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CONSTRUCTIVE EXISTENCE RESULTS FOR SOLUTIONS TO SYSTEMS OF BOUNDARY VALUE PROBLEMS VIA GENERAL LYAPUNOV METHODS

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Abstract. In this work we consider boundary value problems (BVPs) for systems of second-order, ordinary differential equations. *A priori* bounds on solutions are obtained via differential inequalities involving general Lyapunov functions without the need for maximum principles. These bounds are then applied to produce new existence theorems via topological methods. Some constructive results are also developed via A-proper mappings and the Galerkin method, in which solutions to the BVP may be approximated.

1. Introduction

Let $d \in \mathbb{N}$ and consider the system of boundary value problems (BVPs)

$$x'' = f(t,x,x'), \qquad t \in [0,T], \tag{1.1}$$

$$x(0) = A, \quad x(T) = B, \tag{1.2}$$

where $f : [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}^d$ is continuous and $A, B \in \mathbb{R}^d$ with $T > 0$.

Authors such as [1, 2, 3, 4, 5, 6, 7, 9, 12] have studied the existence of solutions to (1.1), (1.2) by employing Lyapunov functions $V(t,x)$ and maximum principles to extract *a priori* bounds on solutions x to (1.1), (1.2). The *a priori* bounds were then applied to obtain existence theorems for solutions to (1.1), (1.2). More recently, [13] analyzed two-point BVPs to establish a firm theorectical foundation for the nonrelativistic Thomas-Fermi equation for heavy atoms in intense magnetic fields.

Motivated by the above works, in this paper we appeal to alternate Lyapunov methods to obtain *a priori* bounds on solutions x to (1.1), (1.2). In particular, the *a priori* bound methods herein do not rely on maximum principles and, in some cases, are less technical and wider ranging than those in the above papers. The *a priori* bound results are then applied to obtain new existence results for solutions to (1.1), (1.2). The method for existence involves an application of Schauder's fixed point theorem [8]. Finally, we

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formulate some "constructive existence" results in which we may approximate solutions to (1.1), (1.2). This constructive theory is based on A-proper mappings and the Galerkin method which is a foundation of modern finite element computation [10].

In what follows if p is a d-dimensional vector then $\|p\|$ represents the Euclidean norm and $\langle\cdot,\cdot\rangle$ is the scalar product with $\langle p,p\rangle = \|p\|^2$.

A solution $x \in C^2([0,T];\mathbb{R}^d)$ to (1.1), (1.2) is a vector-valued function $x = x(t)$ that satisfies (1.1) for all $t \in [0,T]$ and also satisfies (1.2).

Let x be a solution to (1.1), (1.2). Following [4, Hartman], we will call $V \in C^2([0,T]\times\mathbb{R}^d;[0,\infty))$ a Lyapunov function and will denote its first and second derivatives by $\dot{V}(t,x)$ and $\ddot{V}(t,x)$ with each defined by,

$$\begin{aligned}\dot{V}(t,x) &:= V_t(t,x) + \langle V_x(t,x), x'\rangle \\ \ddot{V}(t,x) &:= V_{tt}(t,x) + 2\langle V_{tx}(t,x), x'\rangle + \langle V_{xx}(t,x), x'\rangle + \langle V_x(t,x), x''\rangle \end{aligned} \tag{1.3}$$

where $V_t = \partial V/\partial t$ and $V_x = \partial V/\partial x$ etc.

2. *A priori* bounds

In this section we formulate sufficient conditions under which solutions $x = x(t)$ to (1.1), (1.2) satisfy *a priori* bounds of the type $\|x(t)\| \leqslant M, \|x'(t)\| \leqslant N$ for all $t \in [0,T]$ for some constants M,N and P. These bounds are roughly sufficient in order to prove the existence of solutions via topological methods.

THEOREM 1. *Let $f : [0,T]\times\mathbb{R}^{2d} \to \mathbb{R}^d$ be continuous and let there exist a Lyapunov function V such that*

$$\|f(t,p,q)\| \leqslant \alpha\,[V_{tt}(t,p) + 2\langle V_{tp}(t,p),q\rangle + \langle V_{pp}(t,p),q\rangle + \langle V_p(t,p), f(t,p,q)\rangle] + K, \quad \text{for all } t \in [0,T], (p,q) \in \mathbb{R}^{2d}, \tag{2.1}$$

for some non-negative constants α, K. Then, for all $t \in [0,T]$, all solutions x to (1.1), (1.2) *satisfy*

$$\begin{aligned}\|x(t)\| &\leqslant M \\ &:= \alpha[V(0,A) + V(T,B)] + KT^2/8 + \max\{\|A\|,\|B\|\}.\end{aligned}$$

Proof. Let $x = x(t)$ be a solution to (1.1), (1.2).

