A PROBLEM INVOLVING THE $p$–LAPLACIAN OPERATOR

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Abstract. Using a variational technique we guarantee the existence of a solution to the resonant Lane-Emden problem $-\Delta_p u = \lambda |u|^{q-2} u, \ u|\partial \Omega = 0$ if and only if a solution to $-\Delta_p u = \lambda |u|^{q-2} u + f, \ u|\partial \Omega = 0$, $f \in L^{p'}(\Omega)$ ($p'$ being the conjugate of $p$), exists for $q \in (p, p^*)$ under certain condition on $\lambda$, where $p^*$ is the Sobolev conjugate of $p$.

1. Introduction

The study of partial differential equations involving a $p$-Laplacian differential operator has become a major case of study in the recent times although it is still far from being completely understood, especially when $p = 1$ or $\infty$. A few evidences of the limiting case can be found in [18], [20]. In fact, existence of a positive eigenvector to the eigenvalue problem can be found in [7]. When $p = 2$, the usual Laplacian is obtained for which a vast literature exists ([10], [11] and the references therein). For $p \neq 2$ the $p$-Laplacian operator has physical applications in the study of non-Newtonian fluids (dilatant fluids when $p > 2$) [15]. In practical life most of the problems are non linear by nature for which a numerical solution is sought for, however, unearthing the existence of solution leads to a rich theory hidden behind the partial differential equations. The problems we are going to address in this article are the following. Let $\Omega$ be a bounded subset of $\mathbb{R}^n, n \geq 3$ with a Lipschitz boundary $\partial \Omega$. Given $1 < p < \infty$ and $q \in (p, p^*)$, where $p^* = \frac{np}{n-p}$ if $1 < p < n$ and $p^* = \infty$ if $p \geq n$, we consider the following problems.

1. $-\Delta_p u = \lambda |u|^{q-2} u, \ u|\partial \Omega = 0$. This problem is also known as the resonant Lane-Emden problem.

2. $-\Delta_p u = \lambda |u|^{q-2} u + f, \ f \in L^{p'}(\Omega), \ u|\partial \Omega = 0$.

where $\lambda$ is a real number, $\Delta_p = \nabla \cdot (|\nabla|^{p-2} \nabla \cdot )$. Throughout this paper we shall refer the problems in 1 and 2 as the first and the second problem respectively.

We call the first problem to be of sub-linear type if $1 < q < p < p^*$ and of super-linear type when $1 < p < q < p^*$. In this article, we restrict the first problem to be of
super-linear type. It is found in [5, 6] that a unique solution exists to the first problem for the sub-linear case whereas uniqueness is lost for the super-linear case. Readers interested in knowing more about the first problem can refer to examples found in [8], [16], where the domain is ring shaped for \( q \sim p^* \) and the solution is non-unique. Kawohl [2] showed the same but the domain which was considered is of annulus type with the annulus being sufficiently small in size. Uniqueness of solution is also guaranteed in [9] for the sub-linear case whereas a subdifferential method has been used to prove existence in [13] for both sub and super linear cases. Grumiau and Parini [3] discussed the asymptotic behavior of the ground state solutions as \( q \to p \). In recent times Vérón et al [12] considered a similar problem but with a measure instead of the function \( f \). They have characterized the ‘good’ measures for which the problem - \( \Delta p u + g(x, u) = \mu, u\rvert_{\partial\Omega} = 0 \) - where \( g(\ldots) \) is non-decreasing, \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) - has a solution. Interested readers can also refer to the work of Giri and Choudhuri [19] (and the references therein) who have used the notion of ‘reduced limit’ for problems with measure data.

In this paper we will use a well known variational technique to show the existence of a solution in \( W^{1,p}_0(\Omega) = \{ v \in L^p(\Omega) : \nabla v \in L^p(\Omega), v\rvert_{\partial\Omega} = 0 \} \). A Fredholm type alternative is also proposed thus showing a connection between the first and the second problem. We organize the paper into two sections. In Section 2 we give the Mathematical formulation. In Section 3 we discuss a few preliminary results and the main result.

### 2. Mathematical formulation

The following definitions and theorems will be used in the main result we prove.

