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NONEXISTENCE OF SOLUTIONS FOR SECOND–ORDER INITIAL VALUE PROBLEMS

DANIEL C. BILES

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Abstract. We consider nonexistence of solutions for second-order initial value problems. Two results are given: one in which the problems are singular in the time variable, and one in which the problems are singular in both the time and state variables.

We consider nonexistence of solutions to singular second-order initial value problems. The results and proofs were originally motivated by Proposition 3.2 in [6]. Existence of solutions to singular differential equations has received a great deal of attention – see, for example, the monograph [1].

For more recent results regarding second-order problems, see [2], [4], [7], [9], [10], [12], [13], [16] and [17]. On the other hand, sometimes nonexistence can be trivial: For example, if f is not Lebesgue integrable in a neighborhood of 0, then clearly $x''(t) = f(t)$, $x(0) = x_0$, $x'(0) = x_1$ has no Carathéodory solution. Results in the literature for nonexistence for singular second-order differential equations typically involve boundary conditions, see for example, [3], [5], [11], [14] and [15]. In [8], existence and nonexistence of positive solutions are studied for the problem $x'' = f(t,x,x')$, $x(0) = 0$, $x'(0) = 0$.

We begin with the following definition.

DEFINITION 1. u is a solution to the initial value problem

$$p(t)u''(t) = g(t,u(t),u'(t))$$
$$u(0) = \alpha, \quad u'(0) = \beta$$

if there exists a $T > 0$ such that all of the following are satisfied:

i) u, u' are absolutely continuous on $[0,T]$,
ii) $p(t)u''(t) = g(t,u(t),u'(t))$ a.e. on $[0,T]$,
iii) $u(0) = \alpha$, $u'(0) = \beta$.

We define solution for the problem in Theorem 2 below similarly.

Throughout the paper, we assume a,b,f,p,q and u are real-valued.

Our first result is the following:

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THEOREM 1. *Assume there exists a* $\bar{\delta} > 0$ *such that* $p(t) > 0$ *a.e. on* $(0, \bar{\delta})$ *and at least one of the following "a" options holds with the "b" options holding for the others for all* $0 < \delta \leqslant \bar{\delta}$*:*

1a) $a(t) \leqslant 0$ *a.e. on* $(0, \delta)$ *and* $\int_0^{\delta} [t\, a(t)/p(t)]\, dt = -\infty$

1b) $\int_0^{\delta} [t\, |a(t)|/p(t)]\, dt$ *is finite*

2a) $b(t) \leqslant 0$ *a.e. on* $(0, \delta)$ *and* $\int_0^{\delta} [t\, b(t)/p(t)]\, dt = -\infty$

2b) $\int_0^{\delta} [t\, |b(t)|/p(t)]\, dt$ *is finite*

3a) There exists a $c : [0, \delta] \to \mathbf{R}$ *such that for almost all* $t \in (0, \delta)$*, all* $x \in (0, \delta)$ *and all* $y \in (\beta - \delta, \beta + \delta)$ *we have* $f(t, x, y) \leqslant c(t)$ *and* $\int_0^{\delta} c(t)/p(t)\, dt = -\infty$

3b) There exists a $c : [0, \delta] \to \mathbf{R}$ *such that for almost all* $t \in (0, \delta)$*, all* $x \in (0, \delta)$ *and all* $y \in (\beta - \delta, \beta + \delta)$ *we have* $|f(t, x, y)| \leqslant c(t)$ *and* $\int_0^{\delta} [c(t)/p(t)]\, dt$ *is finite.*

Then, for any $T > 0$ *there is no solution to the problem (IVP1):*

$$p(t)\, u''(t) = a(t)\, t\, u'(t) + b(t)\, u(t) + f(t, u(t), u'(t)), \quad t \in [0, T]$$
$$u(0) = 0, \quad u'(0) = \beta > 0.$$

