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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR KIRCHHOFF TYPE PROBLEMS WITH PARAMETER

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Abstract. In this paper, we study Kirchhoff type problems with parameter on a bounded domain. By using variational methods, we prove the existence and multiplicity of weak solutions.

1. Introduction

In this paper, we are concerned with the existence of solutions for the following Kirchhoff type problem

$$\begin{cases} -M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \lambda f(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, λ is a positive real parameter and $f \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ satisfies the following condition:

(F) there exist $C^* > 0$ and $2 < p < 2^*$ such that $|f(x,u)| \leqslant C^*(1+|u|^{p-1})$, where $2^* = +\infty$ for $n = 1,2$ and $2^* = 2n/(n-2)$ for $n \geqslant 3$.

When $M \not\equiv C$ (C is a constant), problem like (1) is nonlocal problem. Interest of the mathematicians on the nonlocal problems has increased because they represent a variety of relevant physical and engineering situations. For example, the equation($M = a+bt$)

$$\begin{cases} -\Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = g(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

has captured extensive interest only after Lions [12] had proposed an abstract framework for the problem, which was studied widely under various conditions, see, Chen et al [2], Mao and Zhang [14], Perera and Zhang [15], Sun and Tang [20], Sun and Liu [21], Shuai [24] and so on.

Recently, the study of Kirchhoff type equation (1) has been extended to the general form

$$\begin{cases} -M\Big(\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = g(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

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where $\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, for details see [6, 11]. By using the linking theorem and the mountain pass theorem, Liu et al [11] obtained two results about nontrivial solutions for (3). Their results are invalid for (2) since one of the premise in [11] is that M is a bounded function. Other results for problems of Kirchhoff type, we refer the reader to [1, 3, 4, 5, 25, 26] and references therein.

In the present paper, we study (1) by means of the variational principle of Ricceri [16, 17] and the result of Sun [23], and obtain the existence and multiplicity of solutions for (1), which are different from the literature mentioned above.

For the reader's convenience, we call the main tools that we will use, which are due to B.Ricceri and Sun, respectively.

THEOREM 1. [16] *Let X be a reflexive real Banach space, and $\psi,\varphi : X \to \mathbb{R}$ are two sequentially lower semicontinuous functionals. Assume that ψ is (strongly) continuous and $\lim_{\|u\|\to+\infty}\psi(u) = +\infty$. For each $\rho > \inf_X \psi$, put*

$$\phi(\rho) = \inf_{u\in\psi^{-1}(]-\infty,\rho[)} \frac{\varphi(u) - \inf_{cl_w \psi^{-1}(]-\infty,\rho[)}\varphi}{\rho - \psi(u)}, \tag{4}$$

Then for each $\rho > \inf_X \psi$ and each $\mu > \phi(\rho)$, the restriction of the functional $\varphi + \mu\psi$ to $\psi^{-1}(]-\infty,\rho[)$ has a global minimum.

THEOREM 2. [17] *Let X be a reflexive real Banach space, $\psi,\varphi : X \to \mathbb{R}$ are sequentially lower semicontinuous functionals and have continuously Gateaux derivative. For each $\lambda > 0$, $\varphi + \lambda\psi$ satisfies PS condition. Then for each $\rho > \inf_X \psi$ and each $\mu > \phi(\rho)$, where ϕ is defined in (4), either $\varphi + \mu\psi$ has a strict global minimum which lies in $\psi^{-1}(]-\infty,\rho[)$ or it has at least two critical points one of which lies in $\psi^{-1}(]-\infty,\rho[)$.*

THEOREM 3. [23] *Let X be a reflexive real Banach space, $\Phi \in C^1(X,\mathbb{R})$ and $M \in X$ a closed, invariant set of descent flow of Φ. If $\inf \Phi(M) > -\infty$ and Φ satisfies the PS condition on M, then Φ achieves the infimum at a critical point in M.*

2. Main results

In this section, we give the main results. At first, we give the following assumptions:

(M_1) The function $M : [0,+\infty) \to (0,+\infty)$ is continuous, $M(0) > 0$ and there exist constants $0 < \alpha < 2^*/2 - 1$ and $A > 0$ such that $h(u) := M(u) - Au^{\alpha}$ is nondecreasing and

$$\lim_{t\to+\infty} \frac{M(t)}{t^{\alpha}} = A.$$

(M_2) There exist the constants $\beta \geqslant 0$ and $B > 0$ such that

$$\lim_{t\to+\infty} \frac{h(t)}{t^{\beta}} = B.$$

(F_1) There exist nonempty open sets $D \subset \Omega$ and $B \subseteq D$ of positive measure such that

$$\limsup_{u\to 0^+} \frac{\inf_{x\in B} F(x,u)}{u^2} = +\infty, \quad \liminf_{u\to 0^+} \frac{\inf_{x\in D} F(x,u)}{u^2} > -\infty,$$

where $F(x,u) = \int_0^u f(x,s)ds$.