The BVP (1.1), (1.2) has the equivalent integral representation [6, Hartman]

$$x(t) = -\int_0^T G(t,s) f(s,x(s),x'(s))\,ds + \phi(t), \quad t \in [0,T], \tag{2.2}$$

where

$$G(t,s) := \begin{cases} (T-t)s/T, & \text{for } 0 \leqslant s \leqslant t \leqslant T, \\ t(T-s)/T, & \text{for } 0 \leqslant t \leqslant s \leqslant T, \end{cases}$$

and

$$\phi(t) := [TA + (B-A)t]/T.$$

Now taking norms in (2.2) and using $G \geqslant 0$ with (2.1) and (1.3), we have

$$\begin{aligned}
\|x(t)\| &\leqslant \int_0^T G(t,s)\|f(s,x(s),x'(s))\|\,ds + \max\{\|A\|,\|B\|\} \\
&\leqslant \int_0^T G(t,s)[\alpha\ddot{V}(s,x(s)) + K]\,ds + \max\{\|A\|,\|B\|\} \\
&= \alpha\int_0^T G(t,s)\ddot{V}(s,x(s))\,ds + K\int_0^T G(t,s)\,ds + \max\{\|A\|,\|B\|\} \\
&\leqslant \alpha\int_0^T G(t,s)\ddot{V}(s,x(s))\,ds + KT^2/8 + \max\{\|A\|,\|B\|\} \\
&= \alpha\frac{(T-t)}{T}\int_0^t s\ddot{V}(s,x(s))\,ds + \frac{t\alpha}{T}\int_t^T (T-s)\ddot{V}(s,x(s))\,ds \\
&\quad + KT^2/8 + \max\{\|A\|,\|B\|\}
\end{aligned}$$

where we have used the well-known inequality

$$\int_0^T G(t,s)\,ds \leqslant T^2/8, \quad \text{for all } t \in [0,T].$$

Now integrating by parts we obtain

$$\begin{aligned}
I_1(t) &:= \frac{\alpha(T-t)}{T}\int_0^t s\ddot{V}(s,x(s))\,ds \\
&\leqslant \frac{\alpha(T-t)}{T}\{t\dot{V}(t,x(t)) - V(t,x(t)) + V(0,x(0))\} \\
&\leqslant \frac{\alpha(T-t)}{T}\{t\dot{V}(t,x(t)) + V(0,A)\}.
\end{aligned}$$

Similarly, integrating by parts we get

$$\begin{aligned}
I_2(t) &:= \frac{t\alpha}{T}\int_t^T (T-s)\ddot{V}(s,x(s))ds \\
&\leqslant \frac{-\alpha t}{T}\{(T-t)\dot{V}(t,x(t)) - V(T,B)\}.
\end{aligned}$$

So for all $t \in [0,T]$ we have

$$\begin{aligned}
\|x(t)\| &\leqslant I_1(t) + I_2(t) + KT^2/8 + \max\{\|A\|,\|B\|\}, \\
&\leqslant \alpha[V(0,A) + V(T,B)] + KT^2/8 + \max\{\|A\|,\|B\|\}.
\end{aligned}$$

This concludes the proof. $\square$

We note that in the case $V(t,x) = \|x\|^2$ then the proof of Theorem 1 is identical to [6, Hartman].

THEOREM 2. *Let* $f : [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}^d$ *be continuous and let there exist a Lyapunov function* V *such that*

$$\|f(t,p,q)\| \leqslant \alpha\,[V_{tt}(t,p) + 2\langle V_{tp}(t,p),q\rangle + \langle V_{pp}(t,p),q\rangle + \langle V_p(t,p), f(t,p,q)\rangle] + K, \quad \text{for all } t \in [0,T],\ \|p\| \leqslant M,\ q \in \mathbb{R}^d, \tag{2.3}$$

for some non-negative constants α, K *and* M. *If*

$$\max\{\alpha\|V_p(t,p)\| : \quad t \in [0,T],\ \|p\| \leqslant M\} < 1, \tag{2.4}$$

then all solutions x *to* (1.1), (1.2) *that satisfy* $\|x(t)\| \leqslant M$ *for all* $t \in [0,T]$ *also satisfy*

$$\|x'(t)\| \leqslant N := \frac{\alpha\left[T \max_{\substack{t\in[0,T]\\ \|p\|\leqslant M}} V_t(t,p) + V(0,A) + V(T,B)\right] + KT^2/2 + \|B-A\|}{T\left(1 - \max_{\substack{t\in[0,T]\\ \|p\|\leqslant M}} \alpha\|V_p(t,p)\|\right)}.$$

