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SIMILARITY SOLUTIONS OF MIXED CONVECTION BOUNDARY–LAYER FLOWS IN A POROUS MEDIUM

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Abstract. The similarity differential equation $f''' + ff'' + \beta f'(f' - 1) = 0$ with $\beta > 0$ is considered. This differential equation appears in the study of mixed convection boundary-layer flows over a vertical surface embedded in a porous medium. In order to prove the existence of solutions satisfying the boundary conditions $f(0) = a \geqslant 0$, $f'(0) = b \geqslant 0$ and $f'(+\infty) = 0$ or 1, we use shooting and consider the initial value problem consisting of the differential equation and the initial conditions $f(0) = a$, $f'(0) = b$ and $f''(0) = c$. For $0 < \beta \leqslant 1$, we prove that there exists a unique solution such that $f'(+\infty) = 0$, and infinitely many solutions such that $f'(+\infty) = 1$. For $\beta > 1$, we give only partial results and show some differences with the previous case.

1. Introduction

Let $\beta \in \mathbb{R}$. We consider the third order autonomous nonlinear differential equation

$$f''' + ff'' + \beta f'(f' - 1) = 0. \tag{1.1}$$

In fluid mechanics, in the study of mixed convection boundary-layer flows over a vertical surface embedded in a porous medium, such an equation can be derived from the governing partial differential equations in some situations where simplifying assumptions have been made. Any solution of (1.1) provides a *similarity solution* of the initial problem.

A similarity solution is a particular type of solution that reflects the invariance properties of the equation. These solutions are obtained, specifically, by using these properties. Most of the time, the similarity solutions have a particular physical significance.

In the case of mixed convection boundary-layer flows in a porous medium, under some assumptions, the partial differential equation to solve is of the form

$$\frac{\partial^3 \psi}{\partial y^3} + \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial y}\left(\frac{\partial^2 \psi}{\partial x \partial y} - \mu x^{\mu - 1}\right) = 0, \tag{1.2}$$

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where $\mu \in \mathbb{R}$ is some constant; see [3]. It is easy to check that, for any $\ell \neq 0$, the transformation

$$(x,y,\psi) \longmapsto (x/\ell^{\alpha}, y/\ell, \psi/\ell^{\gamma}) \quad \text{with} \quad \alpha = \tfrac{2}{1-\mu} \quad \text{and} \quad \gamma = \tfrac{\mu+1}{\mu-1}$$

is a *symmetry* transformation of equation (1.2). Then it is quite straighforward to verify that any function ψ of the form

$$\psi(x,y) = \kappa x^{\frac{\mu+1}{2}} f(t) \quad \text{where} \quad t = \kappa^{-1} x^{\frac{\mu-1}{2}} y,$$

with an appropriate constant κ, is a solution of (1.2) if and only if f is a solution of the ordinary differential equation (1.1) for some value of β depending on μ. Such a ψ is a so-called similarity solution of (1.2), and the variable t is called the similarity variable.

Equation (1.1) is a particular case of the more general equation

$$f''' + ff'' + \mathbf{g}(f') = 0. \tag{1.3}$$

The most famous equation of this type is certainly the Blasius equation (see [6]), which corresponds to $\mathbf{g} = 0$, and which has been extensively studied over the last hundred years; see for example [10] and the references therein.

For $\mathbf{g}(x) = \beta(x^2 - 1)$, this is the Falkner-Skan equation, introduced in 1931 for studying the boundary layer flow past a semi-infinite wedge, see the original paper [17] and [20] for a overview of mathematical results.

For $\mathbf{g}(x) = \beta x^2$, this corresponds to free convection problems, see for example [16] for the derivation of the model, and [2], [4], [7], [8], [11], [14], [15], [18], [23], [25] for different approaches of the mathematical analysis.

The case where $\mathbf{g}(x) = \beta(x^2 + 1)$ is for the study of the boundary layer separation at a free stream-line, see [1] and [22].

Most of the time, these similarity equations are studied on the half line $[0, +\infty)$ and are associated to boundary conditions as $f(0) = a$, $f'(0) = b$ (or $f''(0) = c$) and a condition at infinity. This condition at infinity can be, either $f'(t) \to \lambda$ as $t \to +\infty$, or $f'(t) \sim At^{\nu}$ as $t \to +\infty$, where A and ν are some positive constants, or also $|f|$ is of polynomial growth at infinity. For more details, we refer to the introduction of [9] and to the references therein.

The boundary value problems associated to the general equation (1.3), with the condition that f' tends to λ at infinity have been studied in [13] and in [9]. Let us notice that, if $\mathbf{g}(\lambda) \neq 0$, then these boundary value problems do not have any solutions, and thus we must assume that $\mathbf{g}(\lambda) = 0$ to have solutions. For example, in the case of mixed convection, i.e. $\mathbf{g}(x) = \beta x(x-1)$, the only relevant conditions are $f'(t) \to 0$ or $f'(t) \to 1$ as $t \to +\infty$. Results about existence, uniqueness and asymptotic behavior of *concave* or *convex* solutions to these boundary value problems are obtained, according to the sign of $\mathbf{g}$ between b and λ. Without further assumptions on $\mathbf{g}$, it is hopeless to have more precise results. Nevertheless, the results of [9] generalize the ones of [12] and some of [19] about mixed convection problems.

Let $a,b \in \mathbb{R}$ and $\lambda \in \{0,1\}$. We associate to equation (1.1) the boundary value problem

$$\begin{cases} f''' + ff'' + \beta f'(f'-1) = 0 \quad \text{on} \quad [0,+\infty) \\ f(0) = a \\ f'(0) = b \\ f'(t) \to \lambda \quad \text{as} \quad t \to +\infty \end{cases} \qquad (\mathscr{P}_{\beta;a,b,\lambda})$$

Usually, the method to investigate such a boundary value problem is the shooting method, which consists of finding the values of a parameter c for which the solution of (1.1) satisfying the initial conditions $f(0) = a$, $f'(0) = b$ and $f''(0) = c$, exists up to infinity and is such that $f'(t) \to \lambda$ as $t \to +\infty$. This approach is used in [12] and [19]. In [12], the problem $(\mathscr{P}_{\beta;a,b,1})$ is considered for $\beta < 0$ and its is shown that this problem has a unique convex solution if $0 < b < 1$, and has a unique concave solution if $b > 1$. In [19], for $\beta \in (0,1)$, $a = 0$ and $b \in (0, \frac{3}{2})$, it is proven that the boundary value problem $(\mathscr{P}_{\beta;a,b,1})$ has infinitely many solutions.

In [21], [26] and [27], some results about the problem $(\mathscr{P}_{\beta;a,b,1})$ are proven by introducing a singular integral equation obtained from (1.1) by a Crocco-type transformation.

In the following, we will study the problems $(\mathscr{P}_{\beta;a,b,0})$ and $(\mathscr{P}_{\beta;a,b,1})$ for $\beta > 0$, $a \geqslant 0$ and $b \geqslant 0$. In the case where $0 < \beta \leqslant 1$, we are able to get complete results (and so we improve the results of [19]), while we only have partial results for $\beta > 1$. On several occasions, we will use the results of [9], that sometimes we re-demonstrate, in our particular case, for the convenience of the reader.

