^{-m_2 z} - e^{-bz}). \qquad (4.38)$$

For finite t, then (4.38) establishes continuous dependence on the coefficients g_α. So, we have proofed the theorem 1.

5. Bounds for $\Psi(0,t)$

To make (4.37) explicit, we have to derive bounds for $\Psi(0,t)$. From (4.35), we have

$$b\Psi(0,t) \leqslant -\frac{\partial}{\partial z}\Psi(0,t) + \tau^2 \frac{1}{a+m_2}\widetilde{k}_3. \tag{5.1}$$

So, we only need to bound $-\frac{\partial}{\partial z}\Psi(0,t)$. In view of (3.31) and (3.34), we have

$$
\begin{aligned}
-\frac{\partial}{\partial z}\Psi(0,t) &= \int_0^t \int_{R_0} \varphi_{,\alpha\beta}\varphi_{,\alpha\beta}dAd\eta + \delta\int_0^t \int_{R_0} \varphi_{,\alpha\eta}\varphi_{,\alpha\eta}dAd\eta + \int_0^t \int_{R_0} T_{,\alpha}T_{,\alpha}dAd\eta \\
&= \int_0^t \int_{R_0} v_{\alpha,\beta}v_{\alpha,\beta}dAd\eta + \delta\int_0^t \int_{R_0} v_{\alpha,\eta}v_{\alpha,\eta}dAd\eta + \int_0^t \int_{R_0} T_{,\alpha}T_{,\alpha}dAd\eta \\
&= \Phi_1(0,t) + \delta\Phi_2(0,t) + \Phi_3(0,t).
\end{aligned}
\tag{5.2}
$$

Now, we began to bound each $\Phi_i(0,t), (i=1,2,3)$. Multiplying (2.26) with v_α and integrating by parts, we have

$$
\int_0^t \int_{R_0} \big[v_{\alpha,\eta} - \Delta v_\alpha + v_\beta w_{\alpha,\beta} + w_\beta^* v_{\alpha,\beta} \\
+ V v_{\alpha,1} + v_2 V_{,2}\delta_{\alpha 1} - \tau_\alpha\theta - g_\alpha^* T + \pi_{,\alpha}\big] v_\alpha dAd\eta = 0. \tag{5.3}
$$

Integrating (5.3) by parts and following the derivation of previous sections, we compute

$$
\begin{aligned}
\frac{1}{2}\int_{R_0} v_\alpha v_\alpha dA\big|_{\eta=t} &+ \int_0^t \int_{R_0} v_{\alpha,\beta}v_{\alpha,\beta}dAd\eta \\
&\leqslant \frac{h}{\pi}\left(\int_0^t \int_{R_0} w_{\alpha,\beta}w_{\alpha,\beta}dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t \int_{R_0} v_{\alpha,\beta}v_{\alpha,\beta}dAd\eta\right) \\
&+ |V_{,2}|_{\max}\frac{h^2}{\pi^2}\left(\int_0^t \int_{R_0} v_{2,2}v_{2,2}dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t \int_{R_0} v_{\alpha,2}v_{\alpha,2}dAd\eta\right)^{\frac{1}{2}} \\
&+ \frac{h^2}{\pi^2}\tau\left(\int_0^t \int_{R_0} \theta_{,2}\theta_{,2}dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t \int_{R_0} v_{\alpha,2}v_{\alpha,2}dAd\eta\right)^{\frac{1}{2}} \\
&+ \frac{h^2}{\pi^2}g^*\left(\int_0^t \int_{R_0} T_{,2}T_{,2}dAd\eta\right)^{\frac{1}{2}}\left(\int_0^t \int_{R_0} v_{\alpha,2}v_{\alpha,2}dAd\eta\right)^{\frac{1}{2}},
\end{aligned}
\tag{5.4}
$$

where we have used the fact

$$
\int_0^t \int_{R_0} w_\beta^* v_\alpha v_{\alpha,\beta}dAd\eta = 0, \quad \int_0^t \int_{R_0} V v_\alpha v_{\alpha,1}dAd\eta = 0, \quad \int_0^t \int_{R_0} \pi_{,\alpha}v_\alpha dAd\eta = 0. \tag{5.5}
$$