Proof. Let x be a solution to (1.1), (1.2) with $\|x(t)\| \leqslant M$ for all $t \in [0,T]$. From (2.2) we have

$$x'(t) = \int_0^t \frac{s}{T} f(s,x(s),x'(s))\,ds - \int_t^T \frac{T-s}{T} f(s,x(s),x'(s))\,ds + \phi'(t).$$

Thus,

$$\begin{aligned}\|x'(t)\| &\leqslant \int_0^t \frac{s}{T}\|f(s,x(s),x'(s))\|\,ds + \int_t^T \frac{T-s}{T}\|f(s,x(s),x'(s))\|\,ds + \frac{\|B-A\|}{T}\\ &\leqslant \int_0^t \frac{s}{T}[\alpha\ddot{V}(s,x(s)) + K]\,ds + \int_t^T \frac{T-s}{T}[\alpha\ddot{V}(s,x(s)) + K]\,ds + \frac{\|B-A\|}{T}.\end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned}\|x'(t)\| &\leqslant \alpha\left[\dot{V}(t,x(t)) + \frac{V(0,A)+V(T,B)}{T}\right] + \frac{KT}{2} + \frac{\|B-A\|}{T}\\ &= \alpha\left[V_t(t,x(t)) + \langle V_x(t,x(t)),x'(t)\rangle + \frac{V(0,A)+V(T,B)}{T}\right] + \frac{KT}{2} + \frac{\|B-A\|}{T}\\ &\leqslant \alpha\left[V_t(t,x(t)) + \|V_x(t,x(t))\|\|x'(t)\| + \frac{V(0,A)+V(T,B)}{T}\right] + \frac{KT}{2} + \frac{\|B-A\|}{T}.\end{aligned}$$

Hence a rearrangement gives $\|x'(t)\| \leqslant N$ for all $t \in [0,T]$. This concludes the proof. □

We now have the *a priori* bounds on solutions that enable us to develop theory for the existence of solutions, but before we progress, let us discuss an example. Inequalities (2.1) and (2.3) are identical, but it is important to note that the sets on which these assumptions hold are different. As such, we now present an example ($d = 2$) where (2.3) holds for $M = 1$.

EXAMPLE 1. Consider the vector-valued f

$$\begin{pmatrix} f_1(t,p,q) \\ f_2(t,p,q) \end{pmatrix} := \begin{pmatrix} p_1^3 + p_1 q_2^2 \\ p_2^3 + p_2 q_1^2 \end{pmatrix}.$$

We claim that this f satsifies (2.3) for $M = 1$ and

$$V(t,x) := \|x\|^2 + 2\int_0^t \int_0^u \|x(s)\|^2 \, ds \, du.$$

Proof. We have

$$\begin{aligned} \dot{V}(t,x) &:= 2\langle x, x' \rangle + 2\int_0^t \|x(s)\|^2 \, ds \\ \ddot{V}(t,x) &:= 2[\langle x, x'' \rangle + \|x'\|^2 + \|x\|^2]. \end{aligned}$$

Let x be a solution to our BVP with our given f that satisfies $\|x(t)\| \leqslant 1$ for all $t \in [0,1]$. If we choose $\alpha = 1/2$ and $K = 4$ then consider, for all $t \in [0,T]$, $\|p\| \leqslant 1$, $q \in \mathbb{R}^2$

$$\begin{aligned} &\alpha[V_{tt}(t,p) + 2\langle V_{tp}(t,p), q\rangle + \langle V_{pp}(t,p), q\rangle + \langle V_p(t,p), f(t,p,q)\rangle] + K \\ &= \langle p, f(t,p,q)\rangle + \|q\|^2 + \|p\|^2 + 4 \\ &= p_1^4 + p_1^2 q_2^2 + p_2^4 + p_2^2 q_1^2 + q_1^2 + q_2^2 + p_1^2 + p_2^2 + 4 \\ &\geqslant q_1^2 + q_2^2 + 4. \end{aligned}$$

On the same set, consider

$$\begin{aligned} \|f(t,p,q)\| &= ((p_1^3 + p_1 q_2^2)^2 + (p_2^3 + p_2 q_1^2)^2)^{1/2} \\ &\leqslant |p_1^3 + p_1 q_2^2| + |p_2^3 + p_2 q_1^2| \\ &\leqslant q_1^2 + q_2^2 + 2 \end{aligned}$$

since $|p_1| \leqslant 1$ and $|p_2| \leqslant 1$.

