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ON GLOBAL CONVERGENCE OF FORCED NONLINEAR DELAY DIFFERENTIAL EQUATIONS AND APPLICATIONS

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Abstract. Consider the following nonlinear delay differential equation with a forcing term $r(t)$:

$$x'(t)+a(t)x(t)+b(t)f(x(t-\tau(t)))=r(t),\ t\geqslant 0,$$

where $a\in C[[0,\infty),[0,\infty)]$, $b,\tau\in C[[0,\infty),(0,\infty)]$, $r\in C[[0,\infty),\mathbb{R}]$, $f\in C[(L,\infty),(L,\infty)]$ with $-\infty\leqslant L\leqslant 0$, and $\lim_{t\to\infty}(t-\tau(t))=\infty$. We establish a sufficient condition for every solution of the equation to converge to zero. By applying the result to some special cases and differential equation models from applications, we obtain several new criteria on the global convergence of solutions.

1. Introduction

Consider the following nonlinear delay differential equation with a forcing term $r(t)$:

$$x'(t)+a(t)x(t)+b(t)f(x(t-\tau(t)))=r(t),\ t\geqslant 0, \tag{1.1}$$

where $a\in C[[0,\infty),[0,\infty)], b,\tau\in C[[0,\infty),(0,\infty)], r\in C[[0,\infty),\mathbb{R}], f\in C[(L,\infty),(L,\infty)]$ with $-\infty\leqslant L\leqslant 0$, and $\lim_{t\to\infty}(t-\tau(t))=\infty$. Our aim in this paper is to study the global convergence of solutions of Eq.(1.1) and its applications.

Let $t_{-1}=\inf_{t\geqslant 0}\{t-\tau(t)\}$. With Eq.(1.1) we associate an initial function of the form

$$x(t)=\phi(t) \text{ for } t_{-1}\leqslant t\leqslant 0 \text{ where } \phi\in C[[t_{-1},0],(L,\infty)]. \tag{1.2}$$

A function $x(t)$ is said to be a solution of Eq.(1.1) if $x(t)$ satisfies (1.2) and Eq.(1.1) for $t\geqslant 0$.

When $a(t)\equiv 0$, Eq.(1.1) becomes

$$x'(t)+b(t)f(x(t-\tau(t)))=r(t),\ t\geqslant 0. \tag{1.3}$$

Furthermore, when $a(t)\equiv 0$ and $f(x)=x$, Eq.(1.1) reduces to the linear equation

$$x'(t)+b(t)x(t-\tau(t))=r(t),\ t\geqslant 0. \tag{1.4}$$

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The global convergence of solutions of Eqs.(1.3) and (1.4) and applications have been studied in [17] and [5] respectively. However, we notice that many equations derived from applications are in the more general form (1.1). For instance, it is well-known that many differential equations derived from mathematical biology (directly or by certain transformations) have the form

$$y'(t) = p(t)[g(y(t-\tau(t))) - y(t)],\ t \geqslant 0 \tag{1.5}$$

where $p, \tau, g \in C[[0,\infty),(0,\infty)]$. Numerous results for a positive equilibrium of the equation, that is, a positive fixed point of g, to be a global attractor of all positive solutions have been obtained. See, for example, [2-5, 7-10, 12-18] and the references cited therein. In applications there are often some unknown factors which might affect the differential models. Hence, it is realistic to add a small forcing term $r(t)$ to Eq.(1.5) which becomes

$$y'(t) = p(t)[g(y(t-\tau(t))) - y(t)] + r(t),\ t \geqslant 0 \tag{1.6}$$

where $r \in C[[0,\infty),\mathbb{R}]$. With Eq.(1.6), an initial condition of the form

$$x(t) = \psi(t) \text{ for } t_{-1} \leqslant t \leqslant 0 \text{ where } \psi \in C[[t_{-1},0],(0,\infty)] \tag{1.7}$$

is associated, where t_{-1} is as defined above. Assume that the function g in Eq.(1.6) has a positive fixed point $\bar{y}$ and let $x(t) = y(t) - \bar{y}$. Then Eq.(1.6) can be written as

$$x'(t) + p(t)x(t) + p(t)[g(\bar{y}) - g(x(t-\tau(t)) + \bar{y})] = r(t),\ t \geqslant 0. \tag{1.8}$$

Eq.(1.8) is in the form (1.1) with $a(t) = b(t) = p(t)$ and $f(x) = g(\bar{y}) - g(x+\bar{y})$ for $x > -\bar{y}$ and with the initial functions of the form

$$x(t) = \psi(t) \text{ for } t_{-1} \leqslant t \leqslant 0 \text{ where } \psi \in C[[t_{-1},0],(-\bar{y},\infty)].$$

In the next section, we will establish a sufficient condition such that every solution $x(t)$ of Eq.(1.1) converges to zero. Then in Section 3, we will apply the main result to some special cases and differential equation models derived from mathematical biology to obtain several new criteria on the global convergence of positive solutions.

