Differential
Equations
& Applications
Volume 8, Number 4 (2016), 521–533
doi:10.7153/dea-08-29

PATTERNS IN A BALANCED BISTABLE EQUATION WITH HETEROGENEOUS ENVIRONMENTS ON SURFACES OF REVOLUTION

MAICON SÔNEGO

(*Communicated by Ulisse Stefanelli*)

Abstract. We use the variational concept of Γ-convergence to obtain sufficient conditions that guarantee existence, stability and the geometric structure of four families of stationary solutions to the singularly perturbed parabolic equation $\partial_t u_\varepsilon = \varepsilon^2 \Delta u_\varepsilon + f(u_\varepsilon, x)$ on surfaces of revolution. We consider the bistable function $f(u,x) = -(u-a(x))(u-b(x))(u-c(x))$ and the conditions found relate the functions a,b,c to the geometry of the surface where such functions are defined.

1. Introduction

In this work we study the following problem

$$\partial_t u_\varepsilon = \varepsilon^2 \Delta_g u_\varepsilon + f(u_\varepsilon, x), \quad (t,x) \in \mathbb{R}^+ \times \mathscr{M} \tag{1}$$

where $\varepsilon > 0$ is a small parameter and $\mathscr{M} \subset \mathbb{R}^3$ is a surface of revolution without boundary with metric g. We consider

$$f(u,x) = -(u-a(x))(u-b(x))(u-c(x)), \tag{2}$$

where $a,b,c \in C^1(\mathscr{M})$ and $a(x) < b(x) = (a(x)+c(x))/2 < c(x)$ for all $x \in \mathscr{M}$. Such $f(u,x)$ is a typical example of the so-called *bistable function*.

There is a vast literature addressing the question of existence as well as nonexistence of nonconstant stable stationary solutions (herein referred to as *patterns*, for short) to (1) in bounded domains of $\mathbb{R}^n$ when the roots of f are constants; we refer to [16, 3, 5, 4] and references therein. For problems on surfaces of revolution see [1, 6, 7, 19]. In these previous results, the effect of domain geometry and/or the effect of diffusivity are considered.

Our concern herein is to find mechanisms of interaction between the functions $a(x)$, $b(x)$, $c(x)$ and the geometry of the domain so as to produce patterns to (1), which develop inner transition layers as $\varepsilon \to 0$.

For one-dimensional domains, i.e., when $\mathscr{M} = (0,1)$ for instance, subjected to zero Neumann boundary conditions there are several results. In [18] it was proved that if $c(x) - a(x)$ is C^2 and attains a nondegenerate local minimum at $x_0 \in (0,1)$

Mathematics subject classification (2010): 35K57, 35B36, 35R01, 35B25, 35B35, 34K20, 58J32.
Keywords and phrases: patterns, Γ-convergence, surface of revolution, stable transition layer.

Paper DEA-08-29

then there exists a stable solution u_ε such that $u_\varepsilon(x) \to c(x)$ on $(0,x_0)$ and $u_\varepsilon(x) \to a(x)$ on (x_0,l). In [17] this result was extended to a degenerate setting. In [9] it was generalized to two-dimensional domains using essentially the same ideas used here, i.e., the variational concept of $\Gamma-$convergence. Indeed, our problem becomes simpler since (1) can be treated as a one-dimensional problem and so our conditions for the existence of patterns appear more naturally. Furthermore, another interesting aspect of the case considered here is the possibility to construct simple examples where our hypothesis for the existence of patterns are satisfied (see Subsection 3.2).

In [8], essentially the same problem was considered for bounded domains in $\mathbb{R}^n$ ($\Omega \subset \mathbb{R}^n$, for instance), supplied with zero Neumann boundary condition, in a degenerate environment. Unlike [9] and of this paper, is precisely this degenerate condition on the roots of f that allows to consider n-dimensional domains and, mainly, ensures the existence of patterns for the problem. To be more specific, in [8] is assumed that the roots of $f(u,x) = -(u-a(x))(u-b(x))(u-c(x))$ are equal in $\Omega \setminus D$, $D = (D_1 \cup D_2) \subset \Omega$, $\overline{D_1} \cap \overline{D_2} = \emptyset$ where D_1 and D_2 are open and connected sets with Lipschitz-continuous boundaries.

There are some works regarding the effect of heterogenous environments (i.e. when the reaction term f, of type bistable or not, depends on the spatial variable x) under different aspects, we cite [13, 14, 15] and references therein.

In order to introduce our results consider a smooth curve $C \subset \mathbb{R}^3$ parametrized by $x = (x_1,x_2,x_3) = (\psi(s),0,\chi(s))$, $s \in [0,l]$ with $\psi(0) = \psi(l) = 0$ and the borderless surface of revolution $\mathscr{M}$ generated by C. We suppose that the functions $a(x)$, $b(x)$ and $c(x)$ do not depend on the angular variable θ, so that, abusing notation, we set $a(x(s,\theta)) = a(s)$, $b(x(s,\theta)) = b(s)$ and $c(x(s,\theta)) = c(s)$.

We find that a sufficient condition for existence of patterns to (1) is that the function $\psi(c-a)^3 : (0,l) \to \mathbb{R}$ has an isolated local minimum in $(0,l)$. In particular, if a and c are constant then the sufficient condition is satisfied as long as, roughly speaking, $\mathscr{M}$ has a neck.

