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DYNAMICS OF A LINK–TYPE INDEPENDENT ADAPTIVE EPIDEMIC MODEL

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Abstract. A link-type-independent adaptive network model of *SIS* epidemic propagation is considered. In the model links can be activated or deleted randomly regardless to the type of nodes. A four-variable pairwise ODE approximation is used to describe how the number of quantities such as number of infected nodes evolves in time. In order to investigate bifurcations in the model an invariant manifold is defined. Using the theory of asymptotically autonomous systems, results obtained for the reduced system on the manifold are extended to the full pairwise model and a non-oscillating behaviour is proven.

1. Introduction

Over the last decade investigation of epidemic propagation on networks has become more and more important [5, 7] and recent researches have also studied adaptive networks [2, 6, 10]. Several works have been motivated by the assumption that propagation of the epidemic affects the structure of the network. In a real-world example it is reasonable that in order to avoid disease susceptibles try to suspend or terminate their connections with infected ones. This has an impact on the network and as a feedback impacts the epidemic spread. In real situations, individuals do not have knowledge about the state of the others, hence they cut their links randomly and in the meantime they create new connections in order not to be separated from the social network. Hence the link creation and deletion may be type independent [8]. There are two main techniques to analyse the evolution of such propagations: (i) individual based Monte Carlo simulations and (ii) approximating ODE models. While, Monte Carlo simulations are easy to carry out, these are not suitable to obtain analytical results. In this paper a mean-field ODE system is considered and the aim is to investigate the epidemic propagation via bifurcation approaches. First, we recall the formulation of an adaptive model coupled with epidemic dynamics and second, bifurcation studies are carried out describing the system from a dynamical perspective. Our main purpose is to prove that the system cannot oscillate and tends to a steady state, regardless to the parameters. This has been conjectured in [8] but to our best knowledge, this is the first work proposing a proper mathematical proof of this.

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In recent years, the number of studies focusing on both the dynamics of the network and on the network has grown rapidly. Gross et. al [3] proposed a re-wiring model where the number of links is preserved. It means that when an SI link is cut then immediately another link is rewired to a susceptible individual. Risau-Gusman and Zanette investigated another re-wiring model where the newly created link can be an SI edge again [11]. Shaw and Schwartz generalised the problem and proposed an SIRS type model [12].

In our previous article [14] we have addressed a detailed study of bifurcations in a link-type-dependent model. It was shown that in some cases the heterogeneity of link activation and deletion parameters gives rise to stable oscillation and Hopf bifurcation. In [13] we investigated how the oscillation of the epidemic effects the network structure.

The paper is structured as follows. In Section 2 the pairwise model is formulated. Then in Section 3 an attempt to investigate the pairwise system in a standard way is made, but it leads to difficulties. In order to overcome this problem an invariant manifold is defined. In Section 4 a reduced system is considered on the manifold and a theorem describing its bifurcations is proven. Finally, in Section 5 using the theory of asymptotically autonomous systems the results of Section 4 are extended to the original system.

2. Model formulation

In this paper SIS (susceptible-infectious/infected-susceptible) epidemic spread is considered on a dynamic network with link activation and deletion. The model can be described as the incorporation of the following independent Poisson processes:

- **Infection:** Infection is passed across between each S and I node, or (SI) link, at rate τ,
- **Recovery:** Each I node recovers at rate γ, and this is independent of the network,
- **Link activation:** A non-existing link between any two of the nodes is created at rate α
- **Link deletion:** An existing link between any two of the nodes is deleted at rate ω,

We note that in this model the link activation or deletion is totally regardless to the type of nodes. This is a bit different approach compared to the general adaptive SIS model, where a link is activated(terminated) between a node of type A and B at rate α_{AB} (ω_{AB}), with $A,B \in S,I$. Moreover edges to be activated or deleted are chosen randomly.

Our main goal is to measure and determine quantities such as the average number of infected nodes or links as a function of time. This will give us the chance to analyse the network process from both graph structural and dynamical aspects. In this paper pairwise ODE approximations of the stochastic process are used.

Let us introduce variables $[I]$ and $[S]$ for the expected values of I and S type nodes and similarly $[SS]$, $[SI]$ and $[II]$ as the expectation of SS, SI and II type links.

We can now formulate the pairwise model for these expected values of the node and pair numbers. As was shown in [8], this leads us to the following system

$$[\dot{I}] = \tau[SI] - \gamma[I], \tag{2.1}$$

$$[\dot{SI}] = \gamma([II] - [SI]) + \tau([SSI] - [ISI] - [SI]) + \alpha([S][I] - [SI]) - \omega[SI], \tag{2.2}$$

$$[\dot{II}] = -2\gamma[II] + 2\tau([ISI] + [SI]) + \alpha([I]([I]-1) - [II]) - \omega[II], \tag{2.3}$$

$$[\dot{SS}] = 2\gamma[SI] - 2\tau[SSI] + \alpha([S]([S]-1) - [SS]) - \omega[SS]. \tag{2.4}$$

From the model it follows that $[S]+[I]=N$ and $[SS]+2[SI]+[II]$ equals to the number of links. In equation (2.3) and (2.4) the terms $\alpha([I]([I]-1)-[II])$ and $\alpha([S]([S]-1)-[SS])$ describe that an II or SS type link is activated at rate α. Of course, these terms aim to positively contribute to $\dot{II}$ and $\dot{SS}$, but if $[I]<1$ or $[S]<1$ these terms yield a negative contribution to the right hand sides of equations (2.3) and (2.4). A reasonable way could be to modify the terms $\alpha[I]([I]-1)$ and $\alpha[S]([S]-1)$ to be zero when $[I]$ or $[S]$ is less than one. It would yield technical difficulties, but does not lead to qualitatively different results, this will be dealt with in Section 4. The derivation can be carried out for the modified system in the same way, hence we consider the original system. There is only one point in the calculation when the original system gives false results. At that point we assume that the parameters satisfy a certain inequality ensuring that the modified system coincides with the original one. We also note that the expected value of triples $[SSI]$, $[ISI]$ is unknown, thus these quantities will be approximated using a well-known closure described below.