**2.1 Definition:** Let \( X \) be a Banach space and \( H : X \to \mathbb{R} \) a \( C^1 \) functional. It is said to satisfy the Palais-Smale condition (PS) if the following holds.

Whenever \( \{ u_n \} \) is a sequence in \( X \) such that \( \{ H(u_n) \} \) is bounded and \( H'(u_n) \to 0 \) strongly in \( X' \) (the dual space), then \( \{ u_n \} \) has a strongly convergent subsequence in \( X \).

The (PS) condition is a strong condition as very “well-behaved” function do not satisfy it (Example: \( f(x) = c, x \in \mathbb{R}, c \) a real constant).

We now state the following important theorem due to Ambrosetti and Rabinowitz [1] which is a common tool used in the theory of modern PDEs.

**Mountain-pass theorem:** Let \( H : X \to \mathbb{R} \) be a \( C^1 \) functional satisfying (PS). Let \( u_0, u_1 \in X, c_0 \in \mathbb{R} \) and \( r > 0 \) such that

1. \( ||u_1 - u_0|| > r \)

2. \( H(u_0), H(u_1) < c_0 \leq H(v) \), \( \forall v \) such that \( ||v - u_0|| = r \). Then \( H \) has a critical value \( c \geq c_0 \) defined by

\[
    c = \inf_{\Gamma \in \mathcal{P}} \max_{t \in [0,1]} H(\Gamma(t)), \tag{2.1}
\]

where \( \mathcal{P} \) is the collection of all continuous paths \( \Gamma : [0, 1] \to X \) such that \( \Gamma(0) = u_0, \Gamma(1) = u_1 \).
2.2 Weak formulation of the problem: We now give the weak formulation of the first problem. We say that \( u \in W^{1,p}_0(\Omega) \) is a weak solution of the first problem if
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} uv dx = 0
\] (2.2)
for every \( v \in W^{1,p}_0(\Omega) \).

The weak solutions of the Lane-Emden problem are the critical points of the energy function defined by
\[
J_q(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{q} \int_{\Omega} |u|^q dx.
\] (2.3)

The following compact embedding theorems, due to Rellich-Kondrachov will be used in our work.

1. if \( p < n \), \( W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega) \), \( 1 \leq q < p^* \),
2. if \( p = n \), \( W^{1,n}_0(\Omega) \hookrightarrow L^q(\Omega) \), \( 1 \leq q < \infty \),
3. if \( p > n \), \( W^{1,p}_0(\Omega) \hookrightarrow C(\overline{\Omega}) \).

We consider the non-homogeneous counterpart of the first problem - which is the second problem - and is as follows.
\[
-\Delta_p u = \lambda |u|^{q-2} u + f,
\]
\( u|_{\partial \Omega} = 0 \), \( p' \) being the conjugate of \( p \), which is equal to \( \frac{p}{p-1} \). Let the corresponding energy functional be denoted by \( J \) which is defined as follows.
\[
J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \int_{\Omega} f u dx.
\] (2.5)

The Fréchet derivative of \( J \), which is in \( W^{-1,p'}_0(\Omega) \) where \( p' = \frac{p}{p-1} \), is
\[
< J'(u), v > = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} uv dx - \int_{\Omega} f v dx,
\] (2.6)
\( \forall v \in W^{1,p}_0(\Omega) \). Thus \( u \in W^{1,p}_0(\Omega) \) is a weak solution of the second problem if
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} uv dx - \int_{\Omega} f v dx = 0.
\]

3. Few preliminary results and the main theorem

The main result of this paper, stated informally, is as follows. The problem \(-\Delta_p u = \lambda |u|^{q-2} u, u|_{\partial \Omega} = 0 \) has a weak solution if and only if the problem \(-\Delta_p u = \lambda |u|^{q-2} u + \)
Since have technical lemmas on which the proof of this result will rely upon. The case of $p \geq n$ follows the same proof as in the case $p < n$ which is based on the results on compact embedding stated after equation (2.3). But first we present a few technical lemmas on which the proof of this result will rely upon.