The geometric profiles of these patterns are also given and moreover, we show that two of four families of patterns found develop internal transition layers as $\varepsilon \to 0$, which are referred to in the literature as stable transition layers. Our approach provides convergence, as $\varepsilon \to 0$, of the stable transition layers in $L^1(\Omega)$ rather than uniform convergence in compact sets outside the interface. All these results remain true for a surface of revolution with border under Neumann boundary condition and this case is also considered in this work.

Note that our results extend [18] to surfaces of revolution. This can be seen by taking $\psi \equiv 1$, which would correspond to a finite right circular cylinder, and then the existence condition for patterns would be $c(x) - a(x)$ having an isolated local minimum in $(0,l)$, as found in [18]. In the end, some simple examples are given to illustrate situations in which our results guarantee the existence of patterns.

2. Preliminaries

We begin with some definitions on BV-functions, Γ-convergence and known results from Differential Geometry which will be used in the following sections.

2.1. Surface of revolution

Consider $M = (\mathscr{M}, g)$ an $n-$dimensional Riemannian manifold with a metric given in local coordinates $x = (x^1, x^2, \cdots, x^n)$ given by (using Einstein summation convention) $dr^2 = g_{ij}dx^i dx^j$, $(g^{ij}) = (g_{ij}^{-1})$, $|g| = \det(g_{ij})$.

We will see how the operator $\Delta_g u$ can be expressed for the particular case where $\mathscr{M}$ is a surface of revolution.

Given a smooth vector field X on $\mathscr{M}$, the divergence operator of X is defined as

$$\mathrm{div}_g X = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} (\sqrt{|g|} X^i)$$

and the Riemannian gradient, denoted by ∇_g, of a sufficiently smooth real function ϕ defined on $\mathscr{M}$, as the vector field

$$(\nabla_g \phi)^i = g^{ij} \partial_j \phi.$$

Let C be the curve of $\mathbb{R}^3$ parametrized by

$$\begin{cases} x_1 = \psi(s) \\ x_2 = 0 \qquad (s \in I := [0,l]) \\ x_3 = \chi(s) \end{cases}$$

where $\psi, \chi \in C^2(I)$, $\psi > 0$ in $(0,l)$ and $(\psi')^2 + (\chi')^2 = 1$ in I. Moreover,

$$\psi(0) = \psi(l) = 0, \tag{3}$$

and

$$\psi'(0) = -\psi'(l) = 1. \tag{4}$$

Let $\mathscr{M}$ be the surface of revolution parametrized by

$$\begin{cases} x_1 = \psi(s)\cos(\theta) \\ x_2 = \psi(s)\sin(\theta) \qquad (s,\theta) \in [0,l] \times [0,2\pi). \\ x_3 = \chi(s) \end{cases} \tag{5}$$

Setting $x^1 = s, x^2 = \theta$ then a surface of revolution in $\mathbb{R}^3$ with the above parametrization is a 2-dimensional Riemannian manifold with metric

$$dr^2 = ds^2 + \psi^2(s) d\theta^2.$$

It follows from (3) and (4) that $\mathscr{M}$ has no boundary and we always assume that $\mathscr{M}$ and the Riemannian metric g on it are smooth (see [2], for instance). The area element on $\mathscr{M}$ is $d\sigma = \psi d\theta ds$ and the gradient of u with respect to the metric g is given by

$$\nabla_g u = \left(u_s, \frac{1}{\psi^2} u_\theta \right).$$

Thus

$$\Delta_g u = u_{ss} + \frac{\psi_s}{\psi} u_s + \frac{1}{\psi^2} u_{\theta\theta}. \tag{6}$$

Although the functions a, b and c may depend on (s,θ), as stated in the Introduction, throughout this work we suppose that they depend only on the variable s. Thus abusing notation, for simplicity's sake, we set

$$f(u,x(s,\theta)) = f(u,s) = -(u-a(s))(u-b(s))(u-c(s)), \tag{7}$$

for $x = (\psi(s)\cos(\theta), \psi(s)\sin(\theta), \chi(s)) \in \mathscr{M}$.

In short, we prove that the local minimum of the energy functional associated with the problem below, are the desired patterns of problem (1)

$$\partial_t u_\varepsilon = \varepsilon^2 \left(\partial^2_{ss} u_\varepsilon + \frac{\psi_s}{\psi} \partial_s u_\varepsilon + \frac{1}{\psi^2} \partial^2_{\theta\theta} u_\varepsilon \right) + f(u,s), \;\; (s,\theta) \in (0,l) \times [0,2\pi). \tag{8}$$

2.2. BV-functions

We say that u is a function of *essential bounded variation in an interval* $I \subset \mathrm{I\!R}$ (and write $u \in BV(I)$) if its partial derivative in the sense of distributions is a measure with finite total variation in I. In the sense of distributions, Du is a vector valued Radon measure with finite total variation in I given by

$$|Du| = \sup \left\{ \int_I u\sigma' ds : \sigma \in C_0^\infty(I),\; |\sigma| \leqslant 1 \right\}.$$

The total variation $|Du|$ is a Radon measure itself.