The study of asymptotic behavior of solutions of various unforced delay differential equations and difference equations has received a lot of attention by many authors. However, results about the behavior of solutions of forced equations are relatively scarce. For some studies of forced delay differential and difference equations, we refer the reader to [5,6,11,17] and the references contained therein.

2. Main Results

In this section, we establish a sufficient condition such that every solution of Eq.(1.1) converges to zero. We define

$$\lim_{f(x)\to 0} x = 0 \text{ (or } x \to 0 \text{ as } f(x) \to 0) \tag{2.1}$$

in the sense that for each number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$|x| < \varepsilon \text{ whenever } |f(x)| < \delta.$$

THEOREM 1. *Assume that* (2.1) *holds and there is a positive constant* β *such that*

$$|f(x)| \leqslant \beta|x|, \text{ and } xf(x) > 0 \text{ for } x \neq 0. \tag{2.2}$$

Suppose also that

$$\int_0^\infty e^{\int_0^t a(s)ds} b(t)dt = \infty, \tag{2.3}$$

$$\beta \limsup_{t\to\infty} \int_{t-\tau(t)}^t e^{-\int_u^t a(s)ds} b(u)du < 1 \tag{2.4}$$

and

$$\lim_{t\to\infty} \frac{r(t)}{b(t)} = 0. \tag{2.5}$$

Then every solution $x(t)$ *of Eq.*(1.1) *tends to zero as* $t \to \infty$.

Proof. First, we assume that $x(t)$ is an eventually monotonic solution. We assume that $x(t)$ is eventually positive; the proof for the case that $x(t)$ is eventually negative is similar and will be omitted. To this end, let $x(t) \to l$ as $t \to \infty$. Then $0 \leqslant l \leqslant \infty$. Clearly, it suffices to show that $l = 0$. Assume, for the sake of contradiction, that $l > 0$. We claim that

$$\liminf_{t\to\infty} f(x(t-\tau(t))) > 0. \tag{2.6}$$

Otherwise, there is a positive sequence $\{s_n\}$ with $s_n \to \infty$ as $n \to \infty$ such that $f(x(s_n - \tau(s_n))) \to 0$ as $n \to \infty$. Then in view of (2.1), it follows that $x(s_n - \tau(s_n)) \to 0$ as $n \to \infty$. This implies that $l = 0$ which is a contradiction. Hence (2.6) holds. By noting (2.5) and (2.6) we see that there are positive numbers λ and t_0 such that

$$\frac{r(t)}{b(t)} \leqslant \lambda \text{ and } f(x(t-\tau(t))) \geqslant 2\lambda,\ t \geqslant t_0.$$

Observe that

$$x'(t) + a(t)x(t) = b(t)\left(\frac{r(t)}{b(t)} - f(x(t-\tau(t)))\right) \leqslant -\lambda b(t),\ t \geqslant t_0$$

which yields

$$\left(e^{\int_0^t a(s)ds} x(t)\right)' \leqslant -\lambda e^{\int_0^t a(s)ds} b(t),\ t \geqslant t_0.$$

Integrating both sides of the above inequality from t_0 to t, we have

$$e^{\int_0^t a(s)ds}x(t) - e^{\int_0^{t_0} a(s)ds}x(t_0) \leqslant -\lambda \int_{t_0}^t e^{\int_0^u a(s)ds}b(u)du$$

and so

$$x(t) \leqslant e^{-\int_0^t a(s)ds}\left[e^{\int_0^{t_0} a(s)ds}x(t_0) - \lambda \int_{t_0}^t e^{\int_0^u a(s)ds}b(u)du\right].$$

By noting (2.3), we see that the right side of the above inequality is negative when t is large. Clearly, this is a contradiction. Hence, $l=0$.

Next, we show that every solution $x(t)$ which is not eventually monotonic converges to zero also. First by noting (2.4), there are positive numbers μ and T_0 with $\mu<1$ such that

$$\beta \int_{t-\tau(t)}^t e^{-\int_u^t a(s)ds}b(u)du < \mu \text{ for } t \geqslant T_0.$$

Since $x(t)$ is not eventually monotonic, there is a sequence $\{t_n\}$ with

$$T_0 \leqslant t_1 < t_2 < \cdots < t_n < \cdots \text{ and } t_n \to \infty \text{ as } n \to \infty$$

such that $x(t)$ has relative extrema at $t_n, n=1,2,\cdots$. Clearly, to show that $x(t)\to 0$ as $t\to\infty$, it suffices to show that $x(t_n)\to 0$ as $n\to\infty$. Since $x'(t_n)=0$, it follows from Eq. (1.1) that

$$a(t_n)x(t_n) + b(t_n)f(x(t_n - \tau(t_n))) = r(t_n),\ n=1,2,\cdots. \tag{2.7}$$

In addition, from Eq. (1.1) we see that

$$\left(e^{\int_0^t a(s)ds}x(t)\right)' = e^{\int_0^t a(s)ds}r(t) - e^{\int_0^t a(s)ds}b(t)f(x(t-\tau(t))),\ t \geqslant 0. \tag{2.8}$$