The paper is organized as follows. In Section 2, general results about the solution of equation (1.1) are given. Section 3 is devoted to the case where $b \geqslant 1$ and to the proofs of results that do not depend on whether $\beta \in (0,1]$ or $\beta > 1$. Section 4 discusses in detail the case $\beta \in (0,1]$ and $b \geqslant 1$. Section 5 considers the case $\beta \in (0,1]$ and $0 \leqslant b < 1$, presents the results and how to prove them. In Section 6, some results in the case $\beta > 1$ are proven.

2. Preliminary results

To any f solution of (1.1) on some interval I, we associate the function $H_f : I \to \mathbb{R}$ defined by

$$H_f = f'' + f(f'-1). \tag{2.1}$$

Then, we have $H_f' = (1-\beta)f'(f'-1)$.

The following lemmas, concerning the solutions of the equation (1.1), will be useful in the next sections. The proofs of some of them can be found in [9].

LEMMA 1. *Let f be a solution of (1.1) on some maximal interval I. If there exists $t_0 \in I$ such that $f'(t_0) \in \{0,1\}$ and $f''(t_0) = 0$, then $I = \mathbb{R}$ and $f''(t) = 0$ for all $t \in \mathbb{R}$.*

Proof. This follows immediatly from the uniqueness of solutions of initial value problem. See [9], Proposition 3.1, item 3. □

LEMMA 2. *Let $\beta > 0$ and f be a solution of equation (1.1) on some interval I, such that f' is not constant.*

1. *If there exists $s < r \in I$ such that $f''(s) \leqslant 0$ and $f'(f'-1) > 0$ on (s,r) then we have $f''(t) < 0$ for all $t \in (s,r]$.*
2. *If there exists $s < r \in I$ such that $f''(s) \geqslant 0$ and $f'(f'-1) < 0$ on (s,r) then we have $f''(t) > 0$ for all $t \in (s,r]$.*
3. *If there exists $s < r \in I$ such that $f'' < 0$ on (s,r) and $f''(r) = 0$, then we have $f'(r)(f'(r)-1) < 0$.*
4. *If there exists $s < r \in I$ such that $f'' > 0$ on (s,r) and $f''(r) = 0$, then we have $f'(r)(f'(r)-1) > 0$.*

Proof. Let F denote any primitive function of f. From (1.1) we deduce the relation

$$(f'' \exp F)' = -\beta f'(f'-1)\exp F.$$

All the assertions 1-4 follow easily from this relation and from Lemma 1. Let us verify the first and the third of these assertions. For the first one, since $\psi = f'' \exp F$ is decreasing on $[s,r]$, we have $f''(t) < f''(s)\exp(F(s) - F(t)) \leqslant 0$ for all $t \in (s,r]$. For the third one, since $\psi < 0$ on (s,r) and $\psi(r) = 0$, one has $\psi'(r) \geqslant 0$. This and Lemma 1 imply that $f'(r)(f'(r)-1) < 0$. □

LEMMA 3. *Let f be a solution of (1.1) on some maximal interval (T_-, T_+). If T_+ is finite, then f' and f'' are unbounded in any neighborhood of T_+.*

Proof. See [9], Proposition 3.1, item 6. □

LEMMA 4. *Let $\beta \neq 0$. If f is a solution of (1.1) on some interval $(\tau, +\infty)$ such that $f'(t) \to \lambda$ as $t \to +\infty$, then $\lambda \in \{0,1\}$. Moreover, if f is of constant sign at infinity, then $f''(t) \to 0$ as $t \to +\infty$.*

Proof. See [9], Proposition 3.1, items 4 and 5. Let us notice that if $\lambda = 1$, then f is necessarily positive at infinity. □

LEMMA 5. *Let $\beta \neq 0$. If f is a solution of (1.1) on some interval $(\tau, +\infty)$ such that $f'(t) \to 0$ as $t \to +\infty$, then $f(t)$ does not tend to plus or minus infinity as $t \to +\infty$.*

Proof. Assume for contradiction that $f(t) \to +\infty$ as $t \to +\infty$. Let $H = H_f$ be defined by (2.1). Since $f'(t) \to 0$ as $t \to +\infty$, we deduce from the second assertion of Lemma 4 that $H(t) \sim -f(t)$ as $t \to +\infty$. This leads to a contradiction if $\beta = 1$. If $\beta \neq 1$, then we have $H'(t) \sim (\beta - 1)f'(t)$ as $t \to +\infty$, and hence $H(t) \sim (\beta - 1)f(t)$ as $t \to +\infty$. This is a contradiction, since $\beta \neq 0$. The proof is the same if we assume that $f(t) \to -\infty$ as $t \to +\infty$. □

LEMMA 6. *Let $\beta > 0$ and f be a solution of equation* (1.1) *on some right maximal interval $I = [\tau, T_+)$. If $f \geqslant 0$ and $f' \geqslant 0$ on I, then $T_+ = +\infty$ and f' is bounded on I.*

Proof. Let $L = L_f$ be the function defined on I by

$$L(t) = 3f''(t)^2 + \beta(2f'(t) - 3)f'(t)^2. \tag{2.2}$$

Easily, using (1.1), we obtain that $L'(t) = -6f(t)f''(t)^2$ for all $t \in I$, and since $f \geqslant 0$ on I, this implies that L is nonincreasing. Hence

$$\forall t \in I, \qquad \beta(2f'(t) - 3)f'(t)^2 \leqslant L(t) \leqslant L(\tau).$$

It follows that f' is bounded on I and, thanks to Lemma 3, that $T_+ = +\infty$. $\square$

LEMMA 7. *Let $\beta > 0$ and f be a solution of equation* (1.1) *on some right maximal interval $I = [\tau, T_+)$. If $f(\tau) \geqslant 0$, $f'(\tau) \geqslant 1$ and $f''(\tau) > 0$, then there exists $t_0 \in (\tau, T_+)$ such that $f'' > 0$ on $[\tau, t_0)$ and $f''(t_0) = 0$.*

Proof. Assume for contradiction that $f'' > 0$ on I. Then, $f'(t) \geqslant 1$ and $f(t) \geqslant 0$ for all $t \in I$. We then have

$$f''' = -ff'' - \beta f'(f' - 1) \leqslant 0. \tag{2.3}$$

It follows that $0 < f''(t) \leqslant c$ for all $t \in I$ and hence, by Lemma 3, we have $T_+ = +\infty$. Next, let $s > \tau$ and $\varepsilon = \beta f'(s)(f'(s) - 1)$. One has $\varepsilon > 0$ and, coming back to (2.3), we obtain $f''' \leqslant -\varepsilon$ on $[s, +\infty)$. After integration, we get

$$\forall t \geqslant s, \qquad f''(t) - f''(s) \leqslant -\varepsilon(t - s)$$

and a contradiction with the fact that $f'' > 0$. Consequently, there exists $t_0 \in (\tau, T_+)$ such that $f'' > 0$ on $[\tau, t_0)$ and $f''(t_0) = 0$. $\square$

The last two lemmas give key results in the case where $\beta \in (0, 1]$. The proofs can be found in [9] (see Lemma 5.16 and Lemma A.11). However, for convenience, we give here proofs corresponding to the particular case that we consider.

LEMMA 8. *Let $\beta \in (0, 1]$ and f be a solution of equation* (1.1) *on some maximal interval $I = (T_-, T_+)$. If there exists $t_0 \in I$ such that*

$$0 < f'(t_0) < 1 \qquad \text{and} \qquad 0 \leqslant f''(t_0) \leqslant f(t_0)(1 - f'(t_0)),$$

then $T_+ = +\infty$ and $f'(t) \to 1$ as $t \to +\infty$. Moreover, $f'' > 0$ on $[t_0, +\infty)$.