(F_2)

$$\lim_{u\to+\infty} \frac{F(x,u)}{u^{2(\alpha+1)}} = +\infty, \quad \lim_{|u|\to+\infty} \frac{uf(x,u) - 2(\alpha+1)F(x,u)}{u^{2+2\beta}} = K$$

uniformly in $x \in \Omega$, where $0 \leqslant K \leqslant +\infty$.

(F_3) $F(t,u) = o(u^2)$ as $u \to 0$, $F(t,u) = o(u^{2\alpha+2})$ as $|u| \to +\infty$ uniformly in $x \in \Omega$ and there is $\hat{v} \in H_0^1(\Omega)$ with $\int_\Omega F(x,\hat{v})dx > 0$.

REMARK 1. The condition (F_1) is a sort of subquadratical growth assumption at zero, which originated by Ricceri in [18]. Similar conditions were usually used, see, [7, 19].

We have the following results.

THEOREM 4. *Assume that (F), (M_1) and (F_1) hold, then there exists $\lambda^* > 0$ such that (1) has at least one nontrivial weak solution u_λ for $\lambda \in (0,\lambda^*)$. Moreover,* $\lim_{\lambda\to 0^+} \|u_\lambda\|_{H_0^1(\Omega)} = 0$.

THEOREM 5. *Assume that (F), (M_1), (M_2) and (F_2) hold, then there exists $\lambda^* > 0$ such that (1) has at least two weak solutions for $\lambda \in (0,\lambda^*)$.*

THEOREM 6. *Assume that $(F),(M_1)$ (F_3) hold, then there exists $\lambda^* > 0$ such that (1) has at least three weak solutions for $\lambda \in (\lambda^*,+\infty)$.*

3. Proofs of theorems

Let $H = H_0^1(\Omega)$ be the Sobolev space equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

Put

$$\psi(u) = \frac{1}{2}\int_0^{\|u\|^2} M(t)dt, \varphi(u) = -\int_\Omega F(x,u)dx,$$

$$J_\lambda(u) = \psi(u) + \lambda\varphi(u).$$

In the sequel, we denote by $|\cdot|_q$ usual L_q norm. Since Ω is a bounded domain, $H \hookrightarrow L^s(\Omega)$ is continuous for $s \in [1,2^*]$, compact for $s \in [1,2^*)$ and there exists ζ_s such that $|u|_s \leqslant \zeta_s\|u\|$ for all $u \in H$, $s \in [1,2^*]$.

The conditions (F) and (M_1) imply that $\varphi : H \to \mathbb{R}$ is sequentially weakly lower semicontinuous and has continuously Gateaux derivative, and J_λ is of class C^1. Moreover,

$$J'_\lambda(u)v = M(\|u\|^2)\int_\Omega \nabla u \nabla v dx - \lambda \int_\Omega f(x,u)v dx.$$

Hence, the critical points of J_λ are the weak solutions of (1) .

LEMMA 1. *Let $(F),(M_1),(F_1)$ hold and $\lambda > 0, \rho > 0$. Assume that u_λ is a global minimum of the restriction of J_λ to $\psi^{-1}(]-\infty,\rho[)$, then $u_\lambda \not\equiv 0$.*

Proof. Thanks to (F_1), there exist $\Lambda \in \mathbb{R}, \delta > 0$ such that

$$F(x,u) \geqslant \Lambda u^2 \text{ for } 0 < u < \delta, x \in D. \tag{5}$$

Fix a set $\Gamma \subseteq B$ of positive measure and $v \in H : \|v\| \neq 0$ such that

$$v = \begin{cases} 0 \leqslant v \leqslant 1, & v \in D, \\ 0, & v \in \Omega/D, \\ 1, & v \in \Gamma. \end{cases} \tag{6}$$