Integrating (2.8) from $t_n - \tau(t_n)$ to t_n, we find that

$$\begin{aligned} x(t_n) &= e^{-\int_0^{t_n} a(s)ds}\left[e^{\int_0^{t_n-\tau(t_n)} a(s)ds}x(t_n-\tau(t_n)) + \int_{t_n-\tau(t_n)}^{t_n} e^{\int_0^t a(s)ds}r(t)dt \right.\\ &\qquad \left. - \int_{t_n-\tau(t_n)}^{t_n} e^{\int_0^t a(s)ds}b(t)f(x(t-\tau(t)))dt\right] \\ &= e^{-\int_{t_n-\tau(t_n)}^{t_n} a(s)ds}x(t_n-\tau(t_n)) + \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds}r(t)dt \\ &\qquad - \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds}b(t)f(x(t-\tau(t)))dt,\ n=1,2,\cdots. \end{aligned} \tag{2.9}$$

Let δ be a positive number such that $2\delta+\mu<1$. We claim that there is a subsequence $\{t_{n_m}\}$ of $\{t_n\}$ such that for any positive integer m,

$$\text{if } x(t_n) \geqslant 0, \text{ then } e^{-\int_{t_n-\tau(t_n)}^{t_n} a(s)ds}x(t_n-\tau(t_n)) < \delta^m \text{ for } n \geqslant n_m \tag{2.10}$$

and

$$\text{if } x(t_n) < 0, \text{ then } e^{-\int_{t_n-\tau(t_n)}^{t_n} a(s)ds} x(t_n - \tau(t_n)) > -\delta^m \text{ for } n \geqslant n_m. \tag{2.11}$$

We now show that (2.10) holds. When $x(t_n - \tau(t_n)) \leqslant 0$, (2.10) is clearly true. Hence, we only need to consider the case that $x(t_n - \tau(t_n)) > 0$. Then $f(x(t_n - \tau(t_n))) > 0$. Since $x(t_n) \geqslant 0$, from (2.7) we see that

$$b(t_n) f(x(t_n - \tau(t_n))) \leqslant r(t_n),\ n = 1, 2, \cdots$$

which yields

$$f(x(t_n - \tau(t_n))) \leqslant \frac{r(t_n)}{b(t_n)},\ n = 1, 2, \cdots.$$

From (2.4) we know that $\frac{r(t_n)}{b(t_n)} \to 0$ as $n \to \infty$. Hence, $f(x(t_n - \tau(t_n))) \to 0$ as $n \to \infty$. Then by the hypotheses on f, it follows that $x(t_n - \tau(t_n)) \to 0$ as $n \to \infty$. This fact implies that (2.10) holds. By a similar argument, we see that (2.11) holds. Clearly, $\{t_{n_m}\}$ could be chosen such that

$$\begin{cases} \text{for any } n \geqslant n_1, t - \tau(t) \geqslant 0 \text{ when } t \geqslant t_n - \tau(t_n), \text{ and} \\ \text{for any } n \geqslant n_{m+1},\ t - \tau(t) \geqslant t_{n_m} \text{ when } t \geqslant t_n - \tau(t_n),\ m = 1, 2, \cdots. \end{cases} \tag{2.12}$$

In addition, observing

$$\begin{aligned} \left| \int_{t-\tau(t)}^{t} e^{-\int_u^t a(s)ds} r(u) du \right| &\leqslant \int_{t-\tau(t)}^{t} e^{-\int_u^t a(s)ds} \left| \frac{r(u)}{b(u)} \right| b(u) du \\ &\leqslant \left(\sup_{u \geqslant t-\tau(t)} \left| \frac{r(u)}{b(u)} \right| \right) \int_{t-\tau(t)}^{t} e^{-\int_u^t a(s)ds} b(u) du, \end{aligned}$$

we see that (2.4) and (2.5) yield

$$\lim_{t \to \infty} \int_{t-\tau(t)}^{t} e^{-\int_u^t a(s)ds} r(u) du = 0.$$

Hence, we may assume that

$$\left| \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds} r(t) dt \right| < \delta^m,\ n \geqslant n_m. \tag{2.13}$$

When (2.10) holds, it follows from (2.9) and (2.13) that

$$\begin{aligned} |x(t_n)| = x(t_n) &\leqslant e^{-\int_{t_n-\tau(t_n)}^{t_n} a(s)ds} x(t_n - \tau(t_n)) + \left| \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds} r(t) dt \right| \\ &\quad + \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds} b(t) |f(x(t - \tau(t)))| dt \\ &\leqslant 2\delta^m + \beta \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds} b(t) |x(t - \tau(t))| dt,\ n \geqslant n_m. \end{aligned}$$

When (2.11) holds, it follows from (2.9) and (2.13) that

$$|x(t_n)| = -x(t_n) \leqslant -e^{-\int_{t_n-\tau(t_n)}^{t_n} a(s)ds} x(t_n-\tau(t_n)) + \left| \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds} r(t)dt \right|$$
$$+ \int_{t_n-\tau(t_n)}^{t_n} e^{\int_t^{t_n} a(s)ds} b(t)|f(x(t-\tau(t)))|dt$$
$$\leqslant 2\delta^m + \beta \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds} b(t)|x(t-\tau(t))|dt, \; n \geqslant n_m.$$