Proof. Let $\tau = \sup A(t_0)$ where

$$A(t_0) = \left\{ t \in [t_0, T_+) \, ; \, f'(t_0) < f' < 1 \text{ and } f'' > 0 \text{ on } (t_0, t) \right\}.$$

The set $A(t_0)$ is not empty. This is clear if $f''(t_0) > 0$, and if $f''(t_0) = 0$ it follows from the fact that $f'''(t_0) = -\beta f'(t_0)(f'(t_0) - 1) > 0$. We claim that $\tau = T_+$. Assume for contradiction that $\tau < T_+$. From Lemma 2, item 2, we get that $f''(\tau) > 0$, which implies, by definition of τ, that $f'(\tau) = 1$. Therefore, since the function H_f defined by (2.1) is nonincreasing on $[t_0, \tau]$, we obtain

$$f''(\tau) = H_f(\tau) \leqslant H_f(t_0) = f''(t_0) + f(t_0)(f'(t_0) - 1) \leqslant 0,$$

a contradiction. Thus, we have $\tau = T_+$. From Lemma 3, it follows that $T_+ = +\infty$. Since $f'' > 0$ on $[t_0, +\infty)$, by virtue of Lemma 4, we get that $f'(t) \to 1$ as $t \to +\infty$. □

REMARK 1. If $f(t_0) > 0$ and $f''(t_0) = 0$, then $f(t) - t \to -\infty$ as $t \to +\infty$ (see [9], Theorem 6.4, item 2.a).

LEMMA 9. *Let $\beta \in (0,1]$ and f be a solution of (1.1) on some maximal interval $I = (T_-, T_+)$. If there exists $t_0 \in I$ such that*

$$f'(t_0) > 1 \qquad \textit{and} \qquad f(t_0)(1 - f'(t_0)) \leqslant f''(t_0) \leqslant 0,$$

then $T_+ = +\infty$ and $f'(t) \to 1$ as $t \to +\infty$. Moreover, $f'' < 0$ on $[t_0, +\infty)$.

Proof. If we set $\tau = \sup B(t_0)$ where

$$B(t_0) = \left\{ t \in [t_0, T_+) \; ; 1 < f' < f'(t_0) \;\text{ and }\; f'' < 0 \;\text{ on }\; (t_0, t) \right\},$$

the conclusion will follow by proceeding in the same way as in the previous proof. □

REMARK 2. If $f(t_0) > 0$ and $f''(t_0) = 0$, then $f(t) - t \to +\infty$ as $t \to +\infty$ (see [9], Theorem 5.19, item 2.a).

3. Description of our approach when $b \geqslant 1$

Let $\beta > 0$, $a \geqslant 0$ and $b \geqslant 1$. As said in the introduction, the method we will use to obtain solutions of the boundary value problems $(\mathscr{P}_{\beta;a,b,0})$ and $(\mathscr{P}_{\beta;a,b,1})$ is the shooting technique. Specifically, for $c \in \mathbb{R}$, let us denote by f_c the solution of equation (1.1) satisfying the initial conditions

$$f_c(0) = a, \quad f_c'(0) = b \quad \text{and} \quad f_c''(0) = c \tag{3.1}$$

and let $[0, T_c)$ be the right maximal interval of existence of f_c. Hence, finding a solution of one of the problems $(\mathscr{P}_{\beta;a,b,0})$ or $(\mathscr{P}_{\beta;a,b,1})$ amounts to finding a value of c such that $T_c = +\infty$ and $f_c'(t) \to 0$ or 1 as $t \to +\infty$.

To this end, let us partition $\mathbb{R}$ into the four sets $\mathscr{C}_0, \ldots, \mathscr{C}_3$ (or less if some of them are empty) defined as follows. Let $\mathscr{C}_0 = (0, +\infty)$ and, according to the notations used in [9], let us set

$$\mathscr{C}_1 = \left\{c \leqslant 0\,;\, 1 \leqslant f_c' \leqslant b \text{ and } f_c'' \leqslant 0 \text{ on } [0,T_c)\right\}$$

$$\mathscr{C}_2 = \Big\{c \leqslant 0\,;\, \exists t_c \in [0,T_c),\ \exists \varepsilon_c > 0 \text{ s.t. } f_c' > 1 \text{ on } (0,t_c),$$
$$f_c' < 1 \text{ on } (t_c, t_c+\varepsilon_c) \text{ and } f_c'' < 0 \text{ on } (0,t_c+\varepsilon_c)\Big\}$$

$$\mathscr{C}_3 = \Big\{c \leqslant 0\,;\, \exists r_c \in [0,T_c),\ \exists \eta_c > 0 \text{ s.t. } f_c'' < 0 \text{ on } (0,r_c),$$
$$f_c'' > 0 \text{ on } (r_c, r_c+\eta_c) \text{ and } f_c' > 1 \text{ on } (0,r_c+\eta_c)\Big\}.$$

This is obvious that $\mathscr{C}_0,\dots,\mathscr{C}_3$ are disjoint sets and that their union is the whole line of real numbers.

Thanks to Lemmas 3 and 4, if $c \in \mathscr{C}_1$ then $T_c = +\infty$ and $f_c'(t) \to 1$ as $t \to +\infty$. In fact, $\mathscr{C}_1$ *is the set of values of* c *for which* f_c *is a concave solution of* $(\mathscr{P}_{\beta;a,b,1})$.

Since $\beta > 0$, the study done in [9] (especially in Section 5.2) says, on the one hand, that $\mathscr{C}_3 = \emptyset$ (which can easily be deduced from Lemma 2, item 1) and, on the other hand, that either $\mathscr{C}_1 = \emptyset$ and $\mathscr{C}_2 = (-\infty, 0]$, or there exists $c^* \leqslant 0$ such that $\mathscr{C}_1 = [c^*, 0]$ and $\mathscr{C}_2 = (-\infty, c^*)$. In addition, if $\beta \in (0,1]$ then we are in the second case and $c^* \leqslant -a(b-1)$. If $\beta > 1$ and $a = 0$ then $\mathscr{C}_1 = \emptyset$, but, for $a > 0$, we do not know if $\mathscr{C}_1$ is empty or not.

In the next sections we will distinguish between the cases $\beta \in (0,1]$ and $\beta > 1$. In the first case, we can give a complete description of the solutions (see Theorem 1), whereas in the second one, we have only partial answers.

We will also consider the case where $b \in [0,1)$, for which we will have to partition $\mathbb{R}$ in a slightly different way.

Before that, and in order to complete the study, let us divide the set $\mathscr{C}_2$ into the following two subsets

$$\mathscr{C}_{2,1} = \{c \in \mathscr{C}_2\,;\, f_c' > 0 \text{ on } [0,T_c)\}$$

$$\mathscr{C}_{2,2} = \{c \in \mathscr{C}_2\,;\, \exists s_c \in (0,T_c) \text{ s.t. } f_c' > 0 \text{ on } [0,s_c) \text{ and } f_c'(s_c) = 0\}$$

and let us give properties of each of them that hold for all $\beta > 0$.

LEMMA 10. *If* $c \in \mathbb{R}$ *is such that* $f_c' > 0$ *on* $[0,T_c)$, *then* $T_c = +\infty$ *and* f_c' *is bounded. Moreover, if* $c \leqslant 0$, *then* $f_c' \leqslant \max\{b; \frac{3}{2}\}$ *on* $[0,+\infty)$.