We first assume that a nontrivial solution exists to the problem

$$-\Delta_p u = \lambda |u|^{q-2}u, \quad u|_{\partial \Omega} = 0,$$

(3.1)

**Theorem 1.** The mapping $J$ defined in (2.5) is a $C^1$-functional over $W^{1,p}_0(\Omega)$.

**Proof.** The functional $J$ is differentiable which can be seen by extending the arguments in [17], Theorem 5.3.1. Thus it is enough to show that $J'\prime$ is continuous. Now from (2.6), we have

$$|J'(u), v| \leq \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + |\lambda| \int_\Omega |u|^{q-2} |v| dx + \int_\Omega |f||v| dx \leq ||\nabla u||_{p^{-1}} ||\nabla v||_p + |\lambda| ||u||_{q^{-1}} ||v||_q + ||f||_{p^{-1}} ||v||_p \leq \left[ ||\nabla u||_{p^{-1}} + C_1 |\lambda|||u||_{q^{-1}} + C_2 ||f||_{p^{-1}} \right] ||\nabla v||_p,$$

(3.2)

$\forall v \in W^{1,p}_0(\Omega)$, where $C_1$, $C_2$ are the constants due to the embedding of $W^{1,p}_0(\Omega)$ in $L^q(\Omega)$ for $q \in [1, p^*)$. From (3.2) one can see that $J$ is a $C^1$ functional over $W^{1,p}_0(\Omega)$.

**Theorem 2.** There exists $u_0, u_1 \in W^{1,p}_0(\Omega)$ and a positive real number $c_0$ such that $J(u_0), J(u_1) < c_0$ and $J(v) \geq c_0$, for every $v$ satisfying $||v-u_0||_{1,p} = r$.

**Proof.** Let $u_0 = 0$. Clearly $u_0$ is a solution of (3.1) $J(0) = 0$. Now let $w \in B(0,1) = \{u \in W^{1,p}_0(\Omega) : ||u||_{1,p} = 1\}$ and consider $v = u_0 + rw$ for $r > 0$ and hence $||v-u_0||_{1,p} = r$. We first show the existence of $r_0$, $c_0$ such that for each $v$ we have $||v-u_0||_{1,p} = r_0$ and for which $J(v) \geq c_0$, where $c_0 > 0$. Since $p < q < p^*$, we have

$$J(u_0 + rw) - J(u_0) = \frac{r^p}{p} \int_\Omega |\nabla w|^p dx - \frac{r^q \lambda}{q} \int_\Omega |w|^q dx - r \int_\Omega fwdx, = \frac{r^p}{p} - \frac{r^q \lambda}{q} \int_\Omega |w|^q dx - r \int_\Omega fwdx.$$

(3.3)

Further, $|w|_{1,p} = 1$ and hence $\int_\Omega w^p dx \leq \int_\Omega |w|^p dx = ||w||_p^p \leq c_1 |w|_{1,p}^p = c_1$. Similarly, $\int_\Omega w^q dx \leq c_2$. Using these arguments leads to

$$J(u_0 + rw) - J(u_0) \geq r \left[ \frac{r^{p-1}}{p} - \frac{r^{q-1} \lambda}{q} c_2 - c_1^{1/p} ||f||_{p'} \right].$$

(3.4)
We first analyze the term

$$
\left[ \frac{r^{p-1}}{p} - \frac{r^{q-1}}{q} \right] c_2 - c_1^{1/p} ||f||_{p'} = F(r)
$$

(say). Clearly $F(0) < 0$ and for $r_0 = \left( \frac{q(p-1)}{p(q-1)} \frac{1}{\lambda c_2} \right)^{\frac{q-p}{p-1}}$ we see that $F'(r_0) = 0$. A bit of calculus guarantees that $F''(r_0) < 0$ and hence $r_0$ is a maximizer of $F$. Note that, if

$$
0 < \lambda < \lambda_1 = \frac{q(p-1)}{c_2 p(q-1)}, \left( \frac{q-p}{p(q-1)} \cdot \frac{1}{c_1^{\frac{1}{p} ||f||_{p'}}} \right)^{\frac{q-p}{p-1}},
$$

then $F(r_0) > 0$. As $r \to \infty$ we have $F(r) \to -\infty$. Hence there exists $r_1, r_2 > 0$, $r_0 > 0$ (this $r_0$ could be different from the above one) such that $r_1 F(r_1) = r_2 F(r_2) = c'$ (say), $r_1 < r_0 < r_2$ and $rF(r) > 0$ for each $r \in (r_1, r_2)$. Thus for $v$ such that $||v - u_0||_{1,p} = r_0$ we have $J(v) \geq c' > 0$ for each $v \in B(0, r_0) \subset W^{1,p}_0(\Omega)$.