Hence, in any case we have

$$|x(t_n)| \leqslant 2\delta^m + \beta \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds} b(t)|x(t-\tau(t))|dt, \; n \geqslant n_m. \tag{2.14}$$

Let

$$M = \max_{0 \leqslant t \leqslant t_{n_1}} \{|x(t)|\}.$$

In the following, we show that for any positive integer m,

$$|x(t_n)| \leqslant (2\delta+\mu)^m (M+1), \; n \geqslant n_m. \tag{2.15}$$

First, we show that (2.15) is true when $m = 1$. In fact, from (2.14) and noting (2.12) we see that

$$|x(t_{n_1})| \leqslant 2\delta + \beta \int_{t_{n_1}-\tau(t_{n_1})}^{t_{n_1}} e^{-\int_t^{t_{n_1}} a(s)ds} b(t)|x(t-\tau(t))|dt$$
$$\leqslant 2\delta + \mu M \leqslant (2\delta+\mu)(M+1).$$

Now, assume that

$$|x(t_n)| \leqslant (2\delta+\mu)(M+1) \text{ for } n_1 \leqslant n \leqslant k. \tag{2.16}$$

Since $t_{k+1} > t_k \geqslant t_{n_1}$ and (2.12) holds,

$$0 \leqslant t-\tau(t) \leqslant t_{k+1} \text{ when } t_{k+1}-\tau(t_{k+1}) \leqslant t \leqslant t_{k+1}.$$

By noting the property of $\{x(t_{n_m})\}$, we see that

$$|x(t-\tau(t))| \leqslant M \text{ if } 0 \leqslant t-\tau(t) \leqslant t_{n_1},$$

(2.16) yields

$$|x(t-\tau(t))| \leqslant (2\delta+\mu)(M+1) \leqslant M+1 \text{ if } t_{n_1} \leqslant t-\tau(t) \leqslant t_k,$$

and

$$|x(t-\tau(t))| \leqslant \max\{|x(t_k)|, |x(t_{k+1})|\} \text{ if } t_k \leqslant t-\tau(t) \leqslant t_{k+1}.$$

Hence,

$$|x(t-\tau(t))| \leqslant \max\{M+1, |x(t_{k+1})|\} \text{ when } t_{k+1}-\tau(t_{k+1}) \leqslant t-\tau(t) \leqslant t_{k+1}$$

and so it follows from (2.14) that

$$\begin{aligned} |x(t_{k+1})| &\leqslant 2\delta + \beta \int_{t_{k+1}-\tau(t_{k+1})}^{t_{k+1}} e^{-\int_t^{t_{k+1}} a(s)ds} b(t)|x(t-\tau(t))|dt \\ &\leqslant 2\delta + \max\{M+1, |x(t_{k+1})|\}\beta \int_{t_{k+1}-\tau(t_{k+1})}^{t_{k+1}} e^{-\int_t^{t_{k+1}} a(s)ds} b(t)dt \\ &\leqslant 2\delta + \max\{M+1, |x(t_{k+1})|\}\mu. \end{aligned} \tag{2.17}$$

We claim that

$$|x(t_{k+1})| \leqslant M+1. \tag{2.18}$$

Otherwise $|x(t_{k+1})| > M+1$. From (2.17) we see that

$$|x(t_{k+1})| \leqslant 2\delta + |x(t_{k+1})|\mu$$

which implies that

$$|x(t_{k+1})| \leqslant \frac{2\delta}{1-\mu}.$$

Then it follows that

$$\frac{2\delta}{1-\mu} > M+1$$

which contradicts the fact that $2\delta+\mu < 1$. Hence, (2.18) holds, and so (2.17) yields

$$|x(t_{k+1})| \leqslant 2\delta + (M+1)\mu \leqslant (2\delta+\mu)(M+1).$$

Therefore, by induction, (2.15) holds when $m = 1$.

Next, assume that

$$|x(t_n)| \leqslant (2\delta+\mu)^k(M+1) \text{ for } n \geqslant n_k. \tag{2.19}$$

We are going to show that

$$|x(t_n)| \leqslant (2\delta+\mu)^{k+1}(M+1) \text{ for } n \geqslant n_{k+1}.$$

In fact, in view of (2.12) and (2.19), it follows from (2.14) that

$$\begin{aligned} |x(t_n)| &\leqslant 2\delta^{k+1} + \beta \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds} b(t)|x(t-\tau(t))|dt \\ &\leqslant 2\delta^{k+1} + (2\delta+\mu)^k(M+1)\beta \int_{t_n-\tau(t_n)}^{t_n} e^{-\int_t^{t_n} a(s)ds} b(t)dt \\ &\leqslant 2\delta^{k+1} + (2\delta+\mu)^k(M+1)\mu, \; n \geqslant n_{k+1}. \end{aligned} \tag{2.20}$$

Since

$$2\delta^{k+1} + (2\delta+\mu)^k \mu \leqslant (2\delta+\mu)^{k+1},$$

(2.20) yields

$$|x(t_n)| \leqslant (2\delta+\mu)^{k+1}(M+1) \text{ for } n \geqslant n_{k+1}.$$

Hence, by induction, we see that (2.15) holds. Clearly, (2.15) implies that $x(t_n) \to 0$ as $n \to \infty$. The proof is complete.