Now we define the closure for system (2.1)–(2.4). For the expected value of triplets the well-known closures [7] are used

$$[SSI] = \frac{n-1}{n}\frac{[SS][SI]}{[S]} \quad \text{and} \quad [ISI] = \frac{n-1}{n}\frac{[SI][SI]}{[S]}, \tag{2.5}$$

where n stands for the average degree of S type nodes, namely

$$n(t) = \frac{[SS](t)+[SI](t)}{[S](t)}. \tag{2.6}$$

Using these closures, the following self-consistent system is obtained from (2.1)–(2.4)

$$[\dot{I}] = \tau[SI] - \gamma[I], \tag{2.7}$$

$$[\dot{SI}] = \gamma([II]-[SI]) + \tau\frac{n-1}{n}\frac{[SI]([SS]-[SI])}{[S]} - \tau[SI] + \alpha([S][I]-[SI]) - \omega[SI], \tag{2.8}$$

$$[\dot{II}] = -2\gamma[II] + 2\tau\frac{n-1}{n}\frac{[SI]^2}{[S]} + 2\tau[SI] + \alpha([I]([I]-1) - [II]) - \omega[II], \tag{2.9}$$

$$[\dot{SS}] = 2\gamma[SI] - 2\tau\frac{n-1}{n}\frac{[SI][SS]}{[S]} + \alpha([S]([S]-1) - [SS]) - \omega[SS], \tag{2.10}$$

where n is given by (2.6).

Our aim is to analyse this system from a dynamical perspective as much as we can. As it can be seen from the next section we have to use different approaches to examine the system around the disease-free and endemic states. First, we investigate the disease-free steady state where the Jacobian matrix can be easily taken and a condition for transcritical bifurcation is obtained. In contrast with this endemic steady states are harder to handle and should be investigated in a different way.

3. Investigation of the full 4D system

We start with the standard way of investigating the ODEs. First, determine the disease-free steady state and investigate its local behaviour, and then turn to the endemic steady state leading to complicated formulas. Hence instead, the problem will be reduced to the investigation of the behaviour on a 2D invariant manifold.

3.1. Disease-free steady state

Now we determine the disease-free steady state and the local behaviour around it. Its coordinates are

$$[I]=0, \quad [SI]=0, \quad [II]=0, \quad [SS]=\frac{\alpha N(N-1)}{\omega+\alpha}. \tag{3.1}$$

In order to obtain the Jacobian J, we calculate its entries in terms of $[I]$, $[SI]$, $[II]$ and $[SS]$,

$$J_{11}=-\gamma, \quad J_{12}=\tau, \quad J_{13}=J_{14}=0,$$

$$J_{21}=\alpha(N-2[I])+\tau\frac{[SI]([SS]-[SI])}{[S]^2},$$

$$J_{22}=-\gamma-\tau-\alpha-\omega+\tau\Big(\frac{[SI]([SS]-[SI])}{([SS]+[SI])^2}+\frac{n-1}{n}\frac{[SS]-2[SI]}{[S]}\Big),$$

$$J_{23}=\gamma, \quad J_{24}=\tau\Big(\frac{[SI]([SS]-[SI])}{([SS]+[SI])^2}+\tau\frac{n-1}{n}\frac{[SI]}{[S]}\Big),$$

$$J_{31}=\alpha(2[I]-1)+2\tau\frac{[SI]^2}{[S]^2}, \quad J_{32}=2\tau+2\tau\Big(\frac{[SI]^2}{([SS]+[SI])^2}+\frac{n-1}{n}\frac{2[SI]}{[S]}\Big),$$

$$J_{33}=-2\gamma-(\alpha+\omega), \quad J_{34}=2\tau\frac{[SI]^2}{([SS]+[SI])^2},$$

$$J_{41}=\alpha(-2N+2[I]+1)-2\tau\frac{[SI][SS]}{[S]^2}, \quad J_{42}=2\gamma-2\tau\Big(\frac{[SI][SS]}{([SS]+[SI])^2}+\frac{n-1}{n}\frac{[SS]}{[S]}\Big),$$

$$J_{43}=0, \quad J_{44}=-(\alpha+\omega)-2\tau\Big(\frac{[SI][SS]}{([SS]+[SI])^2}+\frac{n-1}{n}\frac{[SI]}{[S]}\Big).$$

Substituting the coordinates of the disease-free steady state (3.1) into the Jacobian matrix, we obtain

$$J = \begin{pmatrix} -\gamma & \tau & 0 & 0 \\ \alpha N & \tau(k-2)-\gamma-\alpha-\omega & \gamma & 0 \\ -\alpha & 2\tau & -2\gamma-\alpha-\omega & 0 \\ \alpha(-2N+1) & 2\gamma-2\tau(k-1) & 0 & -\alpha-\omega \end{pmatrix},$$

where

$$k = \frac{\alpha(N-1)}{\alpha+\omega}.$$

The last column has only one non-zero element $-\alpha-\omega$ which is a negative eigenvalue of J. Therefore the remaining three eigenvalues can be calculated from the upper-left 3×3 matrix

$$\tilde{J} = \begin{pmatrix} -\gamma & \tau & 0 \\ \alpha N & \tau(k-2)-\gamma-\alpha-\omega & \gamma \\ -\alpha & 2\tau & -2\gamma-\alpha-\omega \end{pmatrix},$$