We denote by $BV(I,\{\alpha,\beta\})$ the class of all $u \in BV(I)$ which take values α and β only (α and β are constant functions or not). If $u \in BV(I)$, the integral of any positive continuous function h with respect to the measure $|Du|$ can be expressed as

$$\int_I h\,|Du| = \sup \left\{ \int_I u\sigma' ds : \sigma \in C_0^\infty(I),\; |\sigma| \leqslant h \right\}.$$

Given $u \in L^1_{loc}(I)$, the *jump set of* u, denoted by S_u, is the complement of the set of Lebesgue points of u, i.e., the set of points where the upper and lower approximate limits of u differ or are not finite. If $u \in BV(I,\{\alpha,\beta\})$, α and β constants, then $\mathscr{H}^0(S_u) < \infty$ and $(\beta-\alpha)\mathscr{H}^0(S_u)$ agrees with the total variation $|Du|$ of the derivative Du. Here $\mathscr{H}^0$ stands for the Hausdorff counting measure. For details the reader is referred to [10], for instance.

DEFINITION 1. A family $\{E_\varepsilon\}_{\varepsilon>0}$ of real-extended functionals defined in $L^1(I)$ is said to Γ*-converge, as* $\varepsilon \to 0$*, to a functional* E_0 and we write

$$\Gamma - \lim_{\varepsilon\to 0} E_\varepsilon(u) = E_0(u)$$

if:

(i) for each $u \in L^1(I)$ and for any sequence $\{u_\varepsilon\}$ in $L^1(I)$ satisfying $u_\varepsilon \to u$ in $L^1(I)$, as $\varepsilon \to 0$, there holds $E_0(u) \leqslant \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon)$ and

(ii) for each $u \in L^1(I)$ there is a sequence $\{u_\varepsilon\}$ in $L^1(I)$ satisfying $u_\varepsilon \to u$ in $L^1(I)$, as $\varepsilon \to 0$, and $E_0(u) \geqslant \limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon)$.

DEFINITION 2. We shall call $u_0 \in L^1(I)$ a L^1*-local minimizer of* E_0 if there is $\mu > 0$ such that $E_0(u_0) \leqslant E_0(u)$ whenever $0 < \|u-u_0\|_{L^1(I)} < \mu$. Moreover if $E_0(u_0) < E_0(u)$ for $0 < \|u-u_0\|_{L^1(I)} < \mu$, then u_0 is called an *isolated* L^1*-local minimizer of* E_0.

3. Sufficient conditions for existence of patterns

In this section we prove the main theorem of this paper. As usual χ_A denotes the characteristic function of a set A.

THEOREM 1. *If the function* $\psi(c-a)^3 : [0,l] \to \mathbb{R}$ *attains an isolated local minimum at* $s_0 \in (0,l)$ *then exist* $\varepsilon_0 > 0$ *and four families of stable stationary solutions* $\left\{u_\varepsilon^j\right\}_{0<\varepsilon\leqslant\varepsilon_0}$, $j = 1,\ldots,4$ *to* (1) *such that*

(i) $\|u_\varepsilon^1 - u_0^1\|_{L^1(I)} \overset{\varepsilon\to 0}{\longrightarrow} 0$ *where*

$$u_0^1(s) = a(s)\chi_{(0,s_0)}(s) + c(s)\chi_{(s_0,l)}(s);$$

(ii) $\|u_\varepsilon^2 - u_0^2\|_{L^1(I)} \overset{\varepsilon\to 0}{\longrightarrow} 0$ *where*

$$u_0^2(s) = c(s)\chi_{(0,s_0)}(s) + a(s)\chi_{(s_0,l)}(s);$$

(iii) $\|u_\varepsilon^3 - u_0^3\|_{L^1(I)} \overset{\varepsilon\to 0}{\longrightarrow} 0$ *where* $u_0^3(s) = a(s)$ *in* I*;*

(iv) $\|u_\varepsilon^4 - u_0^4\|_{L^1(I)} \overset{\varepsilon\to 0}{\longrightarrow} 0$ *where* $u_0^4(s) = c(s)$ *in* I.

Remember that by a stationary solution of problem (1) we mean a solution to the problem

$$\varepsilon^2\Delta_g u + f(u,x) = 0, \;\; x \in \mathscr{M} \tag{9}$$

and a stationary solution u_ε is called *stable* if for every $\eta > 0$ there exists $\delta > 0$ such that $\|u(\cdot,t) - u_\varepsilon\|_\infty < \eta$ for all $t > 0$, whenever $\|u(\cdot,0) - u_\varepsilon\|_\infty < \delta$, where $\|\cdot\|_\infty$ stands for the norm of the space $L^\infty(\mathscr{M})$. If there exists $\delta_1 > 0$ such that $\|u(\cdot,0) - u_\varepsilon\|_\infty < \delta_1$ implies that $\|u(\cdot,t) - u_\varepsilon\|_\infty \to 0$, as $t \to \infty$, then u_ε is called *asymptotically stable*. We say that u_ε is *unstable* if it is not stable.

Consider the L^1-local minimizers of the family of functionals

$$E_\varepsilon : L^1(I) \to \mathbb{R} \cup \{\infty\}$$

defined by

$$E_\varepsilon(u) = \begin{cases} \int_I \left[\frac{\varepsilon \psi(s)}{2} \left| u' \right|^2 + \frac{\psi(s)}{\varepsilon} F(u,s) \right] ds, \; u \in H^1(I) \\ \infty, \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases} \tag{10}$$

where

$$F(u,s) = -\int_{a(s)}^{u} f(\xi,s)d\xi = \frac{1}{4}(u-a)^2(u-c)^2.$$

We identify such minimizers with stationary solutions of (1) that are independent of the angular variable θ and then we proved its stability. Note that these local minimizers (critical points of $\{E_\varepsilon\}$) are stationary solutions of (8).