Now, we multiplying (2.28) with T to obtain

$$
\int_0^t \int_{R_0} [T_{,\eta} - \Delta T + v_\alpha\theta_{,\alpha} + w_\alpha^* T_{,\alpha} + T_{,1}V]T dAd\eta = 0 \tag{5.6}
$$

from which it follows that

$$\int_0^t \int_{R_0} T_{,\alpha} T_{,\alpha} dA d\eta \leqslant \int_0^t \int_{R_0} v_\alpha \theta_{,\alpha} T dA d\eta$$

$$\leqslant \widetilde{Q}_3^{\frac{1}{2}} \frac{h}{\pi} \left(\int_0^t \int_{R_0} T_{,\alpha} T_{,\alpha} dA d\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_0} v_{\alpha,\beta} v_{\alpha,\beta} dA d\eta \right)^{\frac{1}{2}},$$

$$(5.7)$$

where we have used the fact

$$\int_0^t \int_{R_0} w_\alpha^* T_{,\alpha} T dA d\eta = 0, \ \int_0^t \int_{R_0} V T_{,1} T dA d\eta = 0.$$

So,

$$\int_0^t \int_{R_0} T_{,\alpha} T_{,\alpha} dA d\eta \leqslant \widetilde{Q}_3 \frac{h^2}{\pi^2} \int_0^t \int_{R_0} v_{\alpha,\beta} v_{\alpha,\beta} dA d\eta. \tag{5.8}$$

Inserting this inequality into (5.4), we conclude

$$\Phi_1(0,t) \leqslant \tau^2 C_1, \tag{5.9}$$

where

$$C_1 = \frac{\frac{h^2}{\pi^2}}{1 - \sqrt{\frac{h}{\pi}} Q_1^{\frac{1}{2}} - |V_{,2}|_{\max} \frac{h^2}{\pi^2} - \frac{h^2}{\pi^2} - \frac{h^2}{\pi^2} g^*}. \tag{5.10}$$

In (5.10), we also require h and/or the data small enough such that the denominator of C_1 is positive. Under this condition, it is easy from (5.4) to find that

$$\int_{R_0} v_\alpha v_\alpha dA |_{\eta=t} \leqslant 2\tau^2 \frac{h^2}{\pi^2}. \tag{5.11}$$

In light of (5.8), we have

$$\Phi_3(0,t) \leqslant \tau^2 C_1 \widetilde{Q}_3 \frac{h^2}{\pi^2} \tag{5.12}$$

Now, we derive bound for $\Phi_2(0,t)$. To do this, we multiply (2.26) with $v_{\alpha,\eta}$ to obtain

$$\int_0^t \int_{R_0} \left[v_{\alpha,\eta} - \Delta v_\alpha + v_\beta w_{\alpha,\beta} + w_\beta^* v_{\alpha,\beta} \right.$$

$$\left. + V v_{\alpha,1} + v_2 V_{,2} \delta_{\alpha 1} - \tau_\alpha \theta - g_\alpha^* T + \pi_{,\alpha} \right] v_{\alpha,\eta} dA d\eta = 0. \tag{5.13}$$

We compute

$$-\int_0^t \int_{R_0} v_\beta w_{\alpha,\beta} v_{\alpha,\eta} dA d\eta = \int_0^t \int_{R_0} v_\beta w_\alpha v_{\alpha,\beta\eta} dA d\eta$$

$$= \int_{R_0} v_\beta w_\alpha v_{\alpha,\beta} dA|_{\eta=t} - \int_0^t \int_{R_0} v_{\beta,\eta} w_\alpha v_{\alpha,\beta} dA d\eta$$

$$- \int_0^t \int_{R_0} v_\beta w_{\alpha,\eta} v_{\alpha,\beta} dA d\eta. \tag{5.14}$$

Inserting (5.14) into (5.13) and using the derivation of previous section, we have

$$\frac{1}{2} \int_{R_0} v_{\alpha,\beta} v_{\alpha,\beta} dA|_{\eta=t} + \int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta$$

$$\leqslant \sqrt{\frac{h}{\pi}} \left(\int_{R_0} w_{\alpha,\beta} w_{\alpha,\beta} dA \right)^{\frac{1}{2}} \left(\int_{R_0} v_\beta v_\beta dA \right)^{\frac{1}{4}} \left(\int_{R_0} v_{\alpha,\beta} v_{\alpha,\beta} dA \right)^{\frac{3}{4}} |_{\eta=t}$$