The determinant of $\tilde{J}$ can be expanded along the first row,

$$\begin{aligned} \det(\tilde{J}) =& -\gamma((\tau(k-2)-\gamma-\alpha-\omega)(-2\gamma-\alpha-\omega)-2\tau\gamma) \\ & -\tau(\alpha N(-2\gamma-\alpha-\omega)+\alpha\gamma)+0 \\ =& \tau(\gamma(k-2)(2\gamma+\alpha+\omega)+2\gamma^2+\alpha N(2\gamma+\alpha+\omega)-\alpha\gamma) \\ & -\gamma(\gamma+\alpha+\omega)(2\gamma+\alpha+\omega). \end{aligned}$$

Solving the equation $\det(\tilde{J}) = 0$ for τ yields that the transcritical bifurcation may occur at

$$\tau_c = \frac{\gamma(\gamma+\omega+\alpha)(2\gamma+\omega+\alpha)}{\gamma(k-2)(2\gamma+\alpha+\omega)+2\gamma^2+\alpha N(2\gamma+\alpha+\omega)-\alpha\gamma}. \tag{3.2}$$

Using $\alpha(N-1) = k(\alpha+\omega)$ simple algebra shows

$$\gamma(k-2)(2\gamma+\alpha+\omega)+2\gamma^2+\alpha N(2\gamma+\alpha+\omega)-\alpha\gamma = (\gamma+\alpha+\omega)(\alpha N+2\gamma(k-1)).$$

Thus (3.2) can be simplified to

$$\tau_c = \frac{\gamma(2\gamma+\alpha+\omega)}{\alpha N+2\gamma(k-1)}. \tag{3.3}$$

It is known that $\det(\tilde{J}) = 0$ is not a sufficient condition for transcritical bifurcation. A possible way to prove that a transcritical bifurcation occurs is to apply Theorem 4.1 in [1]. Since the global behaviour will be characterised later we skip the local investigation.

3.2. The existence of a 2D invariant manifold

In this section we introduce a method which helps us characterising the behavior of the system around the endemic steady states. In our case direct calculation of the endemic steady state is not feasible. In order to overcome this difficulty a two-dimensional invariant manifold in the four-dimensional state space will be defined. The behaviour of the system is studied first in this manifold. The manifold is defined by conservation relations for the number of edges. These are dealt with first.

Of course, in general k differs from the average degree of S type nodes (2.6), but as it can be seen from the following statements, these quantities are equal in the invariant manifold. Now an important property of the average degree k will be proven. Let the number of edges at time t be

$$E(t) = [SS](t) + 2[SI](t) + [II](t). \tag{3.4}$$

Then, we have the following statement.

PROPOSITION 1. *The average degree of the network at any steady state is*

$$k = \frac{\alpha(N-1)}{\alpha+\omega}. \tag{3.5}$$

Moreover, $E(t) = (E(0)-kN)e^{-(\alpha+\omega)t} + kN$*, hence if* $E(0) = kN$*, then* $E(t) = kN$ *for all* t*.*

Proof. Differentiating $E(t)$ and using equations (2.8)–(2.10) we have,

$$\begin{aligned}
\dot{E}(t) =& 2\alpha[S][I] - 2\alpha[SI] - 2\omega[SI] + \alpha[S]([S]-1) \\
&- \alpha[SS] - \omega[SS] + \alpha[I]([I]-1) - \alpha[II] - \omega[II] \\
=& \alpha([S]+[I])^2 - \alpha([S]+[I]) - (\alpha+\omega)E(t) \\
=& \alpha N^2 - \alpha N - (\alpha+\omega)E(t).
\end{aligned}$$

Solving this differential equation for E yields

$$E(t) = (E(0)-kN)e^{-(\alpha+\omega)t} + kN.$$

The number of links at the steady state equals to the limit of $E(t)$ at infinity,

$$\lim_{t\to\infty} E(t) = \frac{\alpha N(N-1)}{\alpha+\omega}.$$

Dividing by N the average degree is obtained as $k = \frac{\alpha(N-1)}{\alpha+\omega}$. Finally, using $E(0) = kN$ we have

$$E(t) = \frac{\alpha N(N-1)}{\alpha+\omega}, \quad \forall t \geqslant 0. \quad \square$$

From now we suppose that $k > 1$ holds. We also suppose $E(0) = kN$, i.e. for any $t \geqslant 0$ we have $E(t) = [SS] + 2[SI] + [II] = kN$. For further investigations we follow the same method described in [9], but first we will need the following lemma.

LEMMA 1. *Let us introduce the variable* $A(t) = [SS] + [SI] - k[S]$*, then* A *satisfies*

$$\dot{A} = UA + \gamma(E - kN), \tag{3.6}$$

where $U = -(\alpha + \omega + \gamma + \tau\frac{[SI]}{[S]})$.

Proof. Let us introduce $\tilde{A}(t) = A(t) + k[S] = [SS] + [SI]$. Differentiating $\tilde{A}(t)$ and using (2.7)–(2.10) we have

$$\begin{aligned}\dot{\tilde{A}} =& [\dot{SS}] + [\dot{SI}] \\ =& \gamma([II] - [SI] + 2[SI]) + \tau\frac{n-1}{n}\frac{[SI]}{[S]}([SS] - [SI] - 2[SS]) \\ & - \tau[SI] + \alpha[S][I] + \alpha[S]([S]-1) - \alpha([SI] + [SS]) - \omega([SI] + [SS]).\end{aligned}$$

Since $[II] + [SI] = E - \tilde{A}$ and $[S] + [I] = N$ we have,

$$\dot{\tilde{A}} = \gamma(E - \tilde{A}) - \tau\frac{n-1}{n}\frac{[SI]}{[S]}\tilde{A} - \tau[SI] + \alpha[S](N-1) - (\alpha + \omega)\tilde{A}.$$

Using (3.5) we obtain that $\alpha[S](N-1) = (\alpha + \omega)k[S]$, thus

$$\dot{\tilde{A}} = \gamma(E - \tilde{A}) - \tau\frac{n-1}{n}\frac{[SI]}{[S]}\tilde{A} - \tau[SI] + (\alpha + \omega)(k[S] - \tilde{A}). \tag{3.7}$$