**Choice of $u_1$:** Let $w_q$ be a nontrivial solution to the equation $-\Delta_p w_q = \lambda |w_q|^{q-2} w_q$ in $\Omega$, $w_q = 0$ on $\partial \Omega$. Consider the function $g = k w_q$, $k \in \mathbb{R}$, where we have normalized $w_q$ with respect to the Sobolev norm on $W^{1,p}_0(\Omega)$ without changing its notation. Note that,

$$
J(g) = \left( \frac{k^p}{p} - \frac{\lambda k^q \int_\Omega |w_q|^q dx}{q} \right) - kC,
$$

where $C = \int_\Omega f w_q dx$. Since $p < q < p^*$, we choose $k_0$ to be sufficiently large so that

$$
\frac{k_0^p}{p} - \frac{\lambda k_0^q \int_\Omega |w_q|^q dx}{q} - k_0 C < 0.
$$

Then $J(k_0 w_q) < 0$ and hence $J(k_0 w_q) > J(u_0)$. Thus we can choose $u_1 = k_0 w_q$, where $k_0 > r_0$, due to which $||u_1 - u_0||_{1,p} > r_0$. Hence the result.

**Theorem 3.** The functional $J$ satisfies the Palais-Smale condition.

**Proof.** Let $u_n$ be a sequence in $W^{1,p}_0(\Omega)$ such that $|J(u_n)| \leq M$ and $J'(u_n) \to 0$ as $n \to \infty$ in $W^{-1,p'}_0(\Omega)$, $p'$ being the conjugate of $p$. Now

$$
J(u_n) = \frac{1}{p} \int_\Omega |\nabla u_n|^p dx - \frac{\lambda}{q} \int_\Omega |u_n|^q dx - \int_\Omega f u_n dx, \quad (3.5)
$$

$$
< J'(u_n), v > = \int_\Omega |\nabla u_n|^{p-2} \nabla u_n. \nabla v dx - \lambda \int_\Omega |u_n|^{q-2} u_n v dx - \int_\Omega f v dx, \quad (3.6)
$$
for all \( v \in W_0^{1,p}(\Omega) \). Consider the following.

\[
< J'(u_n), u_n > = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx - \lambda \int_\Omega |u_n|^q dx - \int_\Omega fu_n dx, \tag{3.7}
\]

\[
J(u_n) = \frac{1}{p} \int_\Omega |\nabla u_n|^{p} dx - \frac{\lambda}{q} \int_\Omega |u_n|^q dx - \int_\Omega fu_n dx,
\]

\[
= \frac{1}{p} |u_n|_{1,p}^p - \frac{\lambda}{q} \int_\Omega |u_n|^q dx - \int_\Omega fu_n dx.
\]

\[
\lambda \int_\Omega |u_n|^q dx = \frac{q}{p} |u_n|_{1,p}^p - qJ(u_n) - q \int_\Omega fu_n dx
\]

\[
\frac{p-q}{p} |u_n|_{1,p}^p = < J'(u_n), u_n > - qJ(u_n) - (q-1) \int_\Omega fu_n dx. \tag{3.8}
\]

From (3.8) \( \{u_n\} \) is bounded in \( W_0^{1,p}(\Omega) \) and hence by Eberlein-Šmulian’s theorem (refer Dunford-Schwartz [1; p. 430] [14]) it has a weakly convergent subsequence, say \( \{u_{n_k}\} \), in \( W_0^{1,p}(\Omega) \).