Proof. Assume there exists some $T > 0$ such that (IVP1) has a solution u on $[0, T]$. We first define ϕ_1, $\phi_2 : (0, T] \to \mathbf{R}$ by

$$\phi_1(t) = u'(t) - \beta, \quad \phi_2(t) = u(t)/t - \beta.$$

Note that ϕ_1, ϕ_2 are both continuous and $\lim_{t \to 0^+} \phi_1(t) = \lim_{t \to 0^+} \phi_2(t) = 0$. As a result, given any $\varepsilon \in (0, \min\{\beta, \bar{\delta}\})$ we have that there is a $\delta_1 \in (0, T]$ such that for $t \in (0, \delta_1]$, we have $-\varepsilon \leqslant \phi_1(t) \leqslant \varepsilon$ and $-\varepsilon \leqslant \phi_2(t) \leqslant \varepsilon$. Also, since $u(0) = 0$ and $u'(0) = \beta > 0$, there is a $\delta_2 \in (0, T]$ such that for $t \in (0, \delta_2]$, we have $0 < u(t) < \bar{\delta}$ and $0 < \beta - \varepsilon \leqslant u'(t) \leqslant \beta + \varepsilon$. Choose $\bar{t} = \min\left\{T, \bar{\delta}, \delta_1, \delta_2\right\}$. For almost all $t \in \left(0, \bar{t}\right]$, we have

$$\begin{aligned} p(t)\, u''(t) &= a(t)\, t\, u'(t) + b(t)\, u(t) + f(t, u(t), u'(t)) \Rightarrow \\ u''(t) &= \left[a(t)\, t\, u'(t)\right]/p(t) + [b(t)\, u(t)]/p(t) + f(t, u(t), u'(t))/p(t) \\ &= [a(t) t\, (\phi_1(t) + \beta)]/p(t) + [b(t) t\, (\phi_2(t) + \beta)]/p(t) + f(t, u(t), u'(t))/p(t) \end{aligned} \tag{1}$$

Case 1: Assume 1a, 2b, 3b each hold.

Then, from $a(t) \leqslant 0$ a.e., $0 < \beta - \varepsilon \leqslant \phi_1(t) + \beta$, $0 < \beta - \varepsilon \leqslant \phi_2(t) + \beta \leqslant \varepsilon + \beta$, 3b and (1), we have for almost all $t \in \left(0, \bar{t}\right]$,

$$u''(t) \leqslant [a(t) t\, (\beta - \varepsilon)]/p(t) + [|b(t)| t\, (\beta + \varepsilon)]/p(t) + c(t)/p(t)$$

and hence

$$u'(\bar{t})-u'(t) \leqslant (\beta-\varepsilon)\int_t^{\bar{t}} a(s)\,s/p(s)\,ds+(\beta+\varepsilon)\int_t^{\bar{t}} |b(s)|\,s/p(s)\,ds+\int_t^{\bar{t}} c(s)/p(s)\,ds \tag{2}$$

Letting $t \downarrow 0$ in (2), the left-hand side is finite, while assumptions 1a, 2b and 3b imply the right-hand side is $-\infty$, a contradiction.

Case 2: Assume 1b, 2a, 3b each hold.

Then, from $0 < \beta - \varepsilon \leqslant \phi_1(t) + \beta \leqslant \beta + \varepsilon$, $b(t) \leqslant 0$ a.e., $0 < \beta - \varepsilon \leqslant \phi_2(t) + \beta$, 3b and (1), we have for almost all $t \in \left(0, \bar{t}\right]$,

$$u''(t) \leqslant [|a(t)|\,t\,(\beta+\varepsilon)]/p(t) + [b(t)t\,(\beta-\varepsilon)]/p(t) + c(t)/p(t)$$

and hence

$$u'(\bar{t})-u'(t) \leqslant (\beta+\varepsilon)\int_t^{\bar{t}} |a(s)|\,s/p(s)\,ds+(\beta-\varepsilon)\int_t^{\bar{t}} b(s)\,s/p(s)\,ds+\int_t^{\bar{t}} c(s)/p(s)\,ds,$$

and using assumptions 1b, 2a and 3b, we get a contradiction as in Case 1.

Case 3: Assume 1b, 2b, 3a each hold.

Then, from $0 < \beta - \varepsilon \leqslant \phi_1(t) + \beta \leqslant \beta + \varepsilon$, $0 < \beta - \varepsilon \leqslant \phi_2(t) + \beta \leqslant \varepsilon + \beta$, 3a and (1), we have for almost all $t \in \left(0, \bar{t}\right]$,

$$u''(t) \leqslant [|a(t)|\,t\,(\beta+\varepsilon)]/p(t) + [|b(t)|t\,(\beta+\varepsilon)]/p(t) + c(t)/p(t)$$

and hence

$$u'(\bar{t})-u'(t) \leqslant (\beta+\varepsilon)\int_t^{\bar{t}} |a(s)|\,s/p(s)\,ds+(\beta+\varepsilon)\int_t^{\bar{t}} |b(s)|\,s/p(s)\,ds+\int_t^{\bar{t}} c(s)/p(s)\,ds,$$

and using assumptions 1b, 2b and 3a, we get a contradiction as in Case 1.