Substituting (3.7) into $A = \tilde{A} - k[S]$ yields

$$\dot{A} = \gamma(E - \tilde{A}) - \tau\frac{n-1}{n}\frac{[SI]}{[S]}\tilde{A} - \tau[SI] - (\alpha + \omega)A - k\gamma[I] + k\tau[SI].$$

The term $\gamma(E - \tilde{A}) - k\gamma[I]$ can be written as,

$$\gamma(E - \tilde{A}) - k\gamma[I] = \gamma(E - \tilde{A} - kN + k[S]) = \gamma(E - kN - A).$$

From (2.6) we know that $\frac{\tilde{A}}{[S]} = n$, hence

$$\begin{aligned}k\tau[SI] - \tau[SI] - \tau\frac{n-1}{n}\frac{[SI]}{[S]}\tilde{A} &= \tau[SI]\Big(k - 1 - \frac{n-1}{n}\frac{\tilde{A}}{[S]}\Big) = \tau[SI]\Big(k - 1 - \Big(\frac{\tilde{A}}{[S]} - 1\Big)\Big) \\ &= \tau\frac{[SI]}{[S]}(k[S] - \tilde{A}) = -\tau\frac{[SI]}{[S]}A.\end{aligned}$$

Putting all together we get

$$\dot{A} = -\Big(\alpha + \omega + \gamma + \tau\frac{[SI]}{[S]}\Big)A + \gamma(E - kN). \quad \square$$

We then prove the following conservation relations, which will be essential to determine the invariant manifold mentioned earlier.

PROPOSITION 2. *Assuming* $E(0) = kN$ *where the average degree* k *is given by* (3.5) *and* $[SS](0)+[SI](0) = k[S](0)$, *the solutions of* (2.7)–(2.10) *satisfy*

$$[SS]+[SI] = k[S], \tag{3.8}$$
$$[SI]+[II] = k[I]. \tag{3.9}$$

Proof. Since the sum of the equations, $[SS]+2[SI]+[II] = kN$, is known to hold according to Proposition 1, only one of the relations needs to be proven. We prove (3.8), i.e. show $A(t) = 0$, $t \geqslant 0$. Because $E(0) = kN$ the differential equation (3.6) takes the form

$$\dot{A} = UA.$$

Since U is continuous (supposing that $[S] \neq 0$) the solution of the equation is unique. The initial condition $A(0) = 0$ implies that $A(t) = 0$ for all t. □

It is important to note that (3.8) and (3.9) are related to the average degree formula defined by (2.6). The equations show that the average degree of susceptible nodes and the average degree of infected nodes are the same. It also means that these quantities are equal to the average degree at the steady state. Supposing $E(0) = kN$ and $[SS](0)+[SI](0) = k[S](0)$ we have $E(t) = kN$ and $A(t) = 0$ for all $t \geqslant 0$, i.e. the following equations hold,

$$[SS]+2[SI]+[II] = kN, \tag{3.10}$$
$$[SS]+[SI] = k[S], \tag{3.11}$$

the first relation follows from Proposition 1 and second from Proposition 2. Summarising we have:

THEOREM 1. *The four-dimensional system* (2.7)–(2.10) *has an invariant two-dimensional manifold(in fact a plane) given by equations* (3.10)–(3.11).

In the following section a detailed investigation of the solutions in the invariant manifold is carried out.

4. The behaviour of the 2D reduced system

Let us consider system (2.7)–(2.10) on the manifold given by equations (3.10)–(3.11). The variables in the invariant manifold will be $[S]$ and $[SS]$, all other variables are expressed in the terms of these. Using $[S] = N-[I]$ and $[SI] = k[S]-[SS]$ leads us to the following system,

$$\dot{[S]} = \gamma N - (\gamma + k\tau)[S] + \tau[SS], \tag{4.1}$$
$$\dot{[SS]} = 2(k[S]-[SS])\left(\gamma - \frac{\tau(k-1)[SS]}{k[S]}\right) + \alpha[S]([S]-1) - (\alpha+\omega)[SS]. \tag{4.2}$$

Our aim now is to characterise the behaviour around the steady states of system (4.1)–(4.2). From now we consider this system with zeros on the left hand side. These equations determine curves in the $([S],[SS])$ plane, lying in the domain given by $0 \leqslant [S] \leqslant N$ and $0 \leqslant [SS] \leqslant kN$. The first inequality is trivial and the second one will be clarified soon. Instead of solving the system, the behaviour is studied by using the direction field. The first equation can be easily characterised as a line connecting the points $\left(\frac{\gamma N}{\gamma+k\tau},0\right)$ and (N,kN). Equation (4.2) needs more involved investigation to characterise its properties. It is a second degree polynomial in $[S]$ and this enables us to express it in the form,

$$0 = a[SS]^2 + b[SS] + c, \tag{4.3}$$

where

$$a = 2\frac{\tau(k-1)}{k[S]}, \quad b = -(2\tau(k-1)+2\gamma+\alpha+\omega), \quad c = 2k[S]\gamma+\alpha[S]([S]-1).$$

We note that both a and c depend on $[S]$ but for the sake of simplicity this is not indicated in the notations. Before we start to examine this polynomial form we take a look at its discriminant.