Thus our first step is to find local minimizers of E_ε and for this purpose we will use the following theorem (see [12]) which also provides the behavior of such local minimizers.

THEOREM 2. *Suppose that $\{E_\varepsilon\}$, a sequence of real-extended functionals, Γ-converges to a real-extended functional E_0 and also that the following hypotheses are satisfied:*

(i) *Any sequence $\{u_\varepsilon\}_{\varepsilon>0}$ such that $E_\varepsilon \leqslant C < \infty$ for all $\varepsilon > 0$, is compact in L^1.*

(ii) *There exists an isolated L^1-local minimizer u_0 of E_0.*

Then there exists an $\varepsilon_0 > 0$ and a family $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ such that

(i) *u_ε is an L^1-local minimizer of E_ε and*

(ii) *$\|u_\varepsilon - u_0\|_{L^1} \to 0$, as $\varepsilon \to 0$.*

The next result concerns the computation of the $\Gamma-$limit of the family of functionals $\{E_\varepsilon\}$ (defined in (10)) and its proof can be found in [21] (Theorem 2) for N-dimensionals domains ($N \geqslant 2$) and ψ constant. Essentially the same proof can be adapted to our case (the presence of the positive function ψ adds no significant difficulty, see [7] for instance) thus yielding

THEOREM 3. *Consider $E_0 : L^1(I) \to \mathbb{R} \cup \{\infty\}$ defined by*

$$E_0(u) = \begin{cases} \int_I h(s)\psi(s) \left| D\chi_{\{u=a\}} \right|, \; u \in BV(I,\{a,c\}) \\ \infty, \qquad\qquad\qquad\qquad \textit{otherwise} \end{cases}$$

where

$$h(s) = \sqrt{2}\int_{a(s)}^{c(s)} \sqrt{F(\xi,s)}d\xi = \frac{\sqrt{2}}{12}(c(s)-a(s))^3. \tag{11}$$

Then

$$\Gamma - \lim_{\varepsilon \to 0} E_\varepsilon(u) = E_0(u).$$

The functional E_0 can be thought of as a perimeter functional type with weight $h\psi$.

In order to apply Theorem 2 we need to find an isolated L^1-local minimizer of E_0. The fact that condition (2) of Theorem 2 holds for (10) was proved in [21] under more general conditions than those addressed here. Indeed, this hypothesis would follow if the minimizers were uniformly bounded in L^1 (see [21], Proposition 3, for instance), but this is really the case and it can be accomplished by an application of the maximum principle.

THEOREM 4. *If the function* $\psi(c-a)^3 : [0,l] \to \mathbb{R}$ *attains an isolated local minimum at* $s_0 \in (0,l)$ *then*

(i) $u_0^1(s) = a(s)\chi_{(0,s_0)}(s) + c(s)\chi_{(s_0,l)}(s)$;

(ii) $u_0^2(s) = c(s)\chi_{(0,s_0)}(s) + a(s)\chi_{(s_0,l)}(s)$;

(iii) $u_0^3(s) = a(s)$;

(iv) $u_0^4(s) = c(s)$

are isolated $L^1(I)$*-local minimizers of* E_0.

Proof. We render the proofs for u_0^1 and u_0^3 only since the other cases are similar. By hypothesis there exists $\delta_0 > 0$ such that

$$\psi(c-a)^3(s_0) < \psi(c-a)^3(s)$$

for $0 < |s - s_0| < \delta_0$. Take

$$\delta = \frac{1}{2}\min\left\{\int_{s_0-\delta_0}^{s_0} \{c(s)-a(s)\}\,ds, \int_{s_0}^{s_0+\delta_0} \{c(s)-a(s)\}\,ds\right\} \tag{12}$$

and $u \in BV(I,\{a,c\})$ such that

$$0 < \left\|u - u_0^1\right\|_{L^1(I)} < \delta. \tag{13}$$

Note that if $u \notin BV(I,\{a,c\})$ then $E_0(u) > E_0(u_0^1)$.

Let $S_u \subset (0,l)$ be the jump set of the function u. If $S_u \cap (s_0-\delta_0, s_0+\delta_0) = \emptyset$, then

$$\begin{aligned}
\left\|u - u_0^1\right\|_{L^1(I)} &= \int_I \left|u - u_0^1\right| ds \\
&\geqslant \int_{s_0-\delta_0}^{s_0+\delta_0} \left|u - u_0^1\right| ds \\
&> \tfrac{1}{2}\min\left\{\int_{s_0-\delta_0}^{s_0} \{c(s)-a(s)\}\,ds, \int_{s_0}^{s_0+\delta_0} \{c(s)-a(s)\}\,ds\right\} \\
&= \delta
\end{aligned}$$

which contradicts (13). Thus $S_u \cap (s_0-\delta_0, s_0+\delta_0) \neq \emptyset$ and there exists $s_1 \in S_u \cap (s_0-\delta_0, s_0+\delta_0)$.