Proof. Let $c \in \mathbb{R}$ be such that $f_c' > 0$ on $[0,T_c)$. Then $f_c \geqslant a \geqslant 0$ on $[0,T_c)$, and thanks to Lemma 6, it follows that $T_c = +\infty$ and that f_c' is bounded.

It remains to show that $f_c' \leqslant \max\{b; \frac{3}{2}\}$ in the case where $c \leqslant 0$. As in (2.2), let us define the function L_c on $[0,+\infty)$ by

$$L_c(t) = 3f_c''(t)^2 + \beta(2f_c'(t) - 3)f_c'(t)^2. \tag{3.2}$$

We have $L_c'(t) = -6f_c(t)f_c''(t)^2$ and, since $f_c \geqslant 0$, it implies that L_c is nonincreasing.

If $f_c'' \leqslant 0$ on $(0,+\infty)$, then $f_c' \leqslant b$. Otherwise, there exists t_0 such that $f_c'' < 0$ on $(0,t_0)$ and $f_c''(t_0) = 0$ (which can occur only when $c < 0$, or $c = 0$ and $b > 1$). By Lemma 2, item 3, it follows that $f_c'(t_0) < 1$, and thus $L_c(t_0) < 0$. Then, $L_c < 0$ on $(t_0,+\infty)$ which implies that $f_c' \leqslant \frac{3}{2}$ on $(t_0,+\infty)$. Since $f_c' \leqslant b$ on $(0,t_0)$, the proof is complete. □

PROPOSITION 1. *Let $c_* = \sup(\mathscr{C}_1 \cup \mathscr{C}_{2,1})$. Then c_* is finite.*

Proof. Let $c \in \mathscr{C}_1 \cup \mathscr{C}_{2,1}$. From the definitions of $\mathscr{C}_1$ and $\mathscr{C}_{2,1}$, and thanks to Lemma 10, we have $T_c = +\infty$ and $0 < f_c' \leqslant d$ on $(0,+\infty)$ where $d = \max\{b; \frac{3}{2}\}$.

Since $(f_c'' + f_c f_c')' = -\beta f_c'(f_c' - 1) + f_c'^2 \leqslant \beta f_c' + f_c'^2 \leqslant d(\beta + d)$, by integrating, we then have

$$\forall t \geqslant 0, \qquad f_c''(t) + f_c(t) f_c'(t) \leqslant c + ab + d(\beta + d)t.$$

Integrating once again, we get

$$\forall t \geqslant 0, \qquad 0 < f_c'(t) \leqslant f_c'(t) + \tfrac{1}{2} f_c(t)^2 \leqslant b + \tfrac{1}{2}a^2 + (c + ab)t + \tfrac{1}{2} d(\beta + d)t^2$$

which implies that $c \geqslant -ab - \sqrt{(2b + a^2)(\beta + d)d}$. □

REMARK 3. As we have seen above, if $\mathscr{C}_1 \neq \emptyset$, then $\mathscr{C}_1 = [c^*, 0]$ and thus we have $\mathscr{C}_{2,1} \subset [c_*, c^*)$.

PROPOSITION 2. *We have $(-\infty, c_*) \subset \mathscr{C}_{2,2}$. Moreover, if $c \in \mathscr{C}_{2,2}$ then $T_c < +\infty$ and $f_c'' < 0$ on $(0, T_c)$.*

Proof. The fact that $(-\infty, c_*) \subset \mathscr{C}_{2,2}$ follows from Proposition 1. Let $c \in \mathscr{C}_{2,2}$. Then, there exists $s_c \in (0, T_c)$ such that $f_c' > 0$ on $[0, s_c)$ and $f_c'(s_c) = 0$. Consider the function L_c defined by (3.2). Since $f_c \geqslant 0$ on the interval $[0, s_c]$, then L_c is nonincreasing on $[0, s_c]$.

Suppose first that $c < 0$. Assume for contradiction that there exists $t_0 \in [0, s_c)$ such that $f_c'' < 0$ on $[0, t_0)$ and $f_c''(t_0) = 0$, then $0 < f_c'(t_0) < 1$ (see Lemma 2, item 3), and hence $L_c(t_0) < 0$. Since L_c is nonincreasing on $[0, s_c]$, this contradicts the fact that $L_c(s_c) = 3 f_c''(s_c)^2 \geqslant 0$. Therefore, $f_c'' < 0$ on $[0, s_c]$.

If $c = 0$, which can only happen if $b > 1$, then $f_c'''(0) = -\beta b(b-1) < 0$. Hence there exists $\eta \in (0, s_c)$ such that $f_c'' < 0$ and $f_c' > 1$ on $(0, \eta]$. The arguments above applied to the function $t \mapsto f_c(t + \eta)$ give that $f_c'' < 0$ on $[\eta, s_c]$ and thus on $(0, s_c]$.

To get that $f_c'' < 0$ on $(0, T_c)$, it remains to notice that f_c'' cannot vanish on (s_c, T_c), by virtue of Lemma 2, item 3.

Finally, the fact that $T_c < +\infty$ follows from Proposition 2.11 of [9], which says that, for any $\tau \in \mathbb{R}$, there is no negative (strictly) concave function f of class $\mathscr{C}^3$ such that $f''' + f f'' \leqslant 0$ on $[\tau, +\infty)$. □

REMARK 4. If $c \in \mathscr{C}_{2,2}$ then f_c is strictly concave on $[0, T_c)$, has a global maximum at s_c and $f_c(t) \to -\infty$ as $t \to T_c$. In addition, $f_c'(t)$ and $f_c''(t)$ tend to $-\infty$ as $t \to T_c$.

PROPOSITION 3. *The set* $\mathscr{C}_{2,2}$ *is an open subset of* $(-\infty,0]$ *(for its induced topology).*

Proof. Let $c_0 \in \mathscr{C}_{2,2}$. There exists $\tau \in (0, T_{c_0})$ such that $f'_{c_0}(\tau) < 0$. Let us set $\varepsilon = -\frac{1}{2} f'_{c_0}(\tau)$. By continuity of the function $c \mapsto f'_c(\tau)$, there exists $\alpha > 0$ such that, for all $c \in (-\infty, 0]$, one has

$$|c - c_0| < \alpha \implies f'_c(\tau) < f'_{c_0}(\tau) + \varepsilon.$$

Therefore, $f'_c(\tau) < 0$ and $c \in \mathscr{C}_{2,2}$. □

4. The case $\beta \in (0,1]$ and $b \geqslant 1$

In this section, we assume that $\beta \in (0,1]$, $a \geqslant 0$ and $b \geqslant 1$.

PROPOSITION 4. *If* $c \in \mathscr{C}_0$*, then* $T_c = +\infty$ *and* $f'_c(t) \to 1$ *as* $t \to +\infty$.

Proof. From Lemma 7, we know that there exists $t_0 \in (0, T_c)$ such that $f''_c > 0$ on $[0, t_0)$ and $f''_c(t_0) = 0$. Since $f_c(t_0) > 0$ and $f'_c(t_0) > b > 1$, the conclusion follows from Lemma 9. □

REMARK 5. The previous proposition says that f_c is a *convex-concave* solution of $(\mathscr{P}_{\beta;a,b,1})$ for all $c > 0$. Moreover, we have that $f_c(t) - t \to +\infty$ as $t \to +\infty$ (see Remark 2).