**Claim.** The subsequence \( \{u_{n_k}\} \) is strongly convergent in \( W_0^{1,p}(\Omega) \).

**Proof.** Applying limit \( n_k \to \infty \) to (3.6) (refer Appendix) and using the strong convergence of \( \{u_{n_k}\} \) in \( L^q(\Omega) \) due to compact embedding we obtain

\[
\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx = \lambda \int_\Omega |u|^{q-2}uv dx + \int_\Omega fv dx, \tag{3.9}
\]

We then pass on the limit \( n_k \to \infty \) to (3.7) to get

\[
\lim_{n_k \to \infty} |u_{n_k}|_{1,p}^p = \lambda \int_\Omega |u|^q dx + \int_\Omega fu dx = |u|_{1,p}^p. \tag{3.10}
\]

Since, a weakly convergent sequence which is convergent in norm is strongly convergent, hence \( u_{n_k} \to u \) in \( W_0^{1,p}(\Omega) \) as \( n_k \to \infty \).

So, by the Mountain-pass theorem an extreme point for \( J \) exists in \( W_0^{1,p}(\Omega) \).

We summarize the results proved in Theorems 1, 2 and 3 in the form of a unified theorem as follows.

**THEOREM 4.** Suppose \( -\Delta_p u = \lambda |u|^{q-2}u, \ u|_{\partial\Omega} = 0 \) has a nontrivial solution for some \( \lambda > 0 \), where \( q \in (p, p^*) \). Then the problem \( -\Delta_p u = \lambda |u|^{q-2}u + f, \ f \in L^p(\Omega), \ u|_{\partial\Omega} = 0 \) has a nontrivial solution whenever \( \lambda \in (0, \lambda'] \) where \( \lambda' < \lambda_1 \) and \( \lambda_1 = \frac{q(p-1)}{c_2p(q-1)} \left( \frac{q-p}{p(q-1)} \right)^{\frac{q-p}{p}} \), \( p' = \frac{p}{p-1}, \ c_1^{1/p}, c_2^{1/q} \) are the Sobolev constants corresponding to the embedding of \( W_0^{1,p}(\Omega) \) in \( L^p(\Omega), L^q(\Omega) \) respectively.

Arguing on similar lines, as in Theorems 1, 2 and 3, we conclude the following result.
Theorem 5. If the eigenvalue problem
\[ -\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, \]
\[ u|_{\partial \Omega} = 0 \text{ on } \partial \Omega, \]
has a nontrivial solution, then the non homogeneous Lane-Emden problem
\[ -\Delta_p u = \lambda |u|^{q-2} u + f, \quad f \in L^{p'}(\Omega), \]
\[ u|_{\partial \Omega} = 0, \]
has a nontrivial solution for \( q \in (p, p^*) \) whenever \( \lambda \in (0, \lambda'] \) where \( \lambda' < \lambda_1 \) and
\[ \lambda_1 = \frac{q(p-1)}{c_2 p(q-1)} \left( \frac{q-p}{p(q-1)} \frac{1}{c_1^p ||u||^p} \right)^{\frac{q-p}{p-1}}, \quad p' = \frac{p}{p-1}, \quad c_1^{1/p}, \quad c_2^{1/q} \text{ are the Sobolev constants} \]
corresponding to the embedding of \( W^{1,p}_0(\Omega) \) in \( L^p(\Omega), L^q(\Omega) \) respectively.

Conversely, suppose to each \( f \in L^{p'}(\Omega) \) the problem
\[ -\Delta_p u = \lambda |u|^{q-2} u + f, \]
\[ u|_{\partial \Omega} = 0, \quad (3.11) \]
has a nontrivial solution on the set \( \mathcal{M} = \{ u \in W^{1,p}_0(\Omega) : ||u||_q = 1 \} \) for some \( \lambda > 0 \),
where \( q \in [p, p^*) \). Existence of such solution can be assumed from the weak lower semi continuity and coercivity of the corresponding energy functional \( J \) on the subset \( \mathcal{M} \) of \( W^{1,p}_0(\Omega) \) (refer [17]). In order to prove the existence of nontrivial solution of the first problem for \( q \in [p, p^*) \), we let \( \{ f_n \} \subset L^{p'}(\Omega) \) be a sequence such that \( f_n \rightarrow 0 \) in \( L^{p'}(\Omega) \). Then for each \( f_n \), there exists a solution, say \( u_n \).