It should be clear to the reader how to verify the remaining four cases of the theorem. □

REMARK 1.

1. This initial value problem can be thought of as having a perturbed linear right-hand side.

2. The reader might be tempted to replace the entire right-hand side with $g(t,u(t),u'(t))$ and simply assume 3a (with f replaced by g). However, this actually results in stronger hypotheses: Consider the case $p(t) = t^2$, $a(t) = 0$, $b(t) = -1$, $f = 0$. Then, 2a holds but 3a does not hold for $g(t,x,y) = -x$.

3. We can prove a similar theorem in the cases in which $p(t) < 0$ a.e. and/or $\beta < 0$.

4. In Case 1, instead of assuming $p > 0$ a.e. and $a \leqslant 0$ a.e., we could have assumed $a/p \leqslant 0$ a.e. A similar comment can be made for Case 2.

5. In Case 1, the proof would still have worked in the event that $\int_0^{\delta}[t\,|b(t)|/p(t)]\,dt = \infty$, as long as $(\beta-\varepsilon)\int_t^{\bar{t}} a(s)\,s/p(s)\,ds + (\beta+\varepsilon)\int_t^{\bar{t}} |b(s)|\,s/p(s)\,ds = -\infty$. Similar comments can be made in other places in the proof.

6. A similar theorem can be proven for the differential equation $p(t)\,u^{(n)}(t) = a(t)t\,u'(t) + b(t)\,u(t) + f(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$.

7. The assumption that $\beta \neq 0$ appears to be essential, as evidenced by the problem $t^2u''(t) = -t\,u(t) - u'(t)$, $u(0) = 0$, $u'(0) = 0$, which has the solution $u \equiv 0$. The referee raised an interesting question: Can a result concerning nonexistence of nontrivial solutions be proven for such problems? It does not appear that the proof above can be easily modified to address this question, so a new approach would need to be devised.

8. Proposition 3.2 in [6] concerns initial value problems of the form

$$\begin{gathered} t^2u''(t) = atu'(t) + bu(t) - c(u'(t) - 1)^2 \\ u(0) = 0, \quad u'(0) = 1, \end{gathered}$$

where a, b, c are constants and the condition $u'(0) = 1$ is forced by the differential equation. Note that Theorem 1 above can be applied to a large number of problems that are not of this form, for example,

$$\begin{gathered} t^3u''(t) = -tu(t) - 1 \\ u(0) = 0, \quad u'(0) = 2. \end{gathered}$$

We can also use this approach to prove a nonexistence result for problems singular in both the time and state variables, as follows.

THEOREM 2. *Assume there exists a* $\bar{\delta} > 0$ *such that* $p(t) > 0$ *a.e. on* $(0, \bar{\delta})$, $0 < q(x) \leqslant x$ *for all* $x \in (0, \bar{\delta})$, $a(t) \leqslant 0$ *a.e. on* $(0, \bar{\delta})$, $b(t) \leqslant 0$ *a.e. on* $(0, \bar{\delta})$ *and there exists a* $c : \left[0, \bar{\delta}\right] \to \mathbf{R}$ *such that for almost all* $t \in (0, \bar{\delta})$, *all* $x \in (0, \bar{\delta})$ *and all* $y \in (\beta - \bar{\delta}, \beta + \bar{\delta})$ *we have* $f(t, x, y) \leqslant c(t) \leqslant 0$. *Also, assume at least one of the following "a" options holds with the "b" options holding for the others for all* $0 < \delta \leqslant \bar{\delta}$:

1a) $\int_0^\delta a(t)/p(t)\,dt = -\infty$

1b) $\int_0^\delta a(t)/p(t)\,dt$ *is finite*

2a) $\int_0^\delta b(t)/p(t)\,dt = -\infty$

2b) $\int_0^\delta b(t)/p(t)\,dt$ *is finite*

3a) $\int_0^\delta c(t)/p(t)\,dt = -\infty$

3b) $\int_0^\delta c(t)/p(t)\,dt$ *is finite.*

Then, for any $T > 0$ *there is no solution to the problem (IVP2):*

$$\begin{gathered} p(t)\,q(u(t))\,u''(t) = a(t)\,t\,u'(t) + b(t)\,u(t) + f(t, u(t), u'(t)), \;\; t \in [0, T] \\ u(0) = 0, \quad u'(0) = \beta > 0. \end{gathered}$$