REMARK 1. In the above theorem, we assume that (2.1) and (2.2) hold on f. Clearly, when f is increasing and (2.2) holds, (2.1) holds automatically.

EXAMPLE 1. Consider the following equation

$$x'(t) + a(t)x(t) + b(t)x(t-\tau(t))\left(1 - \frac{1}{2}\sin x(t-\tau(t))\right) = r(t),\ t \geqslant 0. \tag{2.21}$$

Let $f(x) = x(1 - \frac{1}{2}\sin x)$. Clearly, $f(x)$ is not an increasing function, but $x \to 0$ as $f(x) \to 0$. In addition, $|f(x)| \leqslant \frac{3}{2}|x|$ and $xf(x) > 0$ for $x \neq 0$. Hence, by Theorem 1, if (2.3)- (2.5) are satisfied with $\beta = 3/2$, then every solution of Eq. (2.21) tends to zero as $t \to \infty$.

3. Some Special Cases and Applications

In this section, we consider some special cases of Eq. and their applications. When $a(t) \equiv 0$, Eq. (1.1) reduces to

$$x'(t) + b(t)f(x(t-\tau(t))) = r(t),\ t \geqslant 0. \tag{3.1}$$

The following result is a direct consequence of Theorem 1.

COROLLARY 1. *Assume that* (2.1) *holds, and there is a positive constant* β *such that*

$$|f(x)| \leqslant \beta|x|, \text{ and } xf(x) > 0 \text{ for } x \neq 0. \tag{3.2}$$

Suppose also that

$$\int_0^\infty b(t)dt = \infty,\ \ \beta \limsup_{t\to\infty} \int_{t-\tau(t)}^t b(u)du < 1 \tag{3.3}$$

and

$$\lim_{t\to\infty} \frac{r(t)}{b(t)} = 0. \tag{3.4}$$

Then every solution $x(t)$ *of Eq.* (3.1) *tends to zero as* $t \to \infty$.

The global convergence of solutions of Eq.(3.1) with $\tau(t) \equiv \tau$ a positive constant, has been studied in [17] and the following result has been obtained.

Theorem A *Assume that*

$$\int_0^\infty b(t)dt = \infty.$$

Then every solution $x(t)$ *of Eq.(3.1) converges to zero as* $t \to \infty$ *if either*

$$xf(x) > 0, \ |f(x)| < |x| \ for \ x \neq 0$$

and

$$\sup_{t \geqslant 0} \int_{t-\tau}^t b(s)ds = \mu \leqslant 3/2, \int_0^\infty t|r(t)|dt < \infty,$$

or there is a number $\alpha \in (0,1)$ *such that*

$$xf(x) > 0, \ |f(x)| < \alpha|x| \ for \ x \neq 0$$

and

$$\sup_{t \geqslant 0} \int_{t-\tau}^t b(s)ds = \mu \leqslant 3/2, \int_0^\infty |r(t)|dt < \infty.$$

Clearly, Corollary 1 is different from Theorem A. Indeed, when $b(t)$ is a positive constant function, (3.4) becomes

$$\lim_{t \to \infty} r(t) = 0,$$

which is different from either $\int_0^\infty t|r(t)|dt < \infty$ or $\int_0^\infty |r(t)|dt < \infty$ assumed in Theorem A.

When $r(t) \equiv 0$, Eq.(1.1) reduces to

$$x'(t) + a(t)x(t) + b(t)f(x(t - \tau(t))) = 0, \ t \geqslant 0. \tag{3.5}$$

The following result is a direct consequence of Theorem 1.

COROLLARY 2. *Assume that* (2.1) *holds, and there is a positive number* β *such that* (3.2) *holds. Suppose also that*

$$\int_0^\infty e^{\int_0^t a(s)ds} b(t)dt = \infty \tag{3.6}$$

and

$$\beta \limsup_{t \to \infty} \int_{t-\tau(t)}^t e^{-\int_u^t a(s)ds} b(u)du < 1. \tag{3.7}$$

Then every solution of Eq.(3.5) *tends to zero as* $t \to \infty$.

When $f(x) = x$, Eq.(3.5) reduces to the linear equation

$$x'(t) + a(t)x(t) + b(t)x(t-\tau(t)) = 0,\ t \geqslant 0. \tag{3.8}$$

The following conclusion comes from Corollary 2 directly.

COROLLARY 3. *Assume that* (3.6) *and* (3.7) *hold. Then every solution* $x(t)$ *of Eq.*(3.8) *tends to zero as* $t \to \infty$ *and so the equation is asymptotically stable.*

Asymptotic stability of the linear equation (3.8) has been studied by numerous authors, see for example, [1, 3] and the references cited therein. Note that the following result, which is different from Corollary 3, was obtained in [1] recently.