$$+ 3 \sqrt[8]{\frac{72h^7}{\pi^5}} \left(\int_0^t \int_{R_0} w_{\alpha,\beta 2} w_{\alpha,\beta 2} dA d\eta \right)^{\frac{1}{2}}$$

$$\cdot \left(\int_0^t \int_{R_0} v_{\alpha,\beta} v_{\alpha,\beta} dA d\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta \right)^{\frac{1}{2}}$$

$$+ \frac{h}{\pi} \widetilde{Q}_1^{\frac{1}{4}} \widetilde{Q}_5^{\frac{1}{4}} \int_0^t \int_{R_0} v_{\alpha,\beta} v_{\alpha,\beta} dA d\eta$$

$$+ |V|_{\max} \left(\int_0^t \int_{R_0} v_{\alpha,1} v_{\alpha,1} dA d\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta \right)^{\frac{1}{2}}$$

$$+ |V_{,2}|_{\max} \frac{h}{\pi} \left(\int_0^t \int_{R_0} v_{2,2}^2 dA d\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta \right)^{\frac{1}{2}}$$

$$+ \frac{h}{\pi} \tau \left(\int_0^t \int_{R_0} \theta_{,2}^2 dA d\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta \right)^{\frac{1}{2}}$$

$$+ \frac{h}{\pi} g^* \left(\int_0^t \int_{R_0} T_{,2}^2 dA d\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta \right)^{\frac{1}{2}}. \tag{5.15}$$

By Young's inequality (4.14) and the AG mean inequality, from (5.15) we have

$$\frac{1}{2} \int_{R_0} v_{\alpha,\beta} v_{\alpha,\beta} dA|_{\eta=t} + \int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta$$

$$\leqslant \frac{1}{4} \varepsilon_1^{-3} \sqrt{\frac{h}{\pi}} \widehat{Q}_1^{\frac{1}{2}} \int_{R_0} v_\beta v_\beta dA|_{\eta=t}$$

$$+ \frac{3}{4} \varepsilon_1 \sqrt{\frac{h}{\pi}} \widehat{Q}_1^{\frac{1}{2}} \int_{R_0} v_{\alpha,\beta} v_{\alpha,\beta} dA|_{\eta=t} + \frac{3}{2\varepsilon_2} \sqrt[8]{\frac{72h^7}{\pi^5}} E_7^{\frac{1}{2}}(0,t) \int_0^t \int_{R_0} v_{\alpha,\beta} v_{\alpha,\beta} dA d\eta$$

$$+ \frac{3\varepsilon_2}{2} \sqrt[8]{\frac{72h^7}{\pi^5}} E_7^{\frac{1}{2}}(0,t) \int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta + \frac{h}{\pi} \widetilde{Q}_1^{\frac{1}{4}} \widetilde{Q}_5^{\frac{1}{4}} \int_0^t \int_{R_0} v_{\alpha,\beta} v_{\alpha,\beta} dA d\eta$$

$$+ \frac{1}{2\varepsilon_3} |V|_{\max} \int_0^t \int_{R_0} v_{\alpha,1} v_{\alpha,1} dA d\eta + \frac{1}{2} \varepsilon_3 |V|_{\max} \int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta \tag{5.16}$$

$$+ \frac{1}{2\varepsilon_4} |V_{,2}|_{\max} \frac{h}{\pi} \int_0^t \int_{R_0} v_{2,2}^2 dA d\eta + \frac{1}{2} \varepsilon_4 |V_{,2}|_{\max} \frac{h}{\pi} \int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta$$

$$+ \frac{h}{2\varepsilon_5 \pi} \tau^2 \int_0^t \int_{R_0} \theta_{,2}^2 dA d\eta + \frac{h}{2\pi} \varepsilon_5 \int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta$$

$$+ \frac{h}{2\varepsilon_6 \pi} g^* \int_0^t \int_{R_0} T_{,2}^2 dA d\eta + \frac{h}{2\pi} \varepsilon_6 g^* \int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta, \tag{5.17}$$

where $\varepsilon_i, (i = 1, 2, \ldots, 6)$ are positive constants. Choosing

$$\varepsilon_1 = \frac{2}{3} \widehat{Q}_1^{-\frac{1}{2}}, \ \varepsilon_2 = \frac{1}{15} \sqrt[8]{\frac{\pi^5}{72h^7}} E_7^{-\frac{1}{2}}(0,t), \ \varepsilon_3 = \frac{1}{5|V|_{\max}},$$