PROPOSITION 5. *There exists* $c^* \leqslant -a(b-1)$ *such that* $\mathscr{C}_1 = [c^*, 0]$.

Proof. If $b = 1$ then $\mathscr{C}_1 = \{0\}$. If $b > 1$, as we already said in the previous section, this result is proven in [9] (see Corollary 5.13 and Lemma 5.16). For convenience, let us recall briefly the main arguments which were used to get it. On the one hand, from Lemma 9 with $t_0 = 0$ (or Lemma 5.16 of [9]), it follows that $[-a(b-1), 0] \subset \mathscr{C}_1$. On the other hand, Lemma 5.12 of [9] implies that $\mathscr{C}_2$ is an interval of the type $(-\infty, c^*)$. This completes the proof since $\mathscr{C}_1 = (-\infty, 0] \setminus \mathscr{C}_2$. □

REMARK 6. From the previous proposition, we have that $0 \notin \mathscr{C}_{2,2}$. Hence, Proposition 3 implies that $\mathscr{C}_{2,2}$ is an open set.

PROPOSITION 6. *If* $c \in \mathscr{C}_{2,1}$ *then* $T_c = +\infty$ *and* f'_c *has a finite limit at infinity, equal either to* 0 *or to* 1.

Proof. Let $c \in \mathscr{C}_{2,1}$. By Proposition 5, we have $c < 0$. Thanks to Lemma 10, we know that $T_c = +\infty$. Assume first that $f''_c < 0$ on $(0, +\infty)$. Then f'_c is positive and decreasing, and thus f'_c has a finite limit $\lambda \geqslant 0$ at infinity. Moreover, f'_c takes the value 1 at some point, hence $\lambda \in [0, 1)$ and, by Lemma 4, we finally get that $\lambda = 0$.

Assume now that f''_c vanishes on $(0, +\infty)$. Let t_0 be the first point where f''_c vanishes. Thanks to Lemma 2, item 3, we have $0 < f'_c(t_0) < 1$, and the conclusion follows from Lemma 8. □

REMARK 7. If $c \in \mathscr{C}_{2,1}$ then either f_c is a *concave* solution of $(\mathscr{P}_{\beta;a,b,0})$ or f_c is a *concave-convex* solution of $(\mathscr{P}_{\beta;a,b,1})$. In the first case, there exists $l > a$ such that $f_c(t) \to l$ as $t \to +\infty$ (see Lemma 5) and, in the second one, we have that $f_c(t) - t \to -\infty$ as $t \to +\infty$ (see Remark 1).

PROPOSITION 7. *Let $c \in \mathscr{C}_{2,2}$. For all $t \in [0, T_c)$, one has $f_c(t) \leqslant \sqrt{a^2 + 2b}$.*

Proof. Let $c \in \mathscr{C}_{2,2}$ and s_c be as in the definition of $\mathscr{C}_{2,2}$, i.e. such that $f_c' > 0$ on $[0, s_c)$ and $f_c'(s_c) = 0$. For all $t \in [0, s_c]$, we have

$$\begin{aligned}\big(tf_c''(t) - f_c'(t) + tf_c(t)f_c'(t)\big)' &= tf_c'''(t) + tf_c(t)f_c''(t) + tf_c'(t)^2 + f_c(t)f_c'(t) \\ &= (1-\beta)tf_c'(t)^2 + \beta tf_c'(t) + f_c(t)f_c'(t) \geqslant f_c(t)f_c'(t). \end{aligned} \tag{4.1}$$

Integrating between 0 and s_c yields

$$f_c(s_c)^2 \leqslant a^2 + 2\big(s_c f_c''(s_c) + b\big) \leqslant a^2 + 2b$$

and $f_c(s_c) \leqslant \sqrt{a^2 + 2b}$. The conclusion follows from the fact that, for all $t \in [0, T_c)$, we have $f_c(t) \leqslant f_c(s_c)$, as we noticed in Remark 4. □

PROPOSITION 8. *Let c be a point of the boundary of $\mathscr{C}_{2,2}$. Then, $c \in \mathscr{C}_{2,1}$ and $f_c'(t) \to 0$ as $t \to +\infty$. Moreover, f_c is bounded and concave.*

Proof. Let c be a point of the boundary of $\mathscr{C}_{2,2}$ and $(c_n)_{n \geqslant 0}$ be a sequence of $\mathscr{C}_{2,2}$ such that $c_n \to c$ as $n \to +\infty$. For all $n \geqslant 0$, let us set $T_n = T_{c_n}$ and $f_n = f_{c_n}$. Since $\mathscr{C}_{2,2}$ is an open set, then $c \in \mathscr{C}_1 \cup \mathscr{C}_{2,1}$ and hence $T_c = +\infty$. Let $t \geqslant 0$ be fixed. From the lower semicontinuity of the function $d \mapsto T_d$, we get that there exists $n_0 \geqslant 0$ such that $T_n \geqslant t$ for all $n \geqslant n_0$. Since $f_n(t) \to f_c(t)$ as $n \to +\infty$, we deduce from Proposition 7 that f_c is bounded. Therefore, f_c' cannot tend to 1 at infinity and thus, necessarily, we have $c \in \mathscr{C}_{2,1}$ and $f_c'(t) \to 0$ as $t \to +\infty$. Moreover, f_c is concave (see Remark 7). □

PROPOSITION 9. *There exists at most one c such that $f_c'(t) \to 0$ as $t \to +\infty$.*

Proof. From Proposition 5, Proposition 6 and Lemma 5, we see that if c is such that $f_c'(t) \to 0$ as $t \to +\infty$, then $c < 0$, $f_c'' < 0$ and f_c is bounded. For such a c, as done in [9], Section 4, we can define a function $v : (0, b^2] \to \mathbb{R}$ such that

$$\forall t \geqslant 0, \quad v(f_c'(t)^2) = f_c(t). \tag{4.2}$$

By setting $y = f_c'(t)^2$, we get

$$f_c(t) = v(y), \quad f_c'(t) = \sqrt{y}, \quad f_c''(t) = \frac{1}{2v'(y)} \quad \text{and} \quad f_c'''(t) = -\frac{v''(y)\sqrt{y}}{2v'(y)^3}$$

and using (1.1) we obtain

$$\forall y \in (0, b^2], \quad v''(y) = \frac{v(y)\, v'(y)^2}{\sqrt{y}} + 2\beta(\sqrt{y} - 1)\, v'(y)^3. \tag{4.3}$$

From (3.1), we deduce that $v(b^2) = a$ and $v'(b^2) = \frac{1}{2c}$. Moreover, since f_c is bounded, it is so for v.

Assume that there exists $c_1 > c_2$ such that $f'_{c_1}(t) \to 0$ and $f'_{c_2}(t) \to 0$ as $t \to +\infty$, and denote by v_1 and v_2 the functions associated to f_{c_1} and f_{c_2} by (4.2). If we set $w = v_1 - v_2$ then $w(b^2) = 0$ and $w'(b^2) < 0$. We claim that $w' < 0$ on $(0, b^2]$. For contradiction, assume there exists $x \in (0, b^2)$ such that $w' < 0$ on $(0, x)$ and $w'(x) = 0$. Hence we have $w''(x) \leqslant 0$ and $w(x) > 0$. But, thanks to (4.3), we have

$$w''(x) = \frac{w(x)}{\sqrt{x}}\, v_1'(x)^2$$

and a contradiction.