Thus we see that (2.3) holds. □

In the theory of BVPs for second-order differential equations, it is common to bound solutions x and x' through separate approaches. Thus, if we know that solutions x are bounded, then Theorem 2 may be applied to obtain a bound on x'. This shows the significance of Theorem 2 over Theorem 1.

3. Existence Theorems

In this section we apply our results from Section 2 and formulate sufficient conditions under which the system of BVPs (1.1), (1.2) has at least one solution.

THEOREM 3. *Let the conditions of Theorem 1 and (2.4) hold (with M being supplied by Theorem 1). If*

$$V_{tt}(t,p)+2\langle V_{tp}(t,p),q\rangle+\langle V_{pp}(t,p)q,q\rangle \geqslant 0, \quad \text{for all } t\in[0,T],\ (p,q)\in\mathbb{R}^{2d}, \tag{3.1}$$

then the BVP (1.1), (1.2) has at least one solution $x\in C^2([0,T];\mathbb{R}^d)$.

Proof. Define the operator $J: C^1([0,T];\mathbb{R}^d)\to C([0,T];\mathbb{R}^d)$ by

$$[Jx](t) = -\int_0^T G(t,s)f(s,x(s),x'(s))\,ds+\phi(t), \quad t\in[0,T].$$

We see that showing there is at least one $x\in C^1([0,T];\mathbb{R}^d)$ such that $Jx=x$ is equivalent to showing that BVP (1.1), (1.2) has at least one solution $x\in C^2([0,T];\mathbb{R}^d)$.

Now choose a convex, bounded and open set $\Omega\subset C^1([0,T];\mathbb{R}^d)$ defined by

$$\Omega := \{x\in C^1([0,T];\mathbb{R}^d) : \|x\|_1 := \max\{\max_{t\in[0,T]}\|x(t)\|,\ \max_{t\in[0,T]}\|x'(t)\|\}<R\}$$

where

$$R:=\max\{M,N\}+1,$$

and M and N are given in Theorems 1 and 2, respectively.

We show $J: C^1([0,T];\mathbb{R}^d)\to\bar{\Omega}$ and thus $J:\bar{\Omega}\to\bar{\Omega}$. We show that for all $x\in C^1([0,T];\mathbb{R}^d)$ we have $\|Jx\|_1\leqslant R$.

For $x\in C^1([0,T];\mathbb{R}^d)$ consider, for all $t\in[0,T]$,

$$\begin{aligned}\|[Jx](t)\| &\leqslant \int_0^T G(t,s)\|f(s,x(s),x'(s))\|\,ds+\|\phi(t)\|\\ &\leqslant M\end{aligned}$$

where we have followed the same argument as in the proof of Theorem 1.

In addition, for $x\in C^1([0,T];\mathbb{R}^d)$ consider, for all $t\in[0,T]$,

$$\begin{aligned}\|[Jx]'(t)\| &\leqslant \int_0^t \frac{s}{T}\|f(s,x(s),x'(s))\|\,ds+\int_t^T\frac{T-s}{T}\|f(s,x(s),x'(s))\|\,ds+\frac{\|B-A\|}{T}\\ &\leqslant N\end{aligned}$$

where we have applied the same argument as in the proof of Theorem 2.

Thus, we see

$$\|Jx\|_1\leqslant\max\{M,N\}<R.$$

Now $J:\bar{\Omega}\to\bar{\Omega}$ is a compact map by the Arzela-Ascoli Theorem.

We have shown all of the conditions of Schauder's fixed point theorem are satisfied and so we conclude that is at least one $x \in C^1([0,T];\mathbb{R}^d)$ such that $Jx = x$ is equivalent to showing that BVP (1.1), (1.2) has at least one solution $x \in C^2([0,T];\mathbb{R}^d)$.