$$\varepsilon_4 = \frac{\pi}{5h|V_{,2}|_{\max}}, \ \varepsilon_5 = \frac{\pi}{5h}, \ \varepsilon_6 = \frac{\pi}{5g^*h},$$

and making use of (5.9), (5.11) and (5.12), we obtain

$$\int_0^t \int_{R_0} v_{\alpha,\eta} v_{\alpha,\eta} dA d\eta \leqslant \tau^2 C_2, \tag{5.18}$$

where

$$C_2 = \left(\sqrt[8]{\frac{72h^7}{\pi^5}} \frac{3}{\varepsilon_2} E_7^{\frac{1}{2}}(0,t) + \frac{2h}{\pi} \widetilde{Q}_1^{\frac{1}{4}} \widetilde{Q}_5^{\frac{1}{4}} + \frac{1}{\varepsilon_3} |V|_{\max} + \frac{1}{\varepsilon_4} |V_{,2}|_{\max} \frac{h}{\pi} \right) C_1$$

$$+ \frac{h^2}{\pi^2} \varepsilon_1^{-3} \sqrt{\frac{h}{\pi}} \widehat{Q}_1^{\frac{1}{2}} + \frac{h}{\varepsilon_5 \pi} \widetilde{Q}_3 + g^* C_1 Q_3 \frac{h^3}{\varepsilon_6 \pi^3}. \tag{5.19}$$

Combining (5.9), (5.12) and (5.18), we conclude that

$$\Phi(0,t) \leqslant \tau^2 C, \tag{5.20}$$

where $C = C_1 + C_2 \delta + C_1 \widetilde{Q}_3 \frac{h^2}{\pi^2}$. The bound for $\Psi(0,t)$ of data follows (5.1), which makes (4.37) explicit.

6. Bound for $E_7(0,t)$

In this section, we seek bound for $E_7(0,t)$. To do this, we follow the methods which have been used in section 5 of [19]. For convenience, we adopt the following notations:

$$E_1(w_\alpha) = \int_0^t \int_{R_0} w_{\alpha,\beta} w_{\alpha,\beta} dA d\eta, \ E_7(w_\alpha) = \int_0^t \int_{R_0} w_{\alpha,\beta2} w_{\alpha,\beta2} dA d\eta. \tag{6.1}$$

We let

$$\chi_\alpha = u_\alpha - \psi_\alpha, \ \widehat{\chi}_\alpha = \psi_\alpha - V \delta_{\alpha1}, \tag{6.2}$$

$$\sigma_{,\alpha} = p_{,\alpha} - \rho_{,\alpha}, \ \widehat{\sigma}_{,\alpha} = \rho_{,\alpha} - P(t) \delta_{\alpha1}. \tag{6.3}$$

Clearly,

$$w_\alpha = \chi_\alpha + \widehat{\chi}_\alpha, \; q_{,\alpha} = \sigma_{,\alpha} + \widehat{\sigma}_{,\alpha}. \tag{6.4}$$

The functions (ψ_α, ρ) are solutions to the initial boundary value problems

$$
\begin{aligned}
& \psi_{\alpha,t} - \Delta\psi_\alpha + \rho_{,\alpha} = 0, \; in \; \mathbb{R} \times (0,T), \\
& \psi_{\alpha,\alpha} = 0, \; in \; \mathbb{R} \times (0,T), \\
& \psi_\alpha(x_1, x_2, 0) = 0, \; in \; R, \\
& \psi_\alpha(x_1, 0, t) = \psi_\alpha(x_1, h, t) = 0, x > 0, 0 \leqslant t < T, \\
& \psi_\alpha(0, x_2, t) = f_1(x_2, t), x > 0, 0 \leqslant t < T.
\end{aligned}
\tag{6.5}
$$

Then, (χ_α, y, σ) satisfy

$$
\begin{aligned}
& \chi_{\alpha,t} - \Delta\chi_\alpha + u_\beta u_{\alpha,\beta} - g_\alpha\theta + \sigma_{,\alpha} = 0, \; in \; \mathbb{R} \times (0,T), \\
& \chi_{\alpha,\alpha} = 0, \; in \; \mathbb{R} \times (0,T), \\
& \chi_\alpha(x_1, x_2, 0) = 0, \; in \; \mathbb{R}, \\
& \chi_\alpha(x_1, 0, t) = \chi_\alpha(x_1, h, t) = 0, \\
& \chi_\alpha(0, x_2, t) = 0, x > 0, 0 \leqslant t < T.
\end{aligned}
\tag{6.6}
$$