Since $\limsup_{u\to 0^+} u^{-2} \inf_{x\in\Gamma} F(x,u) = +\infty$, there exists a sequence

$$\{\gamma_k\} : \gamma_k \in (0,+\infty), \gamma_k \to 0 \text{ as } k \to \infty$$

such that

$$\lim_{k\to\infty} \frac{\inf_{x\in\Gamma} F(x,\gamma_k)}{\gamma_k^2} = +\infty.$$

For k sufficiently large, from (5) and (6), we have

$$\begin{aligned} -\frac{\varphi(\gamma_k v)}{\psi(\gamma_k v)} &= \frac{\int_\Gamma F(x,\gamma_k v)dx + \int_{D/\Gamma} F(x,\gamma_k v)dx}{\psi(\gamma_k v)} \\ &\geqslant \frac{\int_\Gamma F(x,\gamma_k)dx + \int_{D/\Gamma} \Lambda\gamma_k^2 v^2 dx}{\psi(\gamma_k v)} \\ &= \frac{\int_\Gamma \frac{F(x,\gamma_k)}{\gamma_k^2}dx + \Lambda\int_{D/\Gamma} v^2 dx}{\frac{\psi(\gamma_k v)}{\gamma_k^2}}. \end{aligned}$$

Noting that $\lim_{k\to\infty} \frac{\psi(\gamma_k v)}{\gamma_k^2} = \frac{1}{2}M(0)\|v\|^2$, we have

$$-\frac{\varphi(\gamma_k v)}{\psi(\gamma_k v)} \to +\infty \text{ as } k \to \infty.$$

Set $u_k = \gamma_k v$, then $u_k \to 0$ in H. Noting that for k sufficiently large, $u_k \in \psi^{-1}(-\infty,\rho)$,

$$\frac{J_\lambda(u_\lambda)}{\lambda} = \varphi(u_\lambda) + \frac{1}{\lambda}\psi(u_\lambda)$$

$$\leqslant \varphi(u_k) + \frac{1}{\lambda}\psi(u_k)$$
$$< 0 = \frac{J_\lambda(0)}{\lambda},$$

one easily obtain that $u_\lambda \not\equiv 0$. This ends the proof.

LEMMA 2. *Assume that $(F),(M_1),(M_2),(F_2)$ hold, then J_λ satisfies the PS condition for any $\lambda > 0$.*

Proof. Let $u_m \in H$ such that $|J_\lambda(u_m)| \leqslant C$ and $J'(u_m) \to 0$. Firstly, we show that $\{u_m\}$ is a bounded sequence. From (F_2), for any $\varepsilon > 0$, there exists $r > 0$ with

$$uf(x,u) - 2(\alpha+1)F(x,u) \geqslant (K-\varepsilon)u^{2+2\beta}, |u| \geqslant r,$$

which implies that $uf(x,u)-2(\alpha+1)F(x,u) \geqslant (K-\varepsilon)u^{2+2\beta} - \gamma(\varepsilon)$ for some constant $\gamma(\varepsilon) > 0$ and $\forall u \in \mathbb{R}$. Hence,

$$\begin{aligned}||u_m|| + 2(\alpha+1)C &\geqslant 2(\alpha+1)J_\lambda(u_m) - J'(u_m)u_m \\ &= (\alpha+1)\int_0^{\|u_m\|^2} M(t)dt - M(\|u_m\|^2)\|u_m\|^2 \\ &\quad + \lambda\int_\Omega (u_m f(x,u_m) - 2(\alpha+1)F(x,u_m))dx \\ &\geqslant (\alpha+1)\int_0^{\|u_m\|^2} h(t)dt - h(\|u_m\|^2)\|u_m\|^2 \\ &\quad + \lambda(K-\varepsilon)\int_\Omega u_m^{2+2\beta}dx - \lambda\gamma(\varepsilon)|\Omega|.\end{aligned}$$

By (M_1) and (M_2), one can obtain that

$$\alpha > \beta \geqslant 0, \lim_{t\to\infty}\frac{h(t^2)}{t^{2\beta}} = B, \lim_{t\to\infty}\frac{\int_0^{t^2} h(s)ds}{t^{2\beta+2}} = \frac{B}{\beta+1}.$$

Now, we assume that $\|u_m\| \to \infty$ as $m \to \infty$, then

$$h(\|u_m\|^2)\|u_m\|^2 = B\|u_m\|^{2\beta+2} + o(\|u_m\|^{2\beta+2}),$$
$$\int_0^{\|u_m\|^2} h(t)dt = \frac{B}{\beta+1}\|u_m\|^{2\beta+2} + o(\|u_m\|^{2\beta+2})$$

as $m \to \infty$.