If $s_1 \neq s_0$ we have that

$$\begin{aligned} E_0(u) &= \frac{\sqrt{2}}{12}\int_I (\psi(c-a)^3)(s)\left|D\chi_{\{u=a\}}\right| \\ &= \frac{\sqrt{2}}{12}\int_{S_u} (\psi(c-a)^3)(s)d\mathscr{H}^0 \\ &= \frac{\sqrt{2}}{12}\sum_{s\in S_u} (\psi(c-a)^3)(s) \\ &\geqslant \frac{\sqrt{2}}{12}(\psi(c-a)^3)(s_1) \\ &> \frac{\sqrt{2}}{12}(\psi(c-a)^3)(s_0) = E_0(u_0) \end{aligned}$$

as desired.

If $s_1 = s_0$, there are two possibilities: either $u \equiv u_0^1$ or $u \equiv u_0^2$. Both cases contradict (13).

For u_0^3 we consider δ given by (12) and as $S_{u_0^3} = \emptyset$ we have that

$$\begin{aligned} E_0(u_0^3) &= \int_I (\psi(c-a)^3)(s)\left|D\chi_{\{u_0^3=a\}}\right| \\ &= \int_{S_{u_0^3}} (\psi(c-a)^3)(s)\mathscr{H}^0 = 0. \end{aligned}$$

Then if $u \in BV(I,\{a,c\})$ and satisfies $0 < \left\|u-u_0^3\right\|_{L^1(I)} < \delta$, similarly to the previous case, it is easy to conclude that $E_0(u) > 0$ and therefore $E_0(u) > E_0(u_0^3)$. The theorem is proved. □

REMARK 1. The proof above occurs due to our hypothesis of symmetry on f, making it possible to study the problem in a one-dimensional setting. In fact, in [9] (see Theorem 3.2) – where the problem is considered in open subsets of $\mathbb{R}^2$ – a much more complicated proof is required for a similar result.

Before the next result, we recall that for the linearized problem ((1) around of u_ε)

$$\varepsilon^2\Delta_g\phi + f_u(u_\varepsilon,x)\phi + \lambda\phi = 0 \text{ in } \mathscr{M}, \tag{14}$$

the first eigenvalue λ_1 is given by

$$\lambda_1 = \min\left\{R_{u_\varepsilon}(\phi) : \phi \in H^1(\mathscr{M}), \|\phi\|_{L^2(\mathscr{M})} = 1\right\} \tag{15}$$

where

$$R_{u_\varepsilon}(\phi) = \int_{\mathscr{M}}\left\{\varepsilon\left|\nabla_g\phi\right|^2 - \frac{f_u(u_\varepsilon,x)\phi^2}{\varepsilon}\right\}d\sigma.$$

It is well known that if $\lambda_1 > 0$ then u_ε is asymptotically stable and if $\lambda_1 < 0$ then u_ε is unstable. If $\lambda_1 = 0$ then stability or instability can occur. Moreover, if ϕ_1 is its corresponding eigenfunction then ϕ_1 can be assumed positive in $\mathscr{M}$. The classical argument of linearized stability can be applied to the present situation (e.g., see [20, Chapter 11]).

THEOREM 5. *Let $\{u_\varepsilon^l\}_{0<\varepsilon\leqslant\varepsilon_0}$ ($l=1,\dots,4$) be the family of local minimizers of E_ε provided by Theorems 2, 3 and 4. Then every u_ε^l ($l=1,\dots,4$) is a stable stationary solution to* (1).

Proof. Each local minimizer u_ε of E_ε is a stationary solution to (8) and, by (6), u_ε is also a stationary solution to (1).

Consider the following eigenvalue problem obtained by linearizing problem (8) around the local minimizer u_ε of E_ε

$$\varepsilon^2\left(\partial_{ss}^2\phi+\frac{\psi_s}{\psi}\partial_s\phi+\frac{1}{\psi^2}\partial_{\theta\theta}^2\phi\right)+f_u(u_\varepsilon,s)\phi+\lambda\phi=0. \tag{16}$$

Claim: if ϕ_1 is an eigenfunction corresponding to the first eigenvalue λ_1 of problem (16) then ϕ_1 is independent of θ.

We first observe that for any $\theta_0>0$, $\phi_1(s,\theta+\theta_0)$ is also an eigenfunction corresponding to λ_1. Moreover we have that ϕ_1 is 2π-periodic in θ and

$$\int_0^l\int_0^{2\pi}\phi_1^2(s,\theta)\psi d\theta ds=1. \tag{17}$$

It is well known that λ_1 is a simple eigenvalue, i.e., that the eigenspace corresponding to λ_1 is one-dimensional. We outline the proof for the reader's convenience. We suppose that ϕ_2 $(=\phi_1(s,\theta+\theta_0)$, for instance) also satisfies (16) and prove that ϕ_1 differs from ϕ_2 by a multiplicative constant.