PROPOSITION 3. *The discriminant of equation* (4.3)*, $D = b^2-4ac$, is always non-negative. Moreover it can be zero only if $[S] = N$ and $\tau(k-1) = \gamma + \frac{\alpha+\omega}{2}$.*

Proof. First, we expand b^2 and $4ac$,

$$b^2 = 4\tau^2(k-1)^2 + 4\gamma^2 + (\alpha+\omega)^2 + 8\tau(k-1)\gamma + 4\tau(k-1)(\alpha+\omega) + 4\gamma(\alpha+\omega),$$

$$4ac = 16\tau(k-1)\gamma + 8\tau\frac{k-1}{k}\alpha([S]-1).$$

Using $[S] \leqslant N$ and $\alpha(N-1) = k(\alpha+\omega)$ from (3.5) leads to,

$$f := 16\tau(k-1)\gamma + 8\tau(k-1)(\alpha+\omega) \geqslant 4ac.$$

Hence it follows,

$$\begin{aligned} b^2-4ac \geqslant b^2 - f =& 4\tau^2(k-1)^2 + 4\gamma^2 + (\alpha+\omega)^2 - 8\tau(k-1)\gamma \\ & - 4\tau(k-1)(\alpha+\omega) + 4\gamma(\alpha+\omega) \\ =& (2\tau(k-1) - 2\gamma - (\alpha+\omega))^2 \geqslant 0. \end{aligned}$$

As it can be seen from the inequalities that $b^2 = 4ac$ holds only if $[S] = N$ and $\tau(k-1) = \gamma + \frac{\alpha+\omega}{2}$. $\square$

Using (4.3) the curve can be given as the union of an upper($[SS]_U$) and a lower($[SS]_L$) branch. Both curves are given in terms of $[S]$,

$$[SS]_U([S]) = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad [SS]_L([S]) = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

with $[SS]_U(0) = [SS]_L(0) = 0$. We note that both a and c depends on $[S]$. Furthermore, we would like to ensure that $[SS]_L > 0$ for any $[S] > 0$. Simple algebra shows that if inequality

$$\gamma > \frac{\alpha + \omega}{2(N-1)} \tag{4.4}$$

holds, then $[SS]_L$ is positive. From now we suppose that the parameters satisfy (4.4). This assumption enables us to overcome the technical difficulties caused by the term $\alpha[S]([S]-1)$ in equation (2.4) in the case when $[S] < 1$. We note that for large enough networks, i.e. for large N, the inequality is automatically satisfied.

PROPOSITION 4. $[SS]_U$ $([SS]_L)$ *as a function of* $[S]$ *is concave (convex) in* $[0,N]$. *Moreover* $[SS]_L$ *is monotonically increasing in* $[0,N]$.

Proof. The function $[SS]_U$ can be written as

$$[SS]_U = (-b + \sqrt{b^2 - 4P[S] - 4Q})R[S]$$

where $P = \frac{2\tau(k-1)\alpha}{k} > 0$, $Q = 2\tau(k-1)(2\gamma - \frac{\alpha}{k})$ and $R = \frac{k}{4\tau(k-1)} > 0$.

Taking the second derivative we have,

$$\frac{d^2[SS]_U}{d[S]^2} = -4P^2R[S](b^2 - 4P[S] - 4Q)^{-\frac{3}{2}} - 4PR(b^2 - 4P[S] - 4Q)^{-\frac{1}{2}} < 0.$$

Of course, the same can be applied to $[SS]_L = (-b - \sqrt{b^2 - 4P[S] - 4Q})R[S]$ with positive second derivative.

To prove the monotonicity we take the first derivative of $[SS]_L$,

$$\frac{d[SS]_L}{d[S]} = \frac{2PR[S]}{\sqrt{b^2 - 4P[S] - 4Q}} + (-b - \sqrt{b^2 - 4P[S] - 4Q})R.$$

Simple algebra shows $-b > \sqrt{b^2 - 4P[S] - 4Q}$ and using that P and R are positive we get $\frac{d[SS]_L}{d[S]} > 0$. $\square$

For the further investigation it will be necessary to determine $[SS]_U(N)$, $[SS]_L(N)$.

PROPOSITION 5. *The following statements are true,*
(i) *If* $\tau(k-1) \geqslant \gamma + \frac{\alpha+\omega}{2}$, *then* $[SS]_U(N) = kN$.
(ii) *If* $\tau(k-1) < \gamma + \frac{\alpha+\omega}{2}$, *then* $[SS]_L(N) = kN$.

Proof. Using simple manipulation,

$$(2akN + b)^2 = b^2 - 4ac$$

can be rearranged to

$$k(\alpha + \omega) = \alpha(N-1).$$

If $2akN+b \geqslant 0$, then $2akN+b=\sqrt{b^2-4ac}$ which means that $[SS]_U(N)=kN$. Similarly $2akN+b<0$ implies, that $[SS]_L(N)=kN$. The only thing remained is to expand the term $2akN+b$ as

$$2akN+b=\frac{4\tau(k-1)}{kN}kN-2\tau(k-1)-2\gamma-\alpha-\omega.$$

Thus $2akN+b \geqslant 0$ is equivalent to the given inequalities. $\square$

Summarising the properties proven above about the restricted system (4.1)–(4.2) the following can happen:

- Scenario (A): If $\tau(k-1) \geqslant \gamma+\frac{\alpha+\omega}{2}$, then the upper part crosses the linear function at (N,kN). Since $[SS]_U$ is concave it is always greater than or equal to the isocline given by $[\dot{S}]=0$. Moreover since $[SS]_L$ is convex and monotonically increasing it intersects exactly once the isocline in the endemic steady state, see Figure 1.

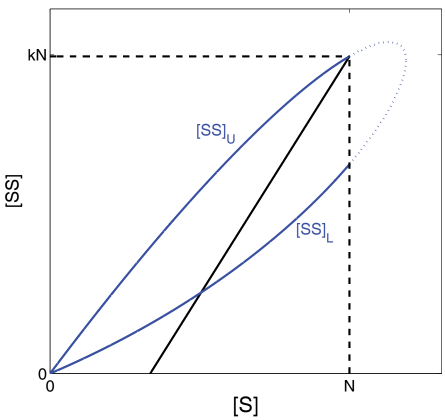

Figure 1: *The phase plane* $([S],[SS])$ *in scenario A, the blue curve corresponds to* $[SS]_U$ *and* $[SS]_L$ *and the black line to* $[\dot{S}]=0$.