If $K > 0$, taking $\varepsilon = K$, we have

$$\begin{aligned}||u_m|| + 2(\alpha+1)C &\geqslant (\alpha+1)\int_0^{\|u_m\|^2} h(t)dt - h(\|u_m\|^2)\|u_m\|^2 - \lambda\gamma(\varepsilon)|\Omega| \\ &\geqslant \frac{B(\alpha-\beta)}{\beta+1}\|u_m\|^{2\beta+2} + o(\|u_m\|^{2\beta+2}) - \lambda\gamma(\varepsilon)|\Omega|.\end{aligned}$$

Hence, $B(\alpha-\beta)/(\beta+1) \leqslant 0$, a contradiction.

If $K=0$, taking

$$\varepsilon = (\alpha-\beta)B/2\lambda(\beta+1)\zeta_{2+2\beta}^{2+2\beta},$$

we have

$$\begin{aligned} ||u_m|| + 2(\alpha+1)C \geqslant (\alpha+1)\int_0^{\|u_m\|^2} & h(t)dt - h(\|u_m\|^2)\|u_m\|^2 \\ & -\lambda\varepsilon\int_\Omega u_m^{2+2\beta}dx - \lambda\gamma(\varepsilon)|\Omega| \\ & \geqslant \frac{B(\alpha-\beta)}{2(\beta+1)}\|u_m\|^{2\beta+2} + o(\|u_m\|^{2\beta+2}) - \lambda\gamma(\varepsilon)|\Omega|, \end{aligned}$$

which implies that $B(\alpha-\beta)/2(\beta+1) \leqslant 0$, a contradiction. Hence, $\{u_m\}$ is bounded in H. If necessary to a subsequence, we assume that $u_m \rightharpoonup u$ in H. Using the condition (F), one can obtain that $\lim_{m\to\infty}\varphi'(u_m)(u_m-u) = 0$. Now, since $J'_\lambda(u_m) \to 0$, there is a sequence $\{\varepsilon_m\}$ with $\varepsilon_m \to 0^+$ such that $|J'_\lambda(u_m)v| \leqslant \varepsilon_m$ for all $v \in H$ and $m \in \mathbb{N}$. Let $v_m = (u_m-u)/\|u_m-u\|$, then $|J'_\lambda(u_m)(u_m-u)| \leqslant \varepsilon_m\|u_m-u\|$. Noting that

$$\begin{aligned} \psi'(u_m)(u_m-u) &= M(\|u_m\|^2)\int_\Omega \nabla u_m\nabla(u_m-u) \\ &= M(\|u_m\|^2)\left(\|u_m\|^2 - \int_\Omega \nabla u_m\nabla u\right) \\ &\geqslant M(\|u_m\|^2)\left(\|u_m\|^2 - \frac{\|u_m\|^2+\|u\|^2}{2}\right) \\ &= \frac{1}{2}M(\|u_m\|^2)(\|u_m\|^2-\|u\|^2), \end{aligned}$$

we obtain that

$$\frac{1}{2}M(\|u_m\|^2)(\|u_m\|^2-\|u\|^2) - \lambda\varphi'(u_m)(u_m-u) \leqslant \varepsilon_m\|u_m-u\|.$$

Hence,

$$\limsup_{m\to\infty}\frac{1}{2}M(\|u_m\|^2)(\|u_m\|^2-\|u\|^2) \leqslant 0,$$

which implies that $\lim_{m\to\infty}\|u_m\|^2 = \|u\|^2$. Hence, u_m strongly converges to u in H. The proof is complete.

LEMMA 3. *Assume that $(F),(M_1),(F_3)$ hold, then the following claim hold:*
(1) J_λ *is coercive.*

(2) J_λ *satisfies the PS condition for any $\lambda > 0$.*

(3)*There are $\rho > 0, 0 < r < \|\hat{v}\|$ such that $J_\lambda \geqslant \rho$ for all $u \in H$ with $\|u\| = r$.*