Note that ϕ_1 and ϕ_2 satisfy the equation

$$\varepsilon^2\Delta_g\phi+f_u(u_\varepsilon,x)\phi+\lambda_1\phi=0 \text{ in } \mathscr{M}. \tag{18}$$

We can assume $\phi_1>0$ and $\phi_2>0$ and is not difficult to see that

$$\begin{aligned}0&=\phi_1\varepsilon^2\Delta_g\phi_2-\phi_2\varepsilon^2\Delta_g\phi_1\\&=\varepsilon^2\nabla_g(\phi_1\nabla_g\phi_2-\phi_2\nabla_g\phi_1)\\&=\varepsilon^2\nabla_g(\phi_1^2\nabla_g(\phi_2/\phi_1)).\end{aligned}$$

Using integration by parts it follows that

$$\begin{aligned}0&=\int_{\mathscr{M}}\varepsilon^2(\phi_2/\phi_1)\nabla_g\left[\phi_1^2\nabla_g(\phi_2/\phi_1)\right]d\sigma\\&=\int_{\mathscr{M}}\varepsilon^2\phi_2\phi_1\Delta(\phi_2/\phi_1)d\sigma+\int_{\mathscr{M}}\varepsilon^2(\phi_2/\phi_1)\nabla_g(\phi_2)\nabla_g(\phi_2/\phi_1)d\sigma\\&=\int_{\mathscr{M}}\varepsilon^2\phi_2\phi_1\Delta(\phi_2/\phi_1)d\sigma-\int_{\mathscr{M}}\varepsilon^2\phi_1^2\nabla_g[(\phi_2/\phi_1)\nabla_g(\phi_2/\phi_1)]d\sigma\\&=-\int_{\mathscr{M}}\varepsilon^2\phi_1^2\left|\nabla_g(\phi_2/\phi_1)\right|^2d\sigma.\end{aligned}$$

This prove that λ_1 is simple.

Hence, there exists a constant k such that

$$\phi_1(s,\theta) = k\phi_1(s,\theta+\theta_0),$$

and by (17)

$$\int_0^l \int_0^{2\pi} \phi_1^2(s,\theta+\theta_0)\psi d\theta ds = \int_0^l \int_{\theta_0}^{2\pi+\theta_0} \phi_1^2(s,\theta)\psi d\theta ds$$

$$= \int_0^l \int_0^{2\pi} \phi_1^2(s,\theta)\psi d\theta ds = 1,$$

then

$$1 = \int_0^l \int_0^{2\pi} \phi_1^2(s,\theta)\psi d\theta ds = k^2 \int_0^l \int_0^{2\pi} \phi_1^2(s,\theta+\theta_0)\psi d\theta ds = k^2.$$

It follows that $k = \pm 1$ for any $\theta_0 > 0$, $0 \leqslant s \leqslant l$ and $0 < \theta < 2\pi$ which proves the claim.

Recall that u_ε is a local minimizer of E_ε, then for all $\phi \in H^1(I)$

$$E_\varepsilon''(u_\varepsilon)(\phi) = \int_0^l \left\{ \varepsilon\psi(\phi')^2 - \frac{\psi f_u(u_\varepsilon,s)\phi^2}{\varepsilon} \right\} ds \geqslant 0. \tag{19}$$

Therefore, if ϕ_1 is the eigenfunction corresponding to the first eigenvalue λ_1, we have that ϕ_1 is independent of θ and thus

$$\begin{aligned}
\lambda_1 &= R_{u_\varepsilon}(\phi_1) \\
&= \int_{\mathscr{M}} \left\{ \varepsilon \left|\nabla_g \phi_1\right|^2 - \frac{f_u(u_\varepsilon,x)\phi_1^2}{\varepsilon} \right\} d\sigma \\
&= 2\pi \int_0^l \left\{ \varepsilon\psi(\phi_1')^2 - \frac{\psi f_u(u_\varepsilon,s)\phi_1^2}{\varepsilon} \right\} ds \\
&= 2\pi E_\varepsilon''(u_\varepsilon)(\phi_1) \geqslant 0.
\end{aligned}$$

If $\lambda_1 > 0$ then u_ε is stable.

Now if $\lambda_1 = 0$, since λ_1 is a simple eigenvalue, there is a local one-dimensional critical manifold $W(u_\varepsilon)$, tangent to the eigenspace spanned by the principal eigenfunction ϕ_1, at u_ε, such that if u_ε is stable in $W(u_\varepsilon)$ then it is also stable in $H^1(\mathscr{M})$. For this matter we refer to Theorem 6.2.1 in [11], which proof can be adapted to fit our case.

But now the stability of u_ε in $W(u_\varepsilon)$ (which is one-dimensional) follows from the fact that the semigroup generated by (1) defines a gradient flow in $H^1(\mathscr{M})$. To be more specific the functionals $E_\varepsilon(u_\varepsilon(x,t))$ defines a Lyapunov function and along each solution $u_\varepsilon(x,t)$ it holds that

$$\frac{d}{dt} E_\varepsilon(u_\varepsilon(x,t)) \leqslant 0, \quad t \geqslant 0.$$

This concludes the proof of Theorem 5. □

Finally we conclude the proof of the main theorem of this work:

Proof. (of Theorem 1) Follows directly from the combination of Theorems 2, 3, 4 and 5. □

3.1. The Neumann boundary condition case

Now, we discuss the case where the domain is a surface of revolution with boundary. Let the surface of revolution $\mathscr{M}$ be as before and let $\mathscr{D} \subset \mathscr{M}$ be the domain delimited by two circles C_{s_1} and C_{s_2}, $0 < s_1 < s_2 < l$, parametrized in the local coordinates (s,θ) as follows:

$$C_{s_1} : \begin{cases} s(t) = s_1 \\ \theta(t) = t \end{cases} \text{ and } \; C_{s_2} : \begin{cases} s(t) = s_2 \\ \theta(t) = t \end{cases}$$

with $t \in [0, 2\pi)$.