Now, let us set $V_i = 1/v_i'$ for $i = 1, 2$ and $W = V_1 - V_2$. On the one hand, we have $W(b^2) = 2(c_1 - c_2) > 0$ and $W(y) \to 0$ as $y \to 0$. In the other hand, from (4.3), we get

$$\forall y \in (0, b^2], \qquad W'(y) = -\frac{w(y)}{\sqrt{y}} - 2\beta(\sqrt{y} - 1)\, w'(y).$$

Therefore, we have

$$\begin{aligned} W(b^2) &= \int_0^{b^2} W'(y)\, dy = -\int_0^{b^2} \left(\frac{w(y)}{\sqrt{y}} + 2\beta(\sqrt{y} - 1)\, w'(y) \right) dy \\ &= -2\left[\sqrt{y}\, w(y) \right]_0^{b^2} + 2\int_0^{b^2} \big((1 - \beta)\sqrt{y} + \beta \big)\, w'(y)\, dy \\ &= 2\int_0^{b^2} \big((1 - \beta)\sqrt{y} + \beta \big)\, w'(y)\, dy, \end{aligned} \tag{4.4}$$

the last equality following from the fact that $w(y)$ tends to a finite limit as $y \to 0$. Since $w' < 0$, we finally obtain $W(b^2) < 0$ and a contradiction. □

REMARK 8. The change of variable (4.2) is particularly efficient to obtain some uniqueness results. In [9], it is used for the general equation $f''' + ff'' + \mathbf{g}(f') = 0$ (see Section 4, Lemma 5.4 and Lemma 5.17). The case we examined in Proposition 9 is part of Lemma 5.17 of [9] with $\lambda = 0$. In this lemma, it is assumed that $0 < \mathbf{g}(x) \leqslant x^2$ for $x \in (0, b]$ to ensure uniqueness. Here, in Proposition 9, we have $\mathbf{g}(x) = \beta x(x - 1)$ with $\beta \in (0, 1]$ and hence $\beta x(x - 1) \leqslant x^2$ for $x \in (0, b]$, but $\beta x(x - 1) \leqslant 0$ for $x \in (0, 1]$. However, the assumption about the positivity of $\mathbf{g}$ is not relevant because not used in the proof of Lemma 5.17 of [9]. In addition, the inequality $\beta x(x - 1) \leqslant x^2$ is still true on $(0, b]$, if $\beta > 1$ and $1 \leqslant b \leqslant \frac{\beta}{\beta - 1}$. Finally, let us notice that, in the latter case, the integral in (4.4) is still negative, and the contradiction occurs there too.

COROLLARY 1. *One has* $\mathscr{C}_{2,2} = (-\infty, c_*)$ *and* $\mathscr{C}_{2,1} = [c_*, c^*)$.

Proof. From Remark 6, Propositions 2, 8 and 9, we see that $\mathscr{C}_{2,2}$ is open, contains $(-\infty, c_*)$ and its boundary is reduced to a single point. Thus, since $c_* = \sup(\mathscr{C}_1 \cup \mathscr{C}_{2,1})$, we necessarily have $\mathscr{C}_{2,2} = (-\infty, c_*)$ and $\mathscr{C}_{2,1} = [c_*, c^*)$. □

To finish this section, let us express the results of Proposition 4, Proposition 5 and Corollary 1 in terms of the boundary problems $(\mathscr{P}_{\beta;a,b,0})$ and $(\mathscr{P}_{\beta;a,b,1})$.

THEOREM 1. *Let* $\beta \in (0,1]$, $a \geqslant 0$ *and* $b \geqslant 1$. *There exists* $c_* < 0$ *such that:*

- ▷ f_c *is not defined on the whole interval* $[0, +\infty)$ *if* $c < c_*$;
- ▷ f_{c_*} *is a* concave *solution of* $(\mathscr{P}_{\beta;a,b,0})$;
- ▷ f_c *is a solution of* $(\mathscr{P}_{\beta;a,b,1})$ *for all* $c \in (c_*, +\infty)$.

Moreover, there exists $c^* \in (c_*, -a(b-1)]$ *such that:*

- ▷ f_c *is a* convex-concave *solution of* $(\mathscr{P}_{\beta;a,b,1})$ *for all* $c \in (0, +\infty)$;
- ▷ f_c *is a* concave *solution of* $(\mathscr{P}_{\beta;a,b,1})$ *for all* $c \in [c^*, 0]$;
- ▷ f_c *is a* concave-convex *solution of* $(\mathscr{P}_{\beta;a,b,1})$ *for all* $c \in (c_*, c^*)$.

REMARK 9. The previous theorem says that problem $(\mathscr{P}_{\beta;a,b,0})$ has one and only one solution, whereas problem $(\mathscr{P}_{\beta;a,b,1})$ has infinite number of solutions.

REMARK 10. We know that f_{c_*} has a finite limit at infinity, denoted by l. By slightly modifying the proof of Proposition 7.2 of [9], one can prove that there exists a positive constant A such that, for all $\varepsilon > 0$, the following hold

$$f''_{c_*}(t) = -l^2 A e^{-lt}\left(1 + \mathrm{o}\left(e^{-(l-\varepsilon)t}\right)\right), \qquad f'_{c_*}(t) = lAe^{-lt}\left(1 + \mathrm{o}\left(e^{-(l-\varepsilon)t}\right)\right)$$

$$f_{c_*}(t) = l - Ae^{-lt}\left(1 + \mathrm{o}\left(e^{-(l-\varepsilon)t}\right)\right) \quad \text{as} \quad t \to +\infty.$$

REMARK 11. Among the concave solutions of $(\mathscr{P}_{\beta;a,b,1})$, only f_{c^*} has a slant asymptote, i.e. there exists $l > a$ such that $f_{c^*}(t) - t \to l$ as $t \to +\infty$. In addition, Proposition 7.5 of [9] implies that we have

$$f''_{c^*}(t) = -e^{-\frac{t^2}{2} - lt + \mathrm{O}(\ln t)}, \qquad f'_{c^*}(t) = 1 + e^{-\frac{t^2}{2} - lt + \mathrm{O}(\ln t)}$$

$$f_{c^*}(t) = t + l - e^{-\frac{t^2}{2} - lt + \mathrm{O}(\ln t)} \quad \text{as} \quad t \to +\infty.$$

If $c^* < 0$, then the function $t \mapsto f_c(t) - t$ is unbounded, for any $c \in (c^*, 0]$.

It is possible to do better and to precise what is the term $\mathrm{O}(\ln t)$. By a method used for the Falkner-Skan equation in [20], Chapter XIV, Theorem 9.1, one can show that there exists a constant $A > 0$ such that

$$f'_{c^*}(t) - 1 \sim At^{\beta - 1} e^{-\frac{t^2}{2} - lt} \quad \text{as} \quad t \to +\infty.$$

Other asymptotic results for f_c (concave, convex-concave or concave-convex) such that the function $t \mapsto f_c(t) - t$ is unbounded, should also be obtained by applying the ideas of [20], Chapter XIV, Theorems 9.1 and 9.2. See also [24].

REMARK 12. The main ingredients used in this section are, one the one hand, Lemmas 8 and 9 that precise the behavior of f_c after a point where f_c'' vanishes and, on the other hand, the fact that the set $\mathscr{C}_{2,2}$ has at most one point on its boundary, implying that it is an interval.