Proof. (1) Since $F(x,u)=o(u^{2(\alpha+1)})$ as $|u|\to\infty$, there is $\tau>0$ such that

$$|F(x,u)|\leqslant \frac{A}{2\lambda(1+\alpha)\zeta_{2(\alpha+1)}^{2(\alpha+1)}}u^{2(\alpha+1)}+\tau$$

for all $x\in\Omega$ and $u\in\mathbb{R}$. Hence,

$$\begin{aligned}
J_\lambda(u) &\geqslant \frac{1}{2}\int_0^{\|u\|^2}M(t)dt-\lambda\int_\Omega|F(x,u)|dx\\
&\geqslant \frac{A}{2(1+\alpha)}\|u\|^{2(\alpha+1)}+\frac{1}{2}\int_0^{\|u\|^2}h(t)dt\\
&\quad -\frac{A}{2(1+\alpha)\zeta_{2(\alpha+1)}^{2(\alpha+1)}}\int_\Omega u^{2(\alpha+1)}dx-\lambda\tau|\Omega|\\
&\geqslant \frac{1}{2}\int_0^{\|u\|^2}h(t)dt-\lambda\tau|\Omega|\geqslant\frac{M(0)}{2}\|u\|^2-\lambda\tau|\Omega|,
\end{aligned}$$

which follows that J_λ is coercive.

(2) The proof is similar to that of Lemma 2.2.

(3) The condition (F_3) implies that

$$|F(x,u)|\leqslant C^*u^{2(\alpha+1)}+M(0)u^2/4\lambda\zeta_2^2, u\in\mathbb{R} \text{ for some } C^*>0.$$

Hence,

$$\begin{aligned}
J_\lambda(u) &\geqslant \frac{1}{2}\int_0^{\|u\|^2}M(t)dt-\lambda\int_\Omega\left(\frac{M(0)}{4\lambda\zeta_2^2}u^2+C^*u^{2(\alpha+1)}\right)dx\\
&= \frac{A}{2(1+\alpha)}\|u\|^{2(\alpha+1)}+\frac{M(0)}{2}\|u\|^2+\frac{1}{2}\int_0^{\|u\|^2}[h(t)-M(0)]dt\\
&\quad -\frac{M(0)}{4}\|u\|^2-\lambda C^*\zeta_{2(\alpha+1)}^{2(\alpha+1)}\|u\|^{2(\alpha+1)}\\
&\geqslant \frac{M(0)}{4}\|u\|^2+\left(\frac{A}{2(1+\alpha)}-\lambda C^*\zeta_{2(\alpha+1)}^{2(\alpha+1)}\right)\|u\|^{2(\alpha+1)},
\end{aligned}$$

where we use the fact that $h(t)\geqslant M(0)$ for $t\geqslant 0$. This shows the existence of r,ρ.

PROOF OF THEOREM 4 Note that ψ,φ are sequentially lower semicontinuous and have continuously Gateaux derivative. Moreover, ψ is coercive and $\inf_H\psi=0$.

Let $\rho>0$ be such that $\phi(\rho)>0$ and put $\lambda^*=1/\phi(\rho)$. Thanks to Theorem 1, for each $\lambda\in(0,\lambda^*)$, there exists $u_\lambda\in\psi^{-1}(]-\infty,\rho[)$ such that $\varphi'(u_\lambda)+\lambda^{-1}\psi'(u_\lambda)=\frac{1}{\lambda}J'_\lambda(u_\lambda)=0$. In particular, u_λ is a global minimum of J_λ to $\psi^{-1}(]-\infty,\rho[)$. By Lemma 1, $u_\lambda\not\equiv 0$. Since ψ is coercive and $u_\lambda\in\psi^{-1}(]-\infty,\rho[)$, there is a positive constant L_1 such that $\|u_\lambda\|<L_1$ for $\lambda\in(0,\lambda^*)$. Therefore, there exists $L_2>0$ such that

$$\left|\int_\Omega f(x,u_\lambda(x))u_\lambda(x)dx\right|\leqslant\|\varphi'(u_\lambda)\|_{H^*}\|u_\lambda\|\leqslant L_1L_2$$

for $\lambda \in (0,\lambda^*)$, since φ' is compact. From $J'_\lambda(u_\lambda)(u_\lambda)=0$, we have

$$M\left(\|u_\lambda\|^2\right)\|u_\lambda\|^2=\lambda\int_\Omega f(x,u_\lambda(x))u_\lambda(x)dx\to 0$$

as $\lambda\to 0^+$, which implies that $\lim_{\lambda\to 0^+}\|u_\lambda\|=0$.