Let ν be the outer normal vector of $\partial\mathscr{D}$ lying in the tangent space $T_p(\mathscr{M})$ for any $p \in \partial\mathscr{D}$. We shall assume that $\partial\mathscr{D}$ is orientable so that the outer normal is well-defined and continuous.

The derivative of u in the direction of ν at $\partial\mathscr{D}$ is given by

$$\frac{\partial u}{\partial \nu} = \langle \nabla_g u, \nu \rangle,$$

where $\nu = \nu_1 \frac{\partial}{\partial s} + \nu_2 \frac{\partial}{\partial \theta}$ and $\left\{ \frac{\partial}{\partial s}, \frac{\partial}{\partial \theta} \right\}$ is the basis of $T_p(\mathscr{M})$.

Suppose furthermore that

$$\chi'(s) \geqslant 0, \;\; s \in (s_1, s_1+\delta) \cup (s_2-\delta, s_2) \tag{20}$$

for some $\delta > 0$. Thus there holds $\nu = (\partial/\partial s)$ on C_{s_2} and $\nu = -(\partial/\partial s)$ on C_{s_1}.

Except for a few natural changes, the proof of the following theorem is similar to that we rendered for domains without boundary.

THEOREM 6. *If $\psi(c-a)^3$ attains an isolated local minimum $s_0 \in (s_1,s_2)$ and f is given by (7), then there exist four families $\left\{u_\varepsilon^j\right\}_{0<\varepsilon\leqslant\varepsilon_0}$ ($j = 1,\dots,4$), for some $\varepsilon_0 > 0$, of nonconstants stables stationary solutions to the problem*

$$\begin{cases} \partial_t u_\varepsilon \; = \varepsilon^2 \Delta_g u_\varepsilon + f(u_\varepsilon, x), \;\; (t,x) \in \mathbb{R}^+ \times \mathscr{D} \\ \partial_\nu u_\varepsilon = 0, \;\; x \in \partial\mathscr{D} \end{cases} \tag{21}$$

satisfying

(i) $\|u_\varepsilon^1 - u_0^1\|_{L^1(s_1,s_2)} \xrightarrow{\varepsilon\to 0} 0$ *where*

$$u_0^1(s) = a(s)\chi_{(s_1,s_0)}(s) + c(s)\chi_{(s_0,s_2)}(s);$$

(ii) $\|u_\varepsilon^2 - u_0^2\|_{L^1(s_1,s_2)} \xrightarrow{\varepsilon\to 0} 0$ *where*

$$u_0^2(s) = c(s)\chi_{(s_1,s_0)}(s) + a(s)\chi_{(s_0,s_2)}(s);$$

(iii) $\|u_\varepsilon^3 - u_0^3\|_{L^1(s_1,s_2)} \xrightarrow{\varepsilon\to 0} 0$ *where* $u_0^3(s) = a(s)$ *in* (s_1,s_2)*;*

(iv) $\|u_\varepsilon^4 - u_0^4\|_{L^1(s_1,s_2)} \xrightarrow{\varepsilon\to 0} 0$ *where* $u_0^4(s) = c(s)$ *in* (s_1,s_2).

REMARK 2. (i) As stated in the Introduction, the families of solutions $\{u_\varepsilon^1\}$ and $\{u_\varepsilon^2\}$ derived from Theorems 1 and 6 develop internal transition layer as $\varepsilon \to 0$.

(ii) Standard bootstrap arguments ensure that each solution u_ε^j, of the problem with or without boundary, is a classical solution.

As a consequence of the Theorems 1 and 6 many examples of existence of patterns can be created. In the sequel we look at some simple examples.

3.2. Examples

For the sake of illustration let us consider the following surfaces and functions $a(x)$, $c(x)$ (recall that $b(x) = (a(x)+c(x))/2$) for which our results guarantee the existence of patterns to (1).

(i) We take $\mathscr{M}$ to be the unit sphere then $\psi(s) = \sin(s)$, $\chi(s) = \cos(s)$ and $I = (0,\pi)$. In this case if $c(s) = \sin^2(2s)$ ($c(x) = 4x_3^2(x_1^2+x_2^2)$, $x \in \mathscr{M}$) and $a(s) = -2$, a simple calculation shows that $\psi(c-a)^3$ attains an isolated local minimum in $(0,\pi)$. The same happens if we take $c(s) = \cos^2(s)$ ($c(x) = x_3^2$, $x \in \mathscr{M}$) and $a(s) = -1$, for instance.

(ii) $\mathscr{D}_1$ a cylindrical surface given by $\psi_1(s) = 1$ and $\chi_1(s) = s+1$, $s \in [0,1]$. If $c(s) = (s-1)^2$ ($c(x) = (x_3-2)^2$, $x \in \mathscr{D}_1$) and $a(s) = -s$ ($a(x) = -x_3+1$, $x \in \mathscr{D}_1$), then $s_0 = 1/2 \in (0,1)$ is an isolated local minimum of $\psi_1(c-a)^3$.