5. The case $\beta \in (0,1]$ and $0 \leqslant b < 1$

Let $\beta \in (0,1]$, $a \geqslant 0$ and $0 < b < 1$. In this situation, it is easy to see that $\mathbb{R}$ can be partitioned into the four sets $\mathscr{C}'_{0,1}$, $\mathscr{C}'_{0,2}$, $\mathscr{C}'_1$ and $\mathscr{C}'_2$ where

$$\mathscr{C}'_{0,1} = \{c < 0\,;\, f_c' > 0 \text{ on } [0,T_c)\}$$

$$\mathscr{C}'_{0,2} = \{c < 0\,;\, \exists s_c \in (0,T_c) \text{ s.t. } f_c' > 0 \text{ on } [0,s_c) \text{ and } f_c'(s_c) = 0\}$$

$$\mathscr{C}'_1 = \left\{c \geqslant 0\,;\, b \leqslant f_c' \leqslant 1 \text{ and } f_c'' \geqslant 0 \text{ on } [0,T_c)\right\}$$

$$\mathscr{C}'_2 = \Big\{c \geqslant 0\,;\, \exists t_c \in [0,T_c),\, \exists \varepsilon_c > 0 \text{ s.t. } f_c' < 1 \text{ on } (0,t_c),$$
$$f_c' > 1 \text{ on } (t_c, t_c+\varepsilon_c) \text{ and } f_c'' > 0 \text{ on } (0, t_c+\varepsilon_c)\Big\}.$$

The fact that any $c \geqslant 0$ belongs to $\mathscr{C}'_1 \cup \mathscr{C}'_2$ is due inter alia to Lemma 2, item 4, which implies that f_c'' remains positive as long as $f_c' \leqslant 1$.

The arguments used in the previous section, and evoked in Remark 12, can be applied here. Some results, as Propositions 7 and 8, are still true. On the other hand, as we will see below, some other results are obtained more easily. For example, the existence and the uniqueness of a concave solution of $(\mathscr{P}_{\beta;a,b,0})$ are already known, and so it is not necessary to argue as in the previous section (see Propositions 8 and 9).

Since $\beta x(x-1) < 0$ for $x \in (0,b]$, it follows from Theorem 5.5 of [9] that there exists a unique c_* such that f_{c_*} is a concave solution of $(\mathscr{P}_{\beta;a,b,0})$. Moreover, we have $c_* < 0$. As in the previous section, this implies that $\mathscr{C}'_{0,2} = (-\infty, c_*)$. Hence $\mathscr{C}'_{0,1} = [c_*, 0)$, and if $c \in (c_*, 0)$, then f_c'' vanishes at a first point where $f_c' < 1$.

Next, proceeding in the same way as in the proof of Proposition 1, we can prove that $c^* = \sup \mathscr{C}'_1$ is finite, and hence that $\mathscr{C}'_1 = [0, c^*]$ and $\mathscr{C}'_2 = (c^*, +\infty)$. Moreover, from Lemma 8, we have $c^* \geqslant a(1-b)$. On the other hand, it follows from Lemma 7 that, if $c \in \mathscr{C}'_2$, then f_c'' vanishes at a first point where $f_c' > 1$.

All this, combined with an appropriate use of Lemmas 8 and 9, allows to state the following theorem. For more details, we refer to [5].

THEOREM 2. *Let $\beta \in (0,1]$, $a \geqslant 0$ and $b \in (0,1)$. There exist two real numbers $c_* < 0$ and $c^* \geqslant a(1-b)$ such that:*

- ▷ *f_c is not defined on the whole interval $[0,+\infty)$ if $c < c_*$;*

- ▷ f_{c_*} *is a* concave *solution of* $(\mathscr{P}_{\beta;a,b,0})$*;*
- ▷ f_c *is a* concave-convex *solution of* $(\mathscr{P}_{\beta;a,b,1})$ *for all* $c \in (c_*, 0)$*;*
- ▷ f_c *is a* convex *solution of* $(\mathscr{P}_{\beta;a,b,1})$ *for all* $c \in [0, c^*]$*;*
- ▷ f_c *is a* convex-concave *solution of* $(\mathscr{P}_{\beta;a,b,1})$ *for all* $c \in (c^*, +\infty)$.

REMARK 13. [*The case* $b = 0$] We can show similar results if $b = 0$. For details of the proof, we refer to [5].

- ▷ If $c < 0$, then $T_c < +\infty$.
- ▷ For $c = 0$, we have $f_0(t) = a$.
- ▷ There exists $c^* \geqslant a$ such that f_c is a convex solution of the problem $(\mathscr{P}_{\beta;a,b,1})$ for all $c \in (0, c^*]$ and is a convex-concave solution of $(\mathscr{P}_{\beta;a,b,1})$ for all $c > c^*$.

6. About the case $\beta > 1$

In this section, we will assume that $\beta > 1$, $a \geqslant 0$ and $b > 0$. The main difference with the case $\beta \in (0,1]$, is that Lemmas 8 and 9 do not necessarily hold anymore. In fact, it is the case if $f(t_0) = 0$, and in particular this implies that, if $a = 0$ and $b > 1$, then we have $\mathscr{C}_1 = \emptyset$ (see [9], Theorem 5.19, item 2.b), and if $a = 0$ and $0 < b < 1$, then $\mathscr{C}_1' = \emptyset$ (see [9], Theorem 6.4, item 2.b).

Another consequence is that, on the contrary to what happens in the case $\beta \in (0,1]$, where for any c the function f_c'' vanishes at most once in $[0, T_c)$, this is not necessarily true if $\beta > 1$, and numerical experimentations indicate that it is so.

Furthermore, nothing indicates whether both problems $(\mathscr{P}_{\beta;a,b,0})$ and $(\mathscr{P}_{\beta;a,b,1})$ have solutions or not.

Nevertheless, some results are still true. We start with a result about the problem $(\mathscr{P}_{\beta;a,b,0})$. Next, we prove that, if f_c' remains positive, then f_c' tends to 0 or 1 at infinity. Finally, we point some situations for which the problem $(\mathscr{P}_{\beta;a,b,1})$ has solutions.

PROPOSITION 10. *If* $b \in (0, \frac{\beta}{\beta-1}]$*, then there exists* $c_* < 0$ *such that* f_{c_*} *is a solution of the problem* $(\mathscr{P}_{\beta;a,b,0})$*. Moreover,* f_{c_*} *is concave and is the unique solution of* $(\mathscr{P}_{\beta;a,b,0})$*.*

Proof. If $b \in (0,1)$, as in the previous section, this follows from [9], Theorem 5.5. If $b \in [1, \frac{\beta}{\beta-1}]$, on the one hand, we remark that inequality (4.1) still holds, and hence it is so for the conclusions of Propositions 7 and 8. Thus, the problem $(\mathscr{P}_{\beta;a,b,0})$ has a solution. On the other hand, as we point out in Remark 8, the uniqueness of the solution of $(\mathscr{P}_{\beta;a,b,0})$ holds true for $b \in [1, \frac{\beta}{\beta-1}]$. □

PROPOSITION 11. *If* $c \in \mathbb{R}$ *is such that* $f_c' > 0$ *on* $(0, T_c)$*, then* $T_c = +\infty$ *and* f_c' *has a finite limit at infinity, equal either to* 0 *or to* 1*.*

Proof. Let $c \in \mathbb{R}$ be such that $f_c' > 0$ on $(0, T_c)$. From Lemma 6, we know that $T_c = +\infty$ and that f_c' is bounded.