This concludes the proof. $\square$

Some corollaries to Theorem 3 are now presented. Most of the proofs are omitted for brevity as they follow very similar lines to the proof of Theorem 3. For the case $V(t,x) = V(x)$, then Theorem 3 reduces to the following.

COROLLARY 1. *Let $f : [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}^d$ be continuous and let there exist a Lyapunov function $V \in C^2(\mathbb{R}^d;[0,\infty))$ with V satisfying:*

$$\|f(t,p,q)\| \leqslant \alpha[\langle V_{pp}(p)q,q\rangle + \langle V_p(p), f(t,p,q)\rangle] + K, \textit{ for all } t \in [0,T], (p,q) \in \mathbb{R}^{2d},$$

and for some non-negative constants α, K with

$$\langle V_{pp}(p)q,q\rangle \geqslant 0, \textit{ for all } t \in [0,T],\ (p,q) \in \mathbb{R}^{2d}.$$

If

$$\max\{\alpha\|V_p(p)\| : t \in [0,T],\ \|p\| \leqslant M\} < 1,$$

(where M is the constant supplied by Theorem 1), then the BVP (1.1), (1.2) has at least one solution $x \in C^2([0,T];\mathbb{R}^d)$.

For the case $V(t,p) = \langle p, Q(t)p\rangle$ where $Q(t)$ is a given symmetric positive definite $d \times d$ matrix which has elements in $C^2([0,T])$, define

$$\ddot{V}(t,x) = 2\langle x', Q(t)x'\rangle + 4\langle x', Q'(t)x\rangle + \langle x, Q''(t)x\rangle + 2\langle Q(t)x, x''\rangle.$$

For this case, Theorem 3 reduces to the following.

COROLLARY 2. *Let $f : [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}^d$ be continuous and let there exist a symmetric positive definite $d \times d$ matrix $Q(t)$ which has elements in $C^2([0,T])$. If Q satisfies:*

$$\|f(t,p,q)\| \leqslant \alpha[2\langle q, Q(t)q\rangle + 4\langle q, Q'(t)p\rangle + \langle p, Q''(t)p\rangle + 2\langle Q(t)p, f(t,p,q)\rangle] + K,$$
$$2\langle q, Q(t)q\rangle + 4\langle q, Q'(t)p\rangle + \langle p, Q''(t)p\rangle \geqslant 0, \textit{ for all } t \in [0,T], (p,q) \in \mathbb{R}^{2d}$$

for some non-negative constants α, K and

$$2\alpha M \max_{t\in[0,T]} \|Q(t)\| < 1,$$

(where M is the constant supplied by Theorem 1 and the matrix norm is compatible with the Euclidean norm), then the BVP (1.1), (1.2) has at least one solution $x \in C^2([0,T];\mathbb{R}^d)$.

For the case $V(t,x) = \|x\|^2$ we have

$$\ddot{V}(t,x) = \langle x, x''\rangle + \|x'\|^2$$

and Theorem 3 reduces to the following result of Hartman's [6].

COROLLARY 3. *Let f be continuous and satisfy*

$$\|f(t,p,q)\| \leqslant 2\alpha[\langle p, f(t,p,q)\rangle + \|q\|^2] + K, \text{ for all } t \in [0,T], (p,q) \in \mathbb{R}^{2d}$$

and for some non-negative constants α, K, with

$$2\alpha M < 1$$

(where M is the constant supplied by Theorem 1). Then the BVP (1.1), (1.2) has at least one solution $x \in C^2([0,T];\mathbb{R}^d)$.

For the case f is bounded, that is,

$$\|f(t,p,q)\| \leqslant L, \quad \text{for all } t \in [0,T],\ (p,q) \in \mathbb{R}^{2d},$$

for some $L \geqslant 0$, Theorem 3 reduces to the following result of classical importance.

COROLLARY 4. *Let f be continuous and bounded for all $t \in [0,T]$, $(p,q) \in \mathbb{R}^{2d}$. Then the BVP (1.1), (1.2) has at least one solution $x \in C^2([0,T];\mathbb{R}^d)$.*

Proof. Choose $\alpha = 0$ and $K = L$ and the result follows from Theorem 3. This concludes the proof. □

For the scalar case, $d = 1$, we have

$$\ddot{V}(t,x) = V_{tt}(t,x) + 2V_{tx}(t,x)x' + V_{xx}(t,x)x' + V_x(t,x)x''$$

and we can reduce Theorem 3 to the following.