Proof. Assume there exists some $T>0$ such that (IVP2) has a solution u on $[0,T]$. We next define $\phi_1,\phi_2:(0,T]\to\mathbf{R}$ as in the proof of Theorem 1. Note that on $(0,T]$, we have $u(t)=t\,(\phi_2(t)+\beta)$ and $u'(t)=\phi_1(t)+\beta$, and hence

$$a(t)\,t\,u'(t)/\left[u(t)\,p(t)\right]=a(t)\,t\,(\phi_1(t)+\beta)/\left[t\,(\phi_2(t)+\beta)\,p(t)\right]=a(t)\,\Phi(t)/p(t), \quad (3)$$

where the continuous function $\Phi:(0,T]\to\mathbf{R}$ is defined by $\Phi(t)=\frac{\phi_1(t)+\beta}{\phi_2(t)+\beta}$ and $\lim\limits_{t\to 0^+}\Phi(t)=1$. Let $\varepsilon\in\left(0,\min\left\{\beta,\bar{\delta},1\right\}\right)$. Then, there exists a $\delta_1>0$ such that for all $t\in(0,\delta_1)$, we have

$$0<1-\varepsilon<\Phi(t)<1+\varepsilon. \quad (4)$$

Also, since $u(0)=0$ and $u'(0)=\beta>0$, there is a $\delta_2\in(0,T]$ such that for $t\in(0,\delta_2]$, we have $0<u(t)\leqslant\bar{\delta}$ and $0<\beta-\varepsilon\leqslant u'(t)\leqslant\beta+\varepsilon$.

Now choose $\bar{t}=\min\left\{T,\bar{\delta},\delta_1,\delta_2\right\}$. Then, for almost all $t\in(0,\bar{t}]$,

$$p(t)\,q(u(t))\,u''(t)=a(t)\,t\,u'(t)+b(t)\,u(t)+f(t,u(t),u'(t))\Rightarrow$$
$$u''(t)=a(t)tu'(t)/\left[p(t)q(u(t))\right]+b(t)u(t)/\left[p(t)q(u(t))\right]+f(t,u(t),u'(t))/\left[p(t)q(u(t))\right]\Rightarrow$$
$$u''(t)\leqslant a(t)\,t\,u'(t)/\left[p(t)\,u(t)\right]+b(t)\,u(t)/\left[p(t)\,u(t)\right]+f(t,u(t),u'(t))/\left[p(t)\,u(t)\right]$$

because $a(t)$, $b(t)$, $f(t,u(t),u'(t))\leqslant 0$ a.e., t, $u(t)$, $u'(t)$, $p(t)\geqslant 0$ a.e. and $0<q(u(t))\leqslant u(t)$. Hence from (3), (4), the assumption on f, $a(t)$, $c(t)\leqslant 0$ a.e. and $0<u(t)<\bar{\delta}$, we have

$$u''(t)\leqslant a(t)\,\Phi(t)/p(t)+b(t)/p(t)+c(t)/\left[p(t)\,u(t)\right]\Rightarrow$$
$$u''(t)\leqslant a(t)\,(1-\varepsilon)/p(t)+b(t)/p(t)+c(t)/\left[p(t)\,\bar{\delta}\right].$$

Thus,

$$u'(\bar{t})-u'(t)\leqslant(1-\varepsilon)\int_t^{\bar{t}}a(s)/p(s)\,ds+\int_t^{\bar{t}}b(s)/p(s)\,ds+\frac{1}{\bar{\delta}}\int_t^{\bar{t}}c(s)/p(s)\,ds.$$

Letting $t\downarrow 0$ and applying 1a, 1b, 2a, 2b, 3a and/or 3b as appropriate, we find that the left-hand side is finite, while the right-hand side approaches $-\infty$, a contradiction. □

REMARK 2.

1. This theorem allows $p\equiv 1$ as a special case.
2. Another special case is $p(t)\,[u(t)]^n u''(t)=a(t)\,t\,u'(t)+b(t)\,u(t)+f(t,u(t),u'(t))$ for $n\geqslant 1$.

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Daniel C. Biles
Department of Math and CS
Belmont University
1900 Belmont Blvd., Nashville, TN 37212, USA
e-mail: daniel.biles@belmont.edu