- Scenario (B): If $\tau(k-1)<\gamma+\frac{\alpha+\omega}{2}$, then the lower part either
  1. crosses the linear function in (N,kN) and another intersection point exists in the domain, see Figure 2 (left panel),
  2. is tangential to or crosses the linear function in (N,kN) but in the latter case the second intersection point is outside the rectangle, see Figure 2 (right panel)

  In this scenario transcritical bifurcation occurs when $[SS]_L$ touches the line $\dot{S}=0$ in (N,kN). It will be proven that the isocline is tangential to SS_L when $\tau=\tau_c$,

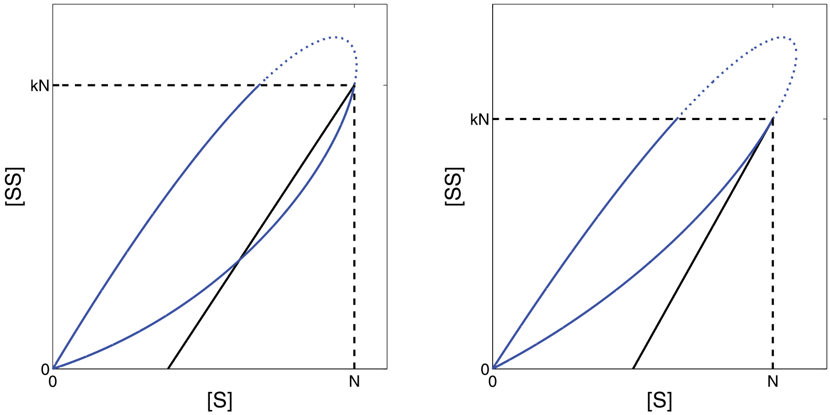

Figure 2: *Phase planes in scenario B, the blue curve corresponds to* $[SS]_U$ *and* $[SS]_L$ *and the black line to* $[\dot{S}] = 0$.

see (3.3). When $\tau > \tau_c$ the line $\dot{S} = 0$ creates an additional endemic steady state. Of course, $[SS]_U$ and the isocline have no intersection points.

Continuing the characterisation of system (4.1)–(4.2) we now investigate the direction field. We have to obtain the signs of the derivatives around the steady states. This enables us to determine the stability of the equilibria and it will be necessary to sketch the phase portrait which describes our system from a dynamical aspect.

The $[SS]_U$ and $[SS]_L$ curves divide the plane into an upper, a middle and a lower part, see Figure 1. It is easy to see that in the upper and lower part $[\dot{SS}] > 0$. In order to prove that it is enough to check the sign of $[\dot{SS}]$ along the $[S] = 0$ and $[SS] = 0$ axes. Moreover, we prove below that $[\dot{SS}]$ is negative along the line given by

$$[SS]_M = -\frac{b}{2a}$$

which lies between $[SS]_U$ and $[SS]_L$. Substituting $[SS]_M$ into

$$[\dot{SS}] = aSS^2 + bSS + c$$

we obtain $[\dot{SS}] = b^2 - 4ac$ which is positive according to Proposition 3. Considering the sign of $[\dot{S}]$, the isocline given by (4.1) divides the plane into a left and a right part. It is obvious that the sign of $[\dot{S}]$ is positive in the left and negative in the right section. Using these statements now we can create a sketch of the direction field. Figure 3 and 4 show scenario A and B, respectively. The arrows represent the direction of the trajectories.

It can be seen from the figures that if only the disease-free steady state exists it is always stable. When the endemic steady state appears as a stable state the disease-free state loses its stability and becomes unstable. The transcritical bifurcation can only happen in scenario B and in this case it occurs exactly when the tangent of $[SS]_L$ gets

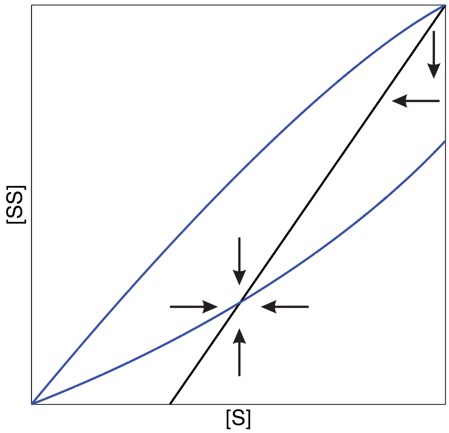

Figure 3: *Direction field of system* (4.1)–(4.2) *in scenario A*

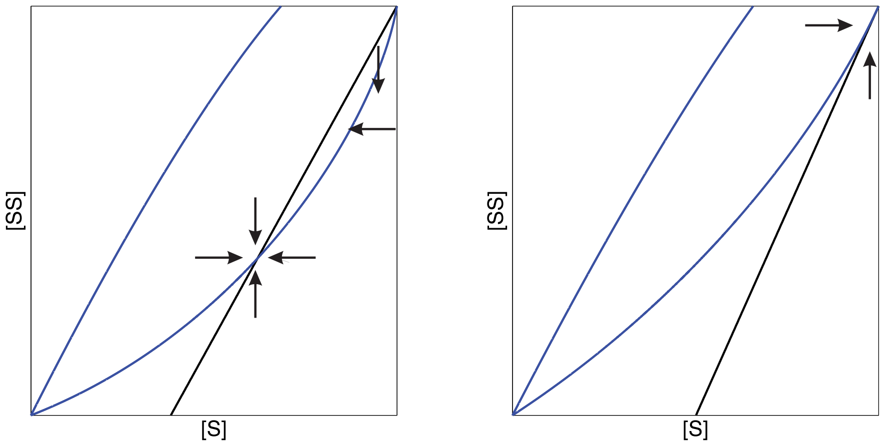

Figure 4: *Direction field of system* (4.1)–(4.2) *in scenario B*

steeper than the tangent of the line $[\dot{S}] = 0$. Since the derivative of the isocline is $\frac{\gamma+k\tau}{\tau}$ (differentiating (4.1) the bifurcation is determined by the following equation,

$$\frac{d[SS]_L}{d[S]}\Big|_{[S]=N} = \frac{\gamma}{\tau} + k \tag{4.5}$$

where the left hand side denotes the derivative of $[SS]_L$ at $[S] = N$. Solving equation (4.5) leads to the same critical value obtained from the Jacobian in (3.3). The behaviour of system (4.1)–(4.2) can be fully characterised as follows.