Theorem B *Assume that* $a(t) \geqslant 0, b(t) \geqslant 0$ *and* $a(t) + b(t) > 0$*,*

$$\int_0^\infty (a(t)+b(t))dt = \infty,\ \limsup_{t\to\infty} \int_{t-\tau(t)}^t (a(s)+b(s))ds < \infty$$

and

$$\limsup_{t\to\infty} \frac{b(t)}{a(t)+b(t)} \int_{t-\tau(t)}^t (a(s)+b(s))ds < 1 + \frac{1}{e}.$$

Then Eq.(3.8) *is asymptotically stable.*

In the following, we show the applications of our results to some equations derived from mathematical biology. As we mentioned in Section 1, many equations derived from mathematical biology have the form

$$y'(t) = p(t)[g(y(t-\tau(t))) - y(t)],\ t \geqslant 0 \tag{3.9}$$

where $p, \tau \in C[[0,\infty),(0,\infty)]$ and $g \in C[(0,\infty),(0,\infty)]$. When a perturbation term $r(t)$ is added, Eq.(3.9) becomes

$$y'(t) = p(t)[g(y(t-\tau(t))) - y(t)] + r(t),\ t \geqslant 0 \tag{3.10}$$

where $r \in C[[0,\infty),\mathbb{R}]$. By Theorem 1, we have the following result.

THEOREM 2. *Let* $\bar{y}$ *be a positive fixed point of* g*. Assume that* $x \to 0$ *as* $g(\bar{y}) - g(x+\bar{y}) \to 0$*, and there is a positive number* β *such that*

$$|g(\bar{y}) - g(x+\bar{y})| \leqslant \beta|x| \text{ and } x(g(\bar{y}) - g(x+\bar{y})) > 0 \text{ for } x > -\bar{y},\ \ x \neq 0. \tag{3.11}$$

Suppose also

$$\int_0^\infty p(t)dt = \infty, \tag{3.12}$$

$$\beta \limsup_{t\to\infty} \left(1 - e^{-\int_{t-\tau(t)}^t p(s)ds}\right) < 1, \tag{3.13}$$

and

$$\lim_{t\to\infty}\frac{r(t)}{p(t)}=0. \tag{3.14}$$

Then every positive solution $y(t)$ of Eq.(3.10) tends to $\bar{y}$ as $t\to\infty$.

Proof. Let $x(t)=y(t)-\bar{y}$. Then Eq.(3.10) can be written as

$$x'(t)+p(t)x(t)+p(t)[g(\bar{y})-g(x(t-\tau(t))+\bar{y})]=r(t),\ t\geqslant 0. \tag{3.15}$$

which is in the form (1.1) with $a(t)=b(t)=p(t)$ and $f(x)=g(\bar{y})-g(x+\bar{y})$, $x>-\bar{y}$. Clearly, f satisfies the conditions assumed in Theorem 1. By noting

$$\int_0^\infty e^{\int_0^t a(s)ds}b(t)dt=\int_0^\infty e^{\int_0^t p(s)ds}p(t)dt=e^{\int_0^\infty p(t)dt}-1$$

and

$$\int_{t-\tau(t)}^t e^{-\int_u^t a(s)ds}b(u)du=\int_{t-\tau(t)}^t e^{-\int_u^t p(s)ds}p(u)du=1-e^{-\int_{t-\tau(t)}^t p(s)ds}$$

we see that all the conditions assumed in Theorem 1 are satisfied. Hence, every solution $x(t)$ of Eq.(3.15) tends to zero as $t\to\infty$. Then it follows that every solution $y(t)$ of Eq.(3.10) tends to $\bar{y}$ as $t\to\infty$. The proof is complete.

In the following, we consider some specific equation models derived from mathematical biology. First, consider the equation

$$y'(t)=q(t)\left[-\alpha y(t)+\frac{c}{1+y^\gamma(t-\tau(t))}\right]+r(t),\ t\geqslant 0 \tag{3.16}$$

where α,γ and c are positive numbers and $q\in C[[0,\infty),(0,\infty)]$. When $r(t)\equiv 0$, asymptotic behavior of positive solutions of this equation has been studied in [2] recently. Rewrite Eq.(3.16) as

$$y'(t)=\alpha q(t)\left[\frac{c/\alpha}{1+y^\gamma(t-\tau(t))}-y(t)\right]+r(t),\ t\geqslant 0$$

which is in the form of (3.9) with $p(t)=\alpha q(t)$ and $g(y)=\frac{c/\alpha}{1+y^\gamma}$, $y>0$. Clearly, g is decreasing and has a unique positive fixed point $\bar{y}$. Let

$$f(x)=g(\bar{y})-g(x+\bar{y}),\ x>-\bar{y}.$$

When $\gamma\leqslant 1$, it has been shown in [2] that

$$0<\frac{f(x)}{x}\leqslant\frac{c\gamma\bar{y}^{\gamma-1}}{\alpha(1+\bar{y}^\gamma)^2}=\frac{\gamma\bar{y}^\gamma}{1+\bar{y}^\gamma}<1.$$