We have
\[ B[u, v] = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} u v dx, \]
\[ = \int_{\Omega} f v dx, \quad \forall v \in W^{1,p}_0(\Omega), \quad (3.12) \]
where \( B \) is a ‘non linear form’ in two variables \( u \) and \( v \). It is easy to check that \( B(\cdot, \cdot) \) is the Fréchet derivative of the \( C^1 \) functional \( \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{\lambda}{q} \int_\Omega |u|^q \) and hence is continuous.

Clearly, for each \( v \in W^{1,p}_0(\Omega) \) we have
\[ B[u_n, v] = \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx - \lambda \int_{\Omega} |u_n|^{q-2} u_n v dx, \]
\[ = \int_{\Omega} f_n v dx, \]
\[ \leq ||f_n||_{p'} ||v||_p \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.13) \]
Hence \( \int_{\Omega} f_n v dx \rightarrow 0 \) as \( n \rightarrow \infty \). Consider \( T_n(v) = \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx \). Then \( T_n \)'s are bounded linear over \( W^{1,p}_0(\Omega) \) and \( ||T_n|| = |||\nabla u_n|^{p-1}||_{p'} \) for \( n \geq 1 \). From the above
In other words, for a fixed \( v \in W_0^{1,p}(\Omega) \) we have the sequence \( \{T_n(v)\} \) to be bounded which implies that \( \{T_n(v)\} \) is pointwise bounded. Thus by the uniform boundedness principle \( \{||T_n||\} \) is bounded. Thus \( \{||\nabla u_n||_p\} \) is bounded. Hence, there exists a subsequence \( \{u_{n_k}\} \) which weakly converges to \( u_\infty \) with respect to the norm \( || \cdot ||_{1,p} \) in \( W_0^{1,p}(\Omega) \). Hence we have

\[
\lim_{n_k \to \infty} \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla u_{n_k} \, dx = \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla u_\infty \, dx, \quad \forall v \in W_0^{1,p}(\Omega).
\]

\[
\Rightarrow \lim_{n_k \to \infty} \int_{\Omega} |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \cdot \nabla u_{n_k} \, dx = \int_{\Omega} |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \cdot \nabla u_\infty \, dx, \quad \forall v \in W_0^{1,p}(\Omega).
\]

(3.14)

for a fixed \( l \). Therefore, since \( u_{n_k} \to u_\infty \) in \( W_0^{1,p}(\Omega) \) implies that \( |\nabla u_{n_k}|^{p-1} \to |\nabla u_\infty|^{p-1} \) (for a subsequence) in \( L^p(\Omega) \) (Refer Appendix). But \( W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W^{-1,p'}(\Omega) \) and hence

\[
\lim_{n_k \to \infty} \int_{\Omega} |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \cdot \nabla v \, dx = \int_{\Omega} |\nabla u_\infty|^{p-2} \nabla u_\infty \cdot \nabla v \, dx, \quad \forall v \in W_0^{1,p}(\Omega),
\]

\[
\Rightarrow \lim_{n_k \to \infty} \int_{\Omega} |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \cdot \nabla u_\infty \, dx = \int_{\Omega} |\nabla u_\infty|^{p} \, dx.
\]

(3.15)

Therefore, \( \lim_{n_k \to \infty} \int_{\Omega} |\nabla u_{n_k}|^p \, dx = \int_{\Omega} |\nabla u_\infty|^p \, dx \). It immediately can be concluded that there exists a \( u_\infty \) such that \( u_{n_k} \to u_\infty \) in \( W_0^{1,p}(\Omega) \). Hence using the continuity of \( B[.,.] \) in (3.12) we have

\[
\lim_{n_k \to \infty} B[u_{n_k}, v] = \lim_{n_k \to \infty} \int_{\Omega} |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \cdot \nabla v \, dx - \lim_{n_k \to \infty} \lambda \int_{\Omega} |u_{n_k}|^{q-2} u_{n_k} \, v \, dx
\]

\[
= \lim_{n_k \to \infty} \int_{\Omega} f_{n_k} \, v \, dx,
\]

\[
\Rightarrow B[u_\infty, v] = 0, \forall v \in W_0^{1,p}(\Omega).
\]