THEOREM 2. *In system* (4.1)–(4.2) *a transcritical bifurcation occurs at*

$$\tau_c = \frac{\gamma(2\gamma+\alpha+\omega)}{\alpha N + 2\gamma(k-1)}. \tag{4.6}$$

(i) *If* $\tau < \tau_c$*, then there is no endemic steady state and the disease-free steady state is globally stable.*

(ii) *If* $\tau > \tau_c$*, then the endemic steady state is globally stable and the disease-free state is unstable.*

Proof. Starting with scenario A we can see that inequality $\tau(k-1) \geqslant \gamma + \frac{\alpha+\omega}{2}$ implies

$$\tau \geqslant \frac{2\gamma+\alpha+\omega}{2(k-1)} = \frac{\gamma(2\gamma+\alpha+\omega)}{2\gamma(k-1)} > \tau_c.$$

Turning to scenario B all we have to prove is that τ_c is a solution of equation (4.5). Using the notation $[SS] = g([S])$ we recall the polynomial form of the $[SS]$ curve (4.3) as

$$0 = a([S])g([S])^2 + bg([S]) + c([S]). \tag{4.7}$$

Differentiating both sides lead us to

$$g'([S]) = -\frac{a'([S])g^2([S]) + c'([S])}{2a([S])g([S]) + b}. \tag{4.8}$$

The derivative should be determined at $[S] = N$ corresponding to (N, kN). Substituting $[S] = N$ and using $g(N) = kN$ and $\alpha(N-1) = k(\alpha+\omega)$ simple algebra shows

$$\left.\frac{d[SS]_L}{d[S]}\right|_{[S]=N} = g'(N) = \frac{\alpha N}{-2\tau(k-1) + 2\gamma + \alpha + \omega} + k. \tag{4.9}$$

Substituting this into (4.5) and expressing τ we arrive at (4.6).

If $\tau > \tau_c$ then $g'(N) > \frac{\gamma}{\tau} + k$, i.e. there is another intersection point of the two curves. Figure 3 and 4 (left hand side) show that for $\tau > \tau_c$ the endemic steady state is globally stable and the disease-free state is unstable.

If $\tau < \tau_c$, then $g'(N) < \frac{\gamma}{\tau} + k$, i.e. the only intersection point is the disease-free steady state and Figure 4 (right hand side) shows that it is globally stable. □

5. The global behaviour of the 4D system

We return to the original system (2.7)–(2.10) and show how it inherits the major properties of the reduced system. Without supposing $E(0) = kN$ and using $[S] = N - [I]$ and $[SI] = A(t) + k[S] - [SS]$ the following system is obtained,

$$\dot{[S]} = \gamma N - (\gamma + k\tau)[S] + \tau[SS] - \tau A(t), \tag{5.1}$$

$$\dot{[SS]} = 2(A(t) + k[S] - [SS])\left(\gamma - \frac{\tau(n(t)-1)[SS]}{n(t)[S]}\right) + \alpha[S]([S]-1) - (\alpha+\omega)[SS]. \tag{5.2}$$

Now the time dependence of average degree $n(t)$ is emphasised because it will play an important role.

First, it will be proven that the right hand sides of the autonomous system (4.1)–(4.2) and system (5.1)–(5.2) are equal as t goes to infinity. System (5.1)–(5.2) is an *asymptotically autonomous system*. A general reference for the theory of asymptotically autonomous systems is [15]. In this paper Thieme showed that under certain conditions the solutions of the non-autonomous system converge to the stable steady state of the autonomous system. Using this theorem the behaviour of the four dimensional system (2.7)–(2.10) can be fully characterised.

PROPOSITION 6. *System* (5.1)–(5.2) *is asymptotically autonomous, i.e. the right hand side of* (5.1)–(5.2) *tends to that of* (4.1)–(4.2) *as* $t \to \infty$.

Proof. It is enough to prove that $n(t) \to k$ and $A(t) \to 0$ as $t \to \infty$. The first convergence follows from Proposition 1 and to obtain the second one equation (3.6) needs to be investigated. It takes the form,

$$\dot{A} = -\Big(\alpha + \omega + \gamma + \tau \frac{[SI]}{[S]}\Big)A + \gamma(E(t) - kN).$$

Using the notation $f(t) = \frac{[SI]}{[S]}(t)$ and the formula of $E(t)$ from Proposition 1, we have

$$\dot{A} = -(\alpha + \omega + \gamma + \tau f(t))A + ce^{-(\alpha+\omega)t} \tag{5.3}$$

where $c = \gamma(E(0) - kN)$. In order to obtain the limit of A at infinity we define two additional systems of equations, the solutions of which are lower and upper bounds of A.