Hence, (3.11) and (3.13) hold with $\beta = \frac{\gamma \bar{y}^\gamma}{1+\bar{y}^\gamma}$. When $\gamma > 1$, observing

$$f'(x) = \frac{c\gamma(x+\bar{y})^{\gamma-1}}{\alpha(1+(x+\bar{y})^\gamma)^2} \text{ and } f''(x) = \frac{c\gamma(x+\bar{y})^{\gamma-2}((\gamma-1)-(\gamma+1)(x+\bar{y})^\gamma)}{\alpha(1+(x+\bar{y})^\gamma)^3},$$

we see that

$$0 < f'(x) \leqslant f'\left(\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}\right) = \frac{c}{4\alpha\gamma}(\gamma-1)^{1-1/\gamma}(1+\gamma)^{1+1/\gamma}.$$

Hence, (3.14) holds with $\beta = \frac{c}{4\alpha\gamma}(\gamma-1)^{1-1/\gamma}(1+\gamma)^{1+1/\gamma}$. Then by Theorem 2, we have the following conclusion.

COROLLARY 4. *Let $\bar{y}$ be the unique positive fixed point of the function $g(y) = \frac{c/\alpha}{1+y^\gamma}$ in Eq.* (3.16) *and assume that*

$$\int_0^\infty q(t)dt = \infty \tag{3.17}$$

and

$$\lim_{t\to\infty} \frac{r(t)}{q(t)} = 0.$$

Then every positive solution of Eq.(3.16) *tends to $\bar{y}$ as $t \to 0$ if either*

$$0 < \gamma \leqslant 1, \tag{3.18}$$

or

$$\gamma > 1 \text{ and } \frac{c}{4\alpha\gamma}(\gamma-1)^{1-\frac{1}{\gamma}}(1+\gamma)^{1+\frac{1}{\gamma}} \limsup_{t\to\infty}\left(1-e^{-\alpha\int_{t-\tau(t)}^t q(s)ds}\right) < 1. \tag{3.19}$$

When $r(t) \equiv 0$, Eq.(3.16) reduces to

$$y'(t) = q(t)\left[-\alpha y(t) + \frac{c}{1+y^\gamma(t-\tau(t))}\right],\ t \geqslant 0. \tag{3.20}$$

In particular, when $q(t) \equiv 1$ and $\tau(t) \equiv \tau$ a positive constant, Eq.(3.20) becomes

$$y'(t) = -\alpha y(t) + \frac{c}{1+y^\gamma(t-\tau)},\ t \geqslant 0. \tag{3.21}$$

Eq.(3.21) was proposed by Mackey and Glass [13] as a model of haematopoiesis (blood cell production) and the global behavior of positive solutions of this equation has been studied by numerous authors.

The following result is a direct consequence of Corollary 4.

COROLLARY 5. *Assume that* (3.17) *holds, and that either* (3.18) *or* (3.19) *holds. Then every positive solution of Eq.*(3.20) *tends to the positive equilibrium as* $t \to \infty$.

The following result on the global convergence of positive solutions of Eq.(3.20) has been obtained in [2].

Theorem C *Assume that one of the following holds:*

(a) $0 < \gamma \leqslant 1$;

(b) $\gamma > 1$ *and* $c\rho \limsup_{t\to\infty} \int_{t-\tau(t)}^{t} q(s)ds < 1 + \frac{1}{e}$;

(c) $\gamma > 1$ *and* $\frac{c\rho}{\alpha} < 1$

where $\rho = \frac{1}{4\gamma}(\gamma-1)^{1-1/\gamma}(1+\gamma)^{1+1/\gamma}$. *Then every positive solution of Eq.*(3.20) *tends to the positive equilibrium as* $t \to \infty$.

Comparing Corollary 5 and Theorem C, we see that (3.19) is an improvement of (c).

Next, consider the equation

$$y'(t) = q(t)[-\alpha y(t) + ce^{-\gamma y(t-\tau(t))}] + r(t),\ t \geqslant 0 \tag{3.22}$$

where α, γ and c are positive numbers, and $g \in C[[0,\infty),(0,\infty)]$. When $r(t) \equiv 0, q(t) \equiv 1$ and $\tau(t) \equiv \tau$ a positive constant, Eq.(3.22) reduces to

$$y'(t) = -\alpha y(t) + ce^{-\gamma y(t-\tau)},\ t \geqslant 0 \tag{3.23}$$

which was used by Wazewska-Czyzewska and Lasota [18] as a model for the survival of red blood cells in an animal.

Rewrite Eq.(3.22) as

$$y'(t) = \alpha q(t)\left[\frac{c}{\alpha}e^{-\gamma y(t-\tau(t))} - y(t)\right] + r(t),\ t \geqslant 0 \tag{3.24}$$

which is in the form of (3.10) with $p(t) = \alpha q(t)$ and $g(y) = \frac{c}{\alpha}e^{-\gamma y}, y > 0$. Clearly, g is decreasing and has a unique positive fixed point $\bar{y}$. Let

$$f(x) = g(\bar{y}) - g(x+\bar{y}),\ x > -\bar{y}.$$

By noting

$$0 < f'(x) = \frac{c}{\alpha}\gamma e^{-\gamma(x+\bar{y})} \leqslant \frac{c}{\alpha}\gamma$$

we see that f is increasing,

$$|f(x)| \leqslant \frac{c}{\alpha}\gamma|x|,\ \text{and}\ xf(x) > 0 \text{ for } x \neq 0.$$

Hence, by Theorem 2, we have the following conclusion.