COROLLARY 5. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. If there exists a Lyapunov function $V \in C^2([0,T] \times \mathbb{R})$ such that*

$$|f(t,p,q)| \leqslant \alpha[V_{tt}(t,p) + 2V_{tp}(t,p)q + V_{pp}(t,p)q^2 + V_p(t,p)f(t,p,q)] + K,$$
$$\text{for all } t \in [0,T], (p,q) \in \mathbb{R}^{2d},$$

for some non-negative constants α, K with

$$V_{tt}(t,p) + 2V_{tp}(t,p)q^2 + V_{pp}(t,p)q \geqslant 0$$

and

$$\max\{\alpha|V_x(t,x)| : t \in [0,T], \quad |x| \leqslant M\} < 1, \tag{3.2}$$

(where M is the constant supplied by Theorem 1) then, for the case $d = 1$, the BVP (1.1), (1.2) has a solution $x \in C^2([0,T];\mathbb{R})$.

We conclude this section with an example that illustrates how to apply our ideas.

EXAMPLE 2. Consider the scalar-valued problem

$$x'' = x^3 - x, \quad t \in [0,1]$$
$$x(0) = 1, \quad x(1) = 0.$$

We claim that this problem has at least one solution.

Proof. Define the Lyapunov function V via

$$V(t,x) := x^2 + 2\int_0^t \int_0^u x^2(s)\, ds\, du$$

so that

$$\dot{V}(t,x) := 2xx' + 2\int_0^t x^2(s)\, ds$$

and

$$\ddot{V}(t,x) := 2[xx'' + (x')^2 + x^2].$$

Now, for all $t \in [0,1]$ and $p \in \mathbb{R}$ consider

$$\begin{aligned} \alpha[2(pf(t,p) + q^2 + p^2)] + K &= p^4 + q^2 + 10 \\ &\geqslant |p^3 - p| \\ &= |f(t,p)| \end{aligned}$$

where we have made the choices $\alpha = 1/2$ and $K = 10$. Since f does not depend on x', we do not require the assumption (3.2). All of the conditions of Corollary 5 hold and we conclude that the problem has at least one solution. We note that other approaches may apply to this problem, such as the method of lower and upper solutions. □

4. Constructive Theorems

In this section we will apply the theory of Sections 2 and 3 to obtain some constructive results for solutions to

$$x'' = f(t,x,x'), \qquad t \in [0,T], \tag{4.1}$$
$$x(0) = 0 = x(T). \tag{4.2}$$

We will need the following notation, also to be found in [10]. Let:

X, Y be real Banach spaces;
$L : X \to Y$ be a Fredholm map of index 0 (ie, $L \in \Phi_0(x,y)$);
Null(L) be the null space of L;
Rank (L) be the rank of L; and
$N : X \to Y$ be a nonlinear map.

Let:

$\{X_n\} \subset X$, $\{Y_n\} \subset Y$ be sequences of finite-dimensional oriented spaces;
$Q_n : Y \to Y_n$ be a linear projection for each $n \in \mathbb{Z}^+$.

Define the scheme $\Gamma = \{X_n, Y_n, Q_n\}$ as admissible for maps $X \to Y$ provided:

$$\dim X_n = \dim Y_n, \quad \text{for each } n;$$
$$\operatorname{dist}(x, X_n) := \inf\{\|x - v\|_X : v \in X_n\} \to 0 \text{ as } n \to \infty, \text{ for each } x \in X.$$

For given maps L and N, the equation

$$Lx = Nx, \qquad x \in D(L) \cap D(N),$$

is said to be *strongly [feebly] a-solvable* with respect to Γ if there exists a $N_0 \in \mathbb{Z}^+$ such that the finite dimensional equation

$$Q_n Lx = Q_n Nx, \qquad x \in (D(L) \cap D(N)) \cap X_n,$$

has a solution $x \in (D(L) \cap D(N) \cap X_n$ for each $n \geqslant N_0$ such that

$$x_n \to x \in D(L) \cap D(N) \qquad \text{in } X$$
$$[x_{n_j} \to x \in D(L) \cap D(N)] \qquad \text{and } Lx = Nx.$$