From (5.29) of [19], we have known that

$$\int_0^t \int_{R_0} \chi_{\alpha,\beta}\chi_{\alpha,\beta} dA d\eta \leqslant Q_1^2, \tag{6.7}$$

which will be used in the following computation. By the triangle inequality, we have

$$E_7^{\frac{1}{2}}(0,t) \leqslant \widetilde{E}_7^{\frac{1}{2}}(0,t) + \widehat{E}_7^{\frac{1}{2}}(0,t), \tag{6.8}$$

where

$$\widetilde{E}_7^{\frac{1}{2}}(0,t) = \int_0^t \int_{R_0} \chi_{\alpha,\beta 2}\chi_{\alpha,\beta 2} dA d\eta, \tag{6.9}$$

$$\widehat{E}_7^{\frac{1}{2}}(0,t) = \int_0^t \int_{R_0} \widehat{\chi}_{\alpha,\beta 2}\widehat{\chi}_{\alpha,\beta 2} dA d\eta. \tag{6.10}$$

The bound for $\widehat{E}_7(0,t)$ may be obtained by following the arguments of [17]. So, we only need to establish bound for $\widetilde{E}_7(0,t)$. To do this, we use (6.8) to have

$$\int_0^t \int_{R_0} \chi_{\alpha,2}[\chi_{\alpha,2\eta} - \Delta\chi_{\alpha,2} + (u_\beta u_{\alpha,\beta})_{,2} - (g_\alpha\theta)_{,2} + \sigma_{,\alpha 2}] dA d\eta = 0. \tag{6.11}$$

Noting that $\chi_{\alpha,2}$ vanish at $x_1 = 0$ and integrating by parts, we obtain

$$\frac{1}{2}\int_{R_0} \chi_{\alpha,2}\chi_{\alpha,2} dA|_{\eta=t} + \int_0^t \int_{R_0} \chi_{\alpha,\beta 2}\chi_{\alpha,\beta 2} dA d\eta$$

$$= \int_0^t \int_{R_0} u_{\beta,2} u_\alpha \chi_{\alpha,\beta 2} dAd\eta + \int_0^t \int_{R_0} u_\beta u_{\alpha,2} \chi_{\alpha,\beta 2} dAd\eta - \int_0^t \int_{R_0} g_\alpha \theta \chi_{\alpha,22} dAd\eta$$

$$= \int_0^t \int_{R_0} (\chi_\beta + \psi_\beta)_{,2} (\chi_\alpha + \psi_\alpha) \chi_{\alpha,\beta 2} dAd\eta$$

$$+ \int_0^t \int_{R_0} (\chi_\beta + \psi_\beta)(\chi_\alpha + \psi_\alpha)_{,2} \chi_{\alpha,\beta 2} dAd\eta - \int_0^t \int_{R_0} g_\alpha \theta \chi_{\alpha,22} dAd\eta. \quad (6.12)$$

Using the Schwarz inequality, (4.1) and (4.8), we have

$$\int_0^t \int_{R_0} (\chi_\beta + \psi_\beta)_{,2} (\chi_\alpha + \psi_\alpha) \chi_{\alpha,\beta 2} dAd\eta$$

$$\leqslant \left(\int_0^t \int_{R_0} (\chi_{\beta,2} \chi_{\beta,2})^2 dAd\eta \right)^{\frac{1}{4}}$$

$$\cdot \left(\int_0^t \int_{R_0} (\chi_\alpha \chi_\alpha)^2 dAd\eta \right)^{\frac{1}{4}} \left(\int_0^t \int_{R_0} \chi_{\alpha,\beta 2} \chi_{\alpha,\beta 2} dAd\eta \right)^{\frac{1}{2}}$$

$$+ \left(\int_0^t \int_{R_0} (\chi_{\beta,2} \chi_{\beta,2})^2 dAd\eta \right)^{\frac{1}{4}} \left(\int_0^t \int_{R_0} (\psi_\alpha \psi_\alpha)^2 dAd\eta \right)^{\frac{1}{4}}$$

$$\cdot \left(\int_0^t \int_{R_0} \chi_{\alpha,\beta 2} \chi_{\alpha,\beta 2} dAd\eta \right)^{\frac{1}{2}}$$