Figures 3 and 4 show that for the steady state value of $[S]$

$$\frac{\gamma N}{\gamma + k\tau} \leqslant [S] \leqslant N$$

holds, thus the limit of $[S]$ is positive at infinity. Moreover since $[S]$ is continuous, positive at $t = 0$ and cannot be zero it is true that on every compact interval $[0,T]$ it has a positive lower bound. Taking into account these properties, it can be easily seen that a time independent lower bound $m_{[S]}$ exists, i.e. $m_{[S]} \leqslant [S](t)$, $\forall t \geqslant 0$. The numerator of $f(t)$ is also bounded, because from Proposition 1 it is trivial that

$$0 \leqslant [SI] \leqslant E(t) \leqslant \max(E(0), kN).$$

Summarising these statements $f(t)$ is bounded as

$$0 \leqslant f(t) \leqslant M, \quad \forall t \geqslant 0$$

with some constant M. Now we define a lower bound of A as the function A_L satisfying,

$$\dot{A}_L = -(\alpha + \omega + \gamma + \tau M)A_L + ce^{-(\alpha+\omega)t} \tag{5.4}$$

and an upper-bound A_U satisfying

$$\dot{A}_U = -(\alpha+\omega+\gamma)A_U + ce^{-(\alpha+\omega)t}. \tag{5.5}$$

It is easy to see that the right hand sides of (5.4) and (5.5) are lower and upper bounds of the right hand side of (5.3). It is known that the solutions A_L and A_U are lower and upper bounds of A, see [4],

$$A_L(t) \leqslant A(t) \leqslant A_U(t), \quad t \geqslant 0.$$

Both A_L and A_U converges to zero at infinity, since

$$A_L(t) = \frac{c}{\gamma+\tau M}e^{-(\alpha+\omega)t} + d_1 e^{-(\alpha+\omega+\gamma+\tau M)t} \to 0 \quad (t\to\infty),$$

$$A_U(t) = \frac{c}{\gamma}e^{-(\alpha+\omega)t} + d_2 e^{-(\alpha+\omega+\gamma)t} \to 0 \quad (t\to\infty)$$

where d_1, d_2 are real constants. Using the squeeze theorem we get $A(t)\to 0$ as t goes to infinity. $\square$

Since the asymptotically autonomous property is proven we can apply the results of [15] and describe system (5.1)–(5.2) from a dynamical perspective. Now we recall Theorem 4.9 from [15].

THEOREM 3. (Thieme) *Let us define an asymptotically autonomous system*

$$\dot{x}(t) = f(t,x), \tag{5.6}$$

with limit equation

$$\dot{y} = g(y), \tag{5.7}$$

i.e. $f(t,x)\to g(x)$, for $t\to\infty$ locally uniformly in $x\in\mathbb{R}^2$. Let X be a subset of $\mathbb{R}^2$ containing no periodic orbits and no cyclically chained equilibria of system (5.7)*, moreover we assume that X contains isolated equilibria only. Then every bounded forward solution of system* (5.7) *and every bounded forward solution of system* (5.6) *in X converges towards an equilibrium of system* (5.7)*.*

Using Theorem 3 the dynamics of system (5.1)–(5.2) can be characterised by the following theorem.

THEOREM 4. *In system* (5.1)–(5.2) *a transcritical bifurcation occurs at*

$$\tau_c = \frac{\gamma(2\gamma+\alpha+\omega)}{\alpha N + 2\gamma(k-1)}. \tag{5.8}$$

(i) *If $\tau<\tau_c$, then there is no endemic steady state and the disease-free steady state is globally stable.*

(ii) *If $\tau>\tau_c$, then the endemic steady state is globally stable and the disease-free state is unstable.*

Proof. We apply Theorem 3 in the context when the autonomous system (4.1)–(4.2) plays the role of system (5.7) and similarly system (5.1)–(5.2) corresponds to system (5.6). From Theorem 2 it is trivial that no periodic orbits or cyclically chained equilibria exists and every equilibria is isolated.

If $\tau < \tau_c$ we choose the subset as $X = [0,N] \times [0,kN]$. Using Theorem 2 the globally stable disease-free steady state is the only equilibria of system (4.1)–(4.2) and using Theorem 3 it follows that every solution of system (5.1)–(5.2) tends to it.

If $\tau > \tau_c$ we define two subsets as $X_1 = (N,kN)$, i.e. the disease-free state and $X_2 = X \setminus X_1$, i.e. the entire rectangle X except the upper-right cornerpoint. Both sets may contain only one steady state of system (4.1)–(4.2). If a solution of system (5.1)–(5.2) is initiated from a point of X_2 we apply Theorem 3 with subset X_2. From Theorem 2 the endemic steady state of system (4.1)–(4.2) is globally stable and because this is the only equilibrium in X_2 Theorem 3 shows that every solution of system (5.1)–(5.2) converges to it. Of course, the disease-free steady state of system (4.1)–(4.2) is also a disease-free state of system (5.1)–(5.2). Hence solutions initiated from the point X_1 remain there. □

Since S has a limit at infinity $I = N - S$ has as well. Similarly, $SS, A(t)$ and $E(t)$ also have a limit and from that it follows $[SI] = A(t) + k[S] - [SS]$ and $[II] = E(t) - 2[SI] - [SS]$ have limits too. Using Theorem 4 also shows that the original four-dimensional system (2.7)–(2.10) cannot have periodic solutions and for $\tau < \tau_c$ it has a globally stable disease-free steady state, while for $\tau > \tau_c$ it has a globally stable endemic steady state.

6. Discussion

In this paper the dynamics of SIS type epidemic propagation was investigated on an adaptive network. Edges can be activated or deleted randomly regardless to the type of nodes. A four-variable pairwise ODE model was considered and an invariant manifold was defined. There are two properties which characterise the manifold. The average degree of S-type nodes and average degree of the entire network are equal and the number of links remain the same at any time. First a two-dimensional system was defined on the manifold and a condition for transcritical bifurcation was proven. The main idea of the proof was to determine the direction field of the system which carried out from graphical properties of the equations. We then turned back to original equations and defined an asymptotically autonomous system. Applying results of [15] we obtained that the condition for the transcritical bifurcation can be extended from the manifold to the entire state space. A topic of further investigation could be the relation of the stochastic model to the pairwise ODE approximation. The stochastic simulation enables us to understand how the network structure is varying in time as the process evolves. It will be the subject of future work.

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