If there exists a point $\tau \geqslant 0$ such that f_c'' does not change of sign on $(\tau, +\infty)$, then f_c' is monotone on this interval. Hence, f_c' has a finite limit at infinity and, by virtue of Lemma 4, this limit is equal to 0 or 1.

If we are not in the previous situation, then there exists an increasing sequence $(\tau_n)_{n\geqslant 0}$ tending to $+\infty$ such that $f_c''(\tau_n) = 0$ and $f_c'''(\tau_n) > 0$, for all $n \geqslant 0$ (notice that Lemma 1 implies that we cannot have $f_c'''(\tau_n) = 0$).

Let L_c be the function defined on $[0, +\infty)$ by (3.2), i.e.

$$\forall t \geqslant 0, \qquad L_c(t) = 3f_c''(t)^2 + \beta(2f_c'(t) - 3)f_c'(t)^2.$$

We know that L_c is decreasing and takes negative value at each τ_n since, by virtue of Lemma 2, item 3, we have $f_c'(\tau_n) < 1$. Therefore, we have $L_c(t) < 0$ for $t \geqslant \tau_0$. Moreover, since $2x^3 - 3x^2 \geqslant -1$ for $x \geqslant 0$, then $L_c(t) \geqslant -\beta$ for all $t \geqslant 0$. Hence $L_c(t)$ tends to some $\alpha < 0$ as $t \to +\infty$.

Inspired by an idea developed in [19] we will show that $f_c(t) \to +\infty$ and $f_c''(t) \to 0$ as $t \to +\infty$.

First, let us prove that $f_c(t) \to +\infty$ as $t \to +\infty$. If it is not the case, then f_c has a finite limit l at infinity (recall that f_c is increasing) and there exists a sequence $(s_n)_{n\geqslant 0}$ in $[\tau_0, +\infty)$ such that $s_n \to +\infty$ and $f_c'(s_n) \to 0$ as $n \to +\infty$.

By passing to the limit as $n \to +\infty$ in the inequalities

$$\beta f_c'(s_n)^2(2f_c'(s_n) - 3) \leqslant L_c(s_n) \leqslant L_c(\tau_0) < 0$$

we get a contradiction. Therefore $f_c(t) \to +\infty$ as $t \to +\infty$.

Next, let us prove that $f_c''(t) \to 0$ as $t \to +\infty$. Let x_n be a point of the interval (τ_n, τ_{n+1}) such that $|f_c''(t)| \leqslant |f_c''(x_n)|$ for all $t \in [\tau_n, \tau_{n+1}]$. We have $f_c'''(x_n) = 0$ and thus, from equation (1.1), one has

$$f_c''(x_n) = \frac{-\beta f_c'(x_n)(f_c'(x_n) - 1)}{f_c(x_n)}.$$

Thus, since f_c' is bounded and that $f_c(x_n) \to +\infty$ as $n \to +\infty$, we get that $f_c''(x_n) \to 0$ as $n \to +\infty$, and hence $f_c''(t) \to 0$ as $t \to +\infty$.

Now we are able to conclude. Since $f_c''(t) \to 0$ and $L_c(t) \to \alpha$ as $t \to +\infty$, we have that $2f_c'^3(t) - 3f_c'^2(t) \to \alpha$ as $t \to +\infty$. Therefore f_c' has a finite limit λ at infinity, that is a root of the polynomial $P(x) = 2x^3 - 3x^2 - \alpha$ (see Remark 14 below). Since $P(0) = -\alpha \neq 0$, by Lemma 4, we get $\lambda = 1$. $\square$

REMARK 14. In the previous proof, we used the fact that for any real polynomial P with real roots $a_1, \ldots, a_s$ and any continuous function $\varphi : [0, +\infty) \to \mathbb{R}$ such that $P(\varphi(t)) \to 0$ as $t \to +\infty$, then $\varphi(t)$ tends to a root of P as $t \to +\infty$. To prove this, note first that, for every ε small enough, the intervals $A_{j,\varepsilon} =]a_j - \varepsilon, a_j + \varepsilon[$ are disjoint. Denote by A_ε their union. On the one hand, since $P(\varphi(t)) \to 0$ as $t \to +\infty$, for all $n \geqslant 1$, there exists t_n such that $\varphi([t_n, +\infty[) \subset P^{-1}([-\frac{1}{n}, \frac{1}{n}])$. On the other hand, since

$$\bigcap_{n\geqslant 1} P^{-1}\big(\big[-\tfrac{1}{n}, \tfrac{1}{n}\big]\big) = P^{-1}(\{0\}) = \{a_1, \ldots, a_s\},$$

by a compactness argument, there exists n_ε such that $P^{-1}([-\frac{1}{n_\varepsilon},\frac{1}{n_\varepsilon}]) \subset A_\varepsilon$. Let us set $t_\varepsilon = t_{n_\varepsilon}$; one has $\varphi([t_\varepsilon,+\infty[) \subset A_\varepsilon$. Due to the continuity of φ the set $\varphi([t_\varepsilon,+\infty[)$ is an interval, and hence there exists $k \in \{1,\dots,s\}$ such that $\varphi([t_\varepsilon,+\infty[) \subset A_{k,\varepsilon}$. In other words, for $t \geqslant t_\varepsilon$ we have $|\varphi(t)-a_k| < \varepsilon$. Finally, $\varphi(t) \to a_k$ as $t \to +\infty$.

REMARK 15. In the proof of Proposition 11, we only use the positivity of β. Thus Proposition 11 implies Proposition 6, but the proof of this latter proposition is simpler and shorter, and says more, i.e. that f_c'' vanishes at most once.

PROPOSITION 12. *If $\beta \in (1,2]$ and $a>0$, then for any value of c such that $2ac \geqslant b^2-(2b-\beta)a^2$, we have $T_c=+\infty$ and $f_c'(t) \to 1$ as $t \to +\infty$.*

Proof. Let $c \in \mathbb{R}$ and denote by K_c the function defined on $[0,T_c)$ by

$$K_c(t) = 2f_c(t)f_c''(t) - f_c'(t)^2 + (2f_c'(t)-\beta)f_c(t)^2.$$

From (1.1), we easily get $K_c'(t) = 2(2-\beta)f_c(t)f_c'(t)^2$.

Assume now that f_c' vanishes, and let s_c be the first point such that $f_c'(s_c)=0$. Then f_c' and f_c are positive on $[0,s_c)$, and hence K_c is nondecreasing on $[0,s_c]$. Since $f''(s_c) \leqslant 0$, we have $K_c(s_c) = 2f_c(s_c)f_c''(s_c) - \beta f_c(s_c)^2 < 0$. This gives $K_c(0)<0$.

Consequently, if $K_c(0) \geqslant 0$, then $f_c'>0$ on $[0,T_c)$. From Proposition 11, it follows that $T_c=+\infty$ and f_c' tends to 0 or 1 at infinity. But, if $f_c'(t) \to 0$ as $t \to +\infty$, then we obtain a contradiction as above, since $K_c(t) \to -\beta l^2$ as $t \to +\infty$, where l is the limit of f_c at infinity (see Lemmas 4 and 5).

The proof is now complete, since $K_c(0) = 2ac-b^2+(2b-\beta)a^2 \geqslant 0$. □

COROLLARY 2. *If $\beta \in (1,2]$, $a>0$ and $b>0$, then the problem $(\mathscr{P}_{\beta;a,b,1})$ has infinitely many solutions.*

Proof. This follows immediately from Proposition 12. □

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