$$+ \left(\int_0^t \int_{R_0} (\psi_{\beta,2} \psi_{\beta,2})^2 dAd\eta \right)^{\frac{1}{4}} \left(\int_0^t \int_{R_0} (\chi_\alpha \chi_\alpha)^2 dAd\eta \right)^{\frac{1}{4}}$$

$$\cdot \left(\int_0^t \int_{R_0} \chi_{\alpha,\beta 2} \chi_{\alpha,\beta 2} dAd\eta \right)^{\frac{1}{2}}$$

$$+ \left(\int_0^t \int_{R_0} (\psi_{\beta,2} \psi_{\beta,2})^2 dAd\eta \right)^{\frac{1}{4}} \left(\int_0^t \int_{R_0} (\psi_\alpha \psi_\alpha)^2 dAd\eta \right)^{\frac{1}{4}}$$

$$\cdot \left(\int_0^t \int_{R_0} \chi_{\alpha,\beta 2} \chi_{\alpha,\beta 2} dAd\eta \right)^{\frac{1}{2}}$$

$$\leqslant \frac{h}{\sqrt{2\pi}} Q_1^{\frac{3}{2}} \widetilde{E}_7^{\frac{3}{4}}(0,t) + \frac{h}{\sqrt{2\pi}} Q_1^{\frac{1}{4}} E_1^{\frac{1}{2}}(\psi_\alpha) \widetilde{E}_7^{\frac{3}{4}}(0,t)$$

$$+ \sqrt{\frac{h}{2\pi}} Q_1^3 E_1^{\frac{1}{4}}(\psi_\alpha) E_7^{\frac{1}{4}}(\psi_\alpha) \widetilde{E}_7^{\frac{1}{2}}(0,t) + \sqrt{\frac{h}{2\pi}} E_1^{\frac{3}{4}}(\psi_\alpha) E_7^{\frac{1}{4}}(\psi_\alpha) \widetilde{E}_7^{\frac{1}{2}}(0,t). \quad (6.13)$$

Similarly, we also have

$$\int_0^t \int_{R_0} (\chi_\beta + \psi_\beta)(\chi_\alpha + \psi_\alpha)_{,2} \chi_{\alpha,\beta 2} dAd\eta$$

$$\leqslant \frac{h}{\sqrt{2\pi}} Q_1^{\frac{1}{4}} E_1^{\frac{1}{2}}(\psi_\alpha) \widetilde{E}_7^{\frac{3}{4}}(0,t) + \sqrt{\frac{h}{2\pi}} Q_1^3 E_1^{\frac{1}{4}}(\psi_\alpha) E_7^{\frac{1}{4}}(\psi_\alpha) \widetilde{E}_7^{\frac{1}{2}}(0,t)$$

$$+ \sqrt{\frac{h}{2\pi}} E_1^{\frac{3}{4}}(\psi_\alpha) E_7^{\frac{1}{4}}(\psi_\alpha) \widetilde{E}_7^{\frac{1}{2}}(0,t), \quad (6.14)$$

and

$$-\int_0^t \int_{R_0} g_\alpha \theta \chi_{\alpha,22} dA d\eta \leqslant \left(\int_0^t \int_{R_0} g_\alpha g_\alpha \theta^2 dA d\eta \right)^{\frac{1}{2}} \left(\int_0^t \int_{R_0} \chi_{\alpha,22}\chi_{\alpha,22} dA d\eta \right)^{\frac{1}{2}}$$

$$\leqslant \frac{h}{\pi} g \widetilde{Q}_3^{\frac{1}{2}} \widetilde{E}_7^{\frac{1}{2}}(0,t), \tag{6.15}$$

where

$$g^2 = \max\{g_\alpha g_\alpha\}, \tag{6.16}$$

and we have used the fact

$$\int_0^t \int_{R_0} \chi_\beta \chi_{\alpha,2} \chi_{\alpha,\beta 2} dA d\eta = 0.$$