PROOF OF THEOREM 5 Let $\rho>0$ be such that $\phi(\rho)>0$ and put $\lambda^*=1/\phi(\rho)$. Thanks to Theorem 2 and Lemma 2, for each $\lambda\in(0,\lambda^*)$, either $\varphi+\frac{1}{\lambda}\psi$ has a strict global minimum which lies in $\psi^{-1}(]-\infty,\rho[)$ or it has at least two critical points one of which lies in $\psi^{-1}(]-\infty,\rho[)$. One only need to verify that J_λ has no global minimum. Fix a $v\in H$ with $v>0$ on Ω. The condition (F_2) follows that there exists $\tau>0$ such that for $\forall x\in\Omega, u\geqslant 0$,

$$F(x,u)\geqslant\left(\frac{A\|v\|^{2(1+\alpha)}}{2\lambda(1+\alpha)|v|_{2(\alpha+1)}^{2(\alpha+1)}}+\frac{1}{\lambda|v|_{2(\alpha+1)}^{2(\alpha+1)}}\right)u^{2(\alpha+1)}-\tau.$$

Noting that

$$\begin{aligned}J_\lambda(tv)&=\frac{1}{2}\int_0^{t^2\|v\|^2}M(t)dt-\lambda\int_\Omega F(x,tv)dx\\&\leqslant\frac{A\|v\|^{2(\alpha+1)}}{2(1+\alpha)}t^{2(\alpha+1)}+\frac{1}{2}\int_0^{t^2\|v\|^2}h(t)dt+\lambda\tau|\Omega|\\&\quad-\lambda\left(\frac{A\|v\|^{2(1+\alpha)}}{2\lambda(1+\alpha)|v|_{2(\alpha+1)}^{2(\alpha+1)}}+\frac{1}{\lambda|v|_{2(\alpha+1)}^{2(\alpha+1)}}\right)t^{2(\alpha+1)}\int_\Omega v^{2(\alpha+1)}dx\\&\leqslant-\left(1+\frac{1}{2t^{2(\alpha+1)}}\int_0^{t^2\|v\|^2}h(t)dt\right)t^{2(\alpha+1)}+\lambda\tau|\Omega|,\end{aligned}$$

$$\alpha>\beta,\ \lim_{t\to+\infty}\frac{\int_0^{t^2\|v\|^2}h(t)dt}{t^{2+2\beta}}=\frac{B\|v\|^2}{\beta+1},$$

we have

$$J_\lambda(tv)\to-\infty\quad\text{as}\ t\to+\infty,$$

which follows that J_λ has no global minimum.

PROOF OF THEOREM 6 Let λ be sufficiently large such that $J_\lambda(\hat{v})<0$, and

$$\Xi=\{u\in H:\|u\|<r\},\quad c=\inf_{h\in\Upsilon}\max_{t\in[0,1]}J_\lambda(h(t)),$$

where $\Upsilon=\{h|h:[0,1]\to H$ is continuous and $h(0)=0,h(1)=\hat{v}\}$, $r,\hat{v}$ are defined in Lemma 3 and (F_3), respectively. Clearly, Ξ is an open set in H, $\hat{v}\notin\Xi$ and

$$\max\{J_\lambda(0),J_\lambda(\hat{v})\}<\inf_{u\in\partial\Xi}J_\lambda.$$

By the Mountain Pass Theorem, c is a critical value of J_λ and hence there exists $u_0 \in H$ such that $J_\lambda(u_0) = c$. Let $0 < \varepsilon < c - \max\{J_\lambda(0), J_\lambda(\hat{v})\}$ and S_1, S_2 path connected component containing $0, \hat{v}$ in the set $\{u \in H : J_\lambda(u) \leqslant c - \varepsilon\}$, respectively. $\inf J_\lambda(S_i) > -\infty, i = 1,2$ since J_λ is coercive. Noting that S_1, S_2 are nonempty closed invariant sets of descent flow of J_λ and J_λ satisfies the PS condition, by Theorem 3, one obtains that there exist the critical points $u_1 \in S_1, u_2 \in S_2$ of J_λ such that $J_\lambda(u_1) = \inf J_\lambda(S_1)$, $J_\lambda(u_2) = \inf J_\lambda(S_2)$. Since $S_1 \cap S_2 = \emptyset$ and $J_\lambda(u_1) < c, J_\lambda(u_2) < c$, we get that $u_i \not\equiv u_j$ for $i \neq j (i, j = 0,1,2)$.

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