COROLLARY 6. *Let* $\bar{y}$ *be the unique positive fixed point of the function* $g(y) = \frac{c}{\alpha}e^{-\gamma y}$ *in Eq.*(3.22) *and assume that*

$$\int_0^{\infty} q(t)dt = \infty,$$

$$\frac{c\gamma}{\alpha}\limsup_{t\to\infty}\left(1-e^{-\alpha\int_{t-\tau(t)}^{t}q(s)ds}\right)<1$$

and

$$\lim_{t\to\infty}\frac{r(t)}{q(t)}=0.$$

Then every positive solution of Eq.(3.22) tends to $\bar{y}$ *as* $t\to\infty$.

REMARK 2. As direct consequences of Corollaries 5 and 6, we see that if

$$\text{either } 0<\gamma\leqslant 1, \text{ or } \gamma>1 \text{ and } \frac{c}{4\alpha\gamma}(\gamma-1)^{1-\frac{1}{\gamma}}(1+\gamma)^{1+\frac{1}{\gamma}}\left(1-e^{-\alpha\tau}\right)<1,$$

then every positive solution of Eq.(3.21) converges to its positive equilibrium; if

$$\frac{c\gamma}{\alpha}(1-e^{-\alpha\tau})<1$$

then every positive solution of Eq.(3.23) converges to its positive equilibrium. These two conditions have been obtained in [14] and [16] by using different approaches. □

Finally, consider the equation

$$\frac{y'(t)}{y(t)}=p(t)\frac{K-y(t-\tau(t))}{K+dy(t-\tau(t))}+r(t),\ t\geqslant 0. \tag{3.25}$$

When $p(t)\equiv p$ and $\tau(t)\equiv\tau$ are positive constants and $d=cp$, Eq.(3.25) reduces to

$$\frac{y'(t)}{y(t)}=p\frac{K-y(t-\tau)}{K+cpy(t-\tau)}+r(t),\ t\geqslant 0. \tag{3.26}$$

In particular, when $r(t)\equiv 0$, Eq.(3.26) was proposed by Gopalsamy et al. [7] as a "food-limited" population model which is a generalization of the well-known delay logistic equation. The global convergence of positive solutions of Eq.(3.26) has been studied in [17] and the following result has been obtained.

Theorem D *Assume that either*

$$p\tau=3/2 \text{ and } \int_0^\infty s|r(s)|ds<\infty$$

or

$$p\tau<3/2 \text{ and } \int_0^\infty |r(s)|ds<\infty$$

holds, and $cp>1/3$. *Then every positive solution of Eq.(3.26) goes to* K *as* $t\to\infty$.

Now, by using Corollary 1, we may get a different condition for K to be a global attractor. Consider Eq.(3.25) and let $y(t)$ be a positive solution. The change of variable $y(t)=Ke^{x(t)}$ transforms Eq.(3.25) to the equation

$$x'(t)+p(t)\frac{e^{x(t-\tau(t))}-1}{1+de^{x(t-\tau(t))}}=r(t),\ t\geqslant 0, \tag{3.27}$$

which is in the form (3.1). To show that $y(t) \to K$ as $t \to \infty$, it suffices to show that $x(t) \to 0$ as $t \to \infty$. Let

$$f(x) = \frac{e^x - 1}{1 + de^x}.$$

By noting

$$f'(x) = \frac{(1+d)e^x}{(1+de^x)^2} \quad \text{and} \quad f''(x) = (1+d)\frac{e^x(1-de^x)}{(1+de^x)^3},$$

we see that

$$0 < f'(x) \leqslant f'\left(\ln\frac{1}{d}\right) = \frac{1+d}{4d}.$$

Hence, f is increasing, $xf(x) > 0$ for $x \neq 0$ and $|f(x)| \leqslant \frac{1+d}{4d}|x|$. The following conclusion follows from Corollary 1 immediately.

COROLLARY 7.

$$\int_0^\infty p(t)dt = \infty,$$

$$\frac{1+d}{4d}\limsup_{t\to\infty}\int_{t-\tau(t)}^t p(u)du < 1$$

and

$$\lim_{t\to\infty}\frac{r(t)}{p(t)} = 0.$$

Then every positive solution of Eq.(3.25) tends to K as $t \to \infty$.

For Eq.(3.26), we have the following result coming from Corollary 7 directly.

COROLLARY 8. *Assume that*

$$\frac{1+cp}{4c}\tau < 1 \textit{ and } \lim_{t\to\infty} r(t) = 0.$$

Then every positive solution of Eq.(3.26) tends to K as $t \to \infty$.

Clearly, Corollary 8 is different from Theorem D. In particular, $cp > 1/3$ is not required in Corollary 8.

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