Inserting (6.13)-(6.15) into (6.12) and using the Young inequality (4.14), we obtain

$$\begin{aligned}
\widetilde{E}_7(0,t) \leqslant {} & \frac{h}{\sqrt{2}\pi} Q_1^{\frac{3}{2}} \widetilde{E}_7^{\frac{3}{4}}(0,t) + \frac{2h}{\sqrt{2}\pi} Q_1^{\frac{1}{4}} E_1^{\frac{1}{2}}(\psi_\alpha) \widetilde{E}_7^{\frac{3}{4}}(0,t) \\
& + 2\sqrt{\frac{h}{2\pi}} Q_1^3 E_1^{\frac{1}{4}}(\psi_\alpha) E_7^{\frac{1}{4}}(\psi_\alpha) \widetilde{E}_7^{\frac{1}{2}}(0,t) \\
& + 2\sqrt{\frac{h}{2\pi}} E_1^{\frac{3}{4}}(\psi_\alpha) E_7^{\frac{1}{4}}(\psi_\alpha) \widetilde{E}_7^{\frac{1}{2}}(0,t) + \frac{h}{\pi} g \widetilde{Q}_3^{\frac{1}{2}} \widetilde{E}_7^{\frac{1}{2}}(0,t) \\
\leqslant {} & \frac{1}{\varepsilon_1^3} \frac{h}{4\sqrt{2}\pi} Q_1^6 + \varepsilon_1 \frac{3h}{4\sqrt{2}\pi} \widetilde{E}_7(0,t) + \frac{1}{\varepsilon_2^3} \frac{h}{2\sqrt{2}\pi} E_1^{\frac{1}{2}}(\psi_\alpha) Q_1 \\
& + \varepsilon_2 \frac{3h}{2\sqrt{2}\pi} E_1^{\frac{1}{2}}(\psi_\alpha) \widetilde{E}_7(0,t) \\
& + \frac{h}{2\pi\varepsilon_3} Q_1^6 E_1^{\frac{1}{2}}(\psi_\alpha) E_7^{\frac{1}{2}}(\psi_\alpha) + \frac{h}{2\pi\varepsilon_4} E_1^{\frac{3}{2}}(\psi_\alpha) E_7^{\frac{1}{2}}(\psi_\alpha) \\
& + \frac{h^2}{2\varepsilon_5 \pi^2} g^2 Q_3 + \frac{1}{2}(\varepsilon_3 + \varepsilon_4 + \varepsilon_5)\widetilde{E}_7(0,t), \tag{6.17}
\end{aligned}$$

where $\varepsilon_i, (i = 1,2,3,4,5)$ are positive constants. Choosing

$$\varepsilon_1 = \frac{\sqrt{2}\pi}{6h}, \quad \varepsilon_2 = \frac{\sqrt{2}\pi}{12h} E_1^{-\frac{1}{2}}(\psi_\alpha), \quad \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \frac{1}{6}, \tag{6.18}$$

inequality (6.17) yields that

$$\begin{aligned}
\widetilde{E}_7(0,t) \leqslant {} & \frac{1}{\varepsilon_1^3} \frac{h}{4\sqrt{2}\pi} Q_1^6 + \frac{1}{\varepsilon_2^3} \frac{h}{\sqrt{2}\pi} E_1^{\frac{1}{2}}(\psi_\alpha) Q_1 \\
& + \frac{h}{\pi\varepsilon_3} Q_1^6 E_1^{\frac{1}{2}}(\psi_\alpha) E_7^{\frac{1}{2}}(\psi_\alpha) + \frac{h}{\pi\varepsilon_4} E_1^{\frac{3}{2}}(\psi_\alpha) E_7^{\frac{1}{2}}(\psi_\alpha) + \frac{h^2 g^2 Q_3}{\varepsilon_5 \pi^2}. \tag{6.19}
\end{aligned}$$

From (6.8), we conclude that $E_7(0,t)$ can be bounded by known data.

REFERENCES

[1] H. ABIDI AND T. HMIDI, *On the global well-posedness for Boussinesq system,* J. Differ. Equ., **233**, (2007), 199–220.

[2] D. ADHIKARI, C. CAO AND J. WU, *The 2-D Boussinesq equations with vertical viscosity and vertical diffusivity,* J. Differ. Equ., **249**, (2010), 1078–1088.

[3] D. ADHIKARI, C. CAO AND J. WU, *Global regularity results for the2DBoussinesq equations with vertical dissipation,* J. Differ. Equ., **251**, (2011), 1637–1655.

[4] B.A. BOLEY, *Some observations on Saint-Venant's principle, in: Proc. 3rd US Nat,* Congr. Appl. Mech., ASME, New York, (1958), 259-264.

[5] J.R. CANNON AND D. ZACHMANN, *Parameter determination in parabolic partial differential equations from overspecified boundary data,* Internat.J.Engrg.Sci., **20**, (1982), 779–788.