(iii) $\mathscr{D}_2$ given by $\psi_2(s) = s^2/4+1/2$ and $\chi_2(s) = s/4\sqrt{4-s^2}+\arcsin(s/2)$, $s \in (0,1)$, unlike $\mathscr{M}$ and $\mathscr{D}_1$, has negative Gaussian curvature (namely, $-\psi_2''/\psi_2 < 0$) and if $c(s) = s^2$ ($c(x) = 4\sqrt{x_1^2+x_2^2}-2$, $x \in \mathscr{D}_2$) and $a(s) = s-1$ ($a(x) = 2((x_1^2+x_2^2)^{1/2} - 1/2)^{1/2}$, $x \in \mathscr{D}_2$) we have that $\psi_2(c-a)^3$ also attains an isolated local minimum in $(0,1)$.

The surfaces $\mathscr{M}$ and $\mathscr{D}_1$ illustrate simple situations where our results can be applied. In particular, if $f(u,x) = f(u)$, was proved in [1, Theorem 3.3] the non existence of patterns in such cases which show the importance of roots $a(x), b(x), c(x)$ of $f(u,x)$. On the other hand, $\mathscr{D}_2$ is just one example with negative Gaussian curvature and, as is well known, the Gaussian curvature has key role in the search for patterns on surfaces of revolution. For more details on this matter see [1, 6, 7, 19] and the references therein.

REFERENCES

[1] C. BANDLE, F. PUNZO AND A. TESEI, *Existence and nonexistence of patterns on Riemannian manifolds*, J. Math. Anal. Appl. **387** (2012), 33–47.

[2] A. L. BESSE, *Manifolds all of whose Geodesics are Closed*, Springer, 1978.

[3] R. G. CASTEN AND C. J. HOLLAND, *Instability results for reaction diffusion equations with Neumann boundary conditions*, J. Diff. Eqns., **2** (1978), 266–273.

[4] A. S. DO NASCIMENTO, *Stable stationary solutions induced by spatial inhomogeneity via Γ-convergence*, Bulletin of the Brazilian Mathematical Society, **29** No.1 (1998), 75–97.

[5] A. S. DO NASCIMENTO, *On the role of diffusivity in some stable equilibria of a diffusion equation*, J. Diff. Eqns., **155** No. 2 (1999), 231–244.

[6] A. S. DO NASCIMENTO AND M. SÔNEGO, *Patterns on surfaces of revolution in a diffusion problem with variable diffusivity*, Electr. J. Diff. Eqns., **2014**, (2014) No. 238, 1–13.

[7] A. S. DO NASCIMENTO AND M. SÔNEGO, *The roles of diffusivity and curvature in patterns on surfaces of revolution*, J. of Math. Anal. and Appl. Vol. **412** (2014), 1084–1096.

[8] A. S. DO NASCIMENTO AND M. SÔNEGO, *Stable equilibria to a singularly perturbed reaction-diffusion equation in a degenerated heterogeneous environment*, J. of Math. Anal. and Appl. Vol. **433** (2016), 1743–1756.

[9] A. S. DO NASCIMENTO AND M. SÔNEGO, *Stable Transition Layers to Singularly Perturbed Spatially Inhomogeneous Allen-Cahn Equation*, Advanced Nonlinear Studies **15** (2015), 363–376.

[10] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkhauser, 1984.

[11] D. HENRY, *Geometric Theory of Semilinear Parabolic Equations*, Springer Lecture Notes in Mathematics, **840**, 1981.

[12] R. V. KOHN AND P. STERNBERG, *Local minimizers and singular pertubations*, Proceedings of the R. Soc. of Edinburg, **111** (1989), 69–84.

[13] K. KURATA, H. MATSUZAWA, *Multiple stable patterns in a balanced bistable equation with heterogeneous environments*, Applicable Analysis, **89** No. 7 (2010), 1023–1035.

[14] F. LI, K. NAKASHIMA, *Transition Layers for a spatially inhomogeneous Allen-Cahn equation in multi-dimensional domains*, Discrete Contin. Dyn. Syst., **32** No. 4 (2012), 1391–1420.

[15] F. MAHMOUDI, A. MALCHIODI AND J. WEI, *Transition layer for the heterogeneous Allen-Cahn equation*, Ann. I. H. Poincare A.N., **25** (2008), 609–631.

[16] H. MATANO, *Asymptotic behavior and stability of solutions of semilinear diffusion equations*, Publ. RIMS, Kyoto Univ., **15** (1979), 401–454.

[17] H. MATSUZAWA, *Stable transition layers in a balanced bistable equation with degeneracy*, Nonlinear Analysis, **58** (2004), 45–67.

[18] K. NAKASHIMA, *Stable transition layers in a balanced bistable equation*, Differential Integral Equations, **13** No. 7-9 (2000), 1025–1038.

[19] F. PUNZO, *The existence of patterns on surfaces of revolution without boundary*, Nonlinear Analysis, **77** (2012), 94–102.

[20] J. SMOLLER, *Shock Waves and Reaction - Diffusion Equations*, Springer New York, **258**, 1994.

[21] P. STERNBERG, *The effect of a singular pertubation on nonconvex variational problems*, Arch. Rat. Mech. Anal., 101 (1988), 209–260.

(Received February 10, 2016)
(Revised June 20, 2016)

Maicon Sônego Universidade Federal de Itajubá - IMC
Itajubá, M.G. Brasil
e-mail: mcn.sonego@unifei.edu.br