The mapping $L - N : D(L) \cap D(N) \subset X \to Y$ is said to be *A-proper* with respect to Γ if

$$Q_n L - Q_n N : (D(L) \cap D(N)) \cap X_n \subset X_n \to Y_n$$

is continuous for each $n \in \mathbb{Z}^+$ and if $\{x_{n_j} : x_{n_j} \in (D(L) \cap D(N)) \cap X_n\}$ is any bounded sequence in X such that

$$Q_{n_j} Lx_{n_j} - Q_{n_j} Nx_{n_j} \to 0 \qquad \text{in } Y,$$

then there is a subsequence $\{x_{n_k}\}$ of $\{x_{n_j}\}$ and $x \in D(L) \cap D(N)$ such that

$$x_{n_k} \to x \quad \text{in } X \qquad \text{and} \qquad Lx = Nx.$$

Since $L \in \Phi_0(X, Y)$ there exist closed subspaces $X_1 \subset X$ and $Y_2 \subset Y$ such that

$$X = \operatorname{Null}(L) \oplus X_1, \qquad Y = Y_2 \oplus \operatorname{Rank}(L)$$

and $\dim \operatorname{Null}(L) = \dim Y_2$.

Let Q be the linear projection of Y onto Y_2 and assume there exists a continuous bilinear form $[\cdot, \cdot]$ on $Y \times X$ mapping (y, x) into $[y, x]$ such that

$$y \in \operatorname{Rank}(L) \quad \text{iff} \quad [y, x] = 0, \quad \text{for all } x \in \operatorname{Null}(L).$$

THEOREM 4. (Theorem A, [10]) *Let $L \in \Phi_0(X, Y)$ and suppose there exists a bounded open ball $G \subset X$ with $0 \in G$ such that*

(a) $L - \lambda N : \overline{G} \to Y$ is A-proper w.r.t. Γ for each $\lambda \in [0, 1]$ with $N(\overline{G})$ bounded.

(b) $Lx \neq \lambda Nx$ for $x \in \partial G$ and $\lambda \in (0, 1]$.

(c) $QNx \neq 0$ *for all* $x \in N(L) \cap \partial G$.

(d) Either (d1): $[QNx, x] \geqslant 0$ *(or (d2):* $[QNx, x] \leqslant 0)$ *for all* $x \in N(L) \cap \partial G$.

Then (4.1), (4.2) is feebly a-solvable w.r.t. Γ *and, in particular, (4.1), (4.2) has a solution* $x \in G$. *If* x *is the unique solution in* G, *then (4.1), (4.2) is strongly a-solvable (i.e., the Galerkin method applies to (4.1), (4.2)).*

THEOREM 5. *Let the conditions of Theorem 3 hold. Then (4.1), (4.2) is feebly a-solvable w.r.t.* Γ *and, in particular, (4.1), (4.2) has a solution* $x \in G$. *If* x *is the unique solution, then (4.1), (4.2) is strongly a-solvable (i.e., the Galerkin method applies to (4.1), (4.2)).*

Proof. Let $Lx = x''$ and $Nx = f(t, x, x')$ with

$$X := \{x \in C^2([0,T]; \mathbb{R}^d),\ x(0) = 0 = x(T)\}$$
$$Y := C([0,T]; \mathbb{R}^d).$$

We need to show that conditions (a)–(d) of Theorem 4 are satisfied. See that the BVP

$$x'' = 0, \qquad t \in [0,T],$$
$$x(0) = 0 = x(T),$$

has only the zero solution and so Null $L = \{0\}$ with $L : X \to Y$ a linear homeomorphism. Thus by [10] L is A-proper with respect to Γ.

Since $N : X \to Y$ is continuous, we see that N is also completely continuous because X is compactly embedded into $C^1([0,T]; \mathbb{R}^d)$ and therefore N is A-proper with respect to Γ (see [10]).

Hence $L - \lambda N$ is A-proper with respect to Γ for all $\lambda \in [0,1]$ and hence (a) holds.

Now choose $G := B(0,R) \subset X$, with R defined in the proof of Theorem 3. Since Theorems 1 and 2 hold, we see that

$$Lx \neq \lambda Nx \qquad \text{for all } x \in \partial\Omega \text{ and all } \lambda \in [0,1],$$

and hence (b) holds.

Since Null $\{L\} \cap \partial G = \{0\} \cap \partial G = \emptyset$ we have (c) and (d) holding vacuously.

All of the conditions of Theorem 4 hold and the conclusion follows. This concludes the proof. □

REMARK 1. Under the conditions of any of Corollaries 1–5, the constructive conclusion of Theorem 5 will hold. The statements of the results are omitted for brevity.

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