[6] CH. CAO AND J. WU, *Global Regularity for the Two-Dimensional Anisotropic Boussinesq Equations with Vertical Dissipation,* Arch. Rational Mech. Anal., **208**,(2013), 985–1004.

[7] D. CHAE, *Global regularity for the 2-D Boussinesq equations with partial viscosity terms,* Adv.Math., **203**, (2006), 497–513.

[8] D. CHAE, S. K. KIM AND H.S. NAM,*Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations,* Nagoya Math. J., **155**, (1999), 55–80.

[9] J. CHENG AND S. LI, *Global weak solutions and regularity for two-dimensional Boussinesq equations with partial disspation and thermal diffusivity,* J. Math. Anal.Appl., **428**, (2015), 794–803.

[10] C.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities, Cambridge Univ,* Press, 1953.

[11] C.O. HORGAN, *Plane steady flows and energy astimates for the Niver-stokes equations,* Arch.Rat.Mech.Anal.,**68**(1978),359-381.

[12] T.Y. HOU AND C. LI, *Global well-posedness of the viscous Boussinesq equations,* Discrete Contin.Dyn.Syst., **12**, (2005), 1–12.

[13] J. LEE AND J.C. SONG, *Spatial decay bounds in a linearized magnetohydrodynamics channesl flow,* Communications on Pure and Applied analysis, **12**, (2013), 1349–1361.

[14] C. LIN, *Spatial decay estimates and energy bounds for the Stokes flow equations,* Stability Appl. Anal. Contin. Media., **2**, (1992), 249–264.

[15] C. LIN AND L. E. PAYNE,*Spatial decay bounds in the channel flow of an incompressible viscous fluid,* Math. Model. Meth., **6**, (2004), 795–818.

[16] C. LIN AND L. E. PAYNE,*Phragmen-Lindeöf type results for second order quasilinear parabolic equations in $\mathbb{R}^2$,* A angew Math Phys., **45**, (1994), 294–311.

[17] C. LIN AND L. E. PAYNE,*Spatial decay estimates and energy bounds for the Stokes flow equation,* SAACM.2, (1992), 249–264.

[18] Y.LIU AND C. LIN,*Spatial decay bounds in time dependent pipe flow of an incompressible viscous fluid,* SIAM J. Appl. Math., **65**, (2005), 458–475.

[19] Y. LIU, YF. LI, YW. LIN AND ZH. YAO, *Spatial decay bounds for the channel flow of the Boussinesq equations,* J.Math.Anal.Appl., **381**, (2011), 87–109.

[20] A.J. MAJDA AND A.L. BERTOZZI, *Vorticity and Incompressible Flow,* Cambridge University Press, Cambridge., 2001.

[21] A.J. MAJDA, *Introduction to PDEs andWaves for the Atmosphere and Ocean,* Courant Lecture Notes in Mathematics.,**9**.AMS/CIMS,(2003).

[22] D.S. MITRONOVIC,*Analytical Inequalities,* Springer-Verlag., 1970.

[23] L.E. PAYNEN,*Uniqueness criteria for steady state solutions of the Navier-Stokes equation, in: Simpos. Internaz,* Appl. Anal. Fis. Mat., Cagliari-Sassari, 1964, Edizioni Cremonse, Rome, (1965), 130–153.

[24] L.E. PAYNE, *Uniqueness and continuous dependence criteria for the Navier-Stokes equations,* Rocky Mountain J. Math. **2**, (1971), 641–660.

[25] L.E. PAYNE AND G.A. PHILIPPIN, *Pointwise bounds and spatial decay estimates in heat conduction problems,* Math. Models Methods Appl. Sci., **5**, (1995), 755–775.

[26] L.E. PAYNE, P.W. SCHAEFER AND J.C. SONG, *Growth and decay results in heat conduction problems with nonlinear boundary conditions,* Nonlinear Anal., **33**, (1999), 269–286.

[27] J.PEDLOSKY,*Geophysical Fluid Dyanmics,* Springer, New York., 1987.

[28] J. C. SONG, *Improved decay estimates in time-dependent Stokes flow,* J. Math. Anal. Appl., **288**, (2003), 505–517.

[29] J. C. SONG, *Improved spatial decay bounds in the plane Stokes flow,* Appl. Math. Mech., **30**, (2009), 833–838.

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