In other words,

\[
\int_{\Omega} |\nabla u_\infty|^{p-2} \nabla u_\infty \cdot \nabla v \, dx - \lambda \int_{\Omega} |u_\infty|^{q-2} u_\infty \, v \, dx = 0, \forall v \in W_0^{1,p}(\Omega).
\]

(3.16)

Assume that \( \lambda \in (0, \inf_{u \neq 0 \in W_0^{1,p}(\Omega)} \left\{ \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} |u|^q} \right\} \) (infimum exists and is strictly greater than zero which follows from the embedding result for \( p < N \)). Since \( ||u_{n_k}||_q = 1 \) and \( u_{n_k} \to u_\infty \) in \( W_0^{1,p}(\Omega) \), hence we have

\[
0 < \lambda \leq \liminf \frac{\int_{\Omega} |\nabla u_{n_k}|^p}{\int_{\Omega} |u_{n_k}|^q}
\]

\[
= \liminf \int_{\Omega} |\nabla u_{n_k}|^p
\]

\[
= \liminf ||\nabla u_{n_k}||_p^p = ||\nabla u_\infty||_p^p = ||u_\infty||_{1,p}^p
\]

This implies that \( u_\infty \) is a nontrivial solution of the first problem. Thus we summarize the result proved as follows.
Theorem 6. Suppose to each \( f \in L^q(\Omega) \), \( p' = \frac{p}{p-1} \), the problem \(-\Delta_p u = \lambda |u|^{q-2}u + f, u|_{\partial \Omega} = 0 \) has a solution in \( M \subset W_0^{1,p}(\Omega) \) for some \( \lambda > 0 \), then the problem \(-\Delta_p u = \lambda |u|^{q-2}u, u|_{\partial \Omega} = 0 \), has a nontrivial solution in \( W_0^{1,p}(\Omega) \) for \( q \in [p, p^*], \) whenever \( \lambda \in \left( 0, \inf_{u \neq 0 \in W_0^{1,p}(\Omega)} \frac{\|f\|_{\Omega} |\nabla u|^p}{\|u\|^q} \right) \).

We end this section with a small observation from Theorem 6 that if to each \( f \in L^p(\Omega) \), the problem \(-\Delta_p u = \lambda |u|^{p-2}u + f, u|_{\partial \Omega} = 0 \) has a solution in \( M \subset W_0^{1,p}(\Omega) \) for some \( \lambda > 0 \) and \( q \in [p, p^*], \) then the eigenvalue problem \(-\Delta_p u = \lambda |u|^{p-2}u, u|_{\partial \Omega} = 0 \), has a nontrivial solution in \( W_0^{1,p}(\Omega) \), whenever \( \lambda \in \left( 0, \inf_{u \neq 0 \in W_0^{1,p}(\Omega)} \frac{\|f\|_{\Omega} |\nabla u|^p}{\|u\|^q} \right) \).

4. Appendix

We show that

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{4.1}
\]

We divide the explanation into two cases:

Case 1: When \( p > 2 \).

This implies that \( p' \), the conjugate of \( p \), should be lesser than 2, i.e., \( 1 < p' < 2 < p \). Thus we have \( W_0^{1,p}(\Omega) \hookrightarrow compact L^{p'}(\Omega) \) (since \( W_0^{1,p}(\Omega) \hookrightarrow compact L^q(\Omega) \) for \( q \in [1, p^*] \)). Since \( \nabla u_n \) converges weakly to, say \( \nabla u \), in \( L^p(\Omega) \), hence \( \langle \nabla u_n - \nabla u, \nabla v \rangle \to 0 \) for each \( v \in L^p(\Omega) \). Thus \( \langle \nabla u_n - |\nabla u|, |\nabla u| - |\nabla v| \rangle \to 0 \), i.e., \( ||\nabla u_n||_2 \to ||\nabla u||_2 \). Hence \( ||\nabla u_n||_{p'} \to ||\nabla u||_{p'} \) because \( p' < 2 < p \). By the Riesz-Fischer theorem [4], there exists a subsequence of \( \{\nabla u_n\} \) which converges pointwise a.e., i.e., \( |\nabla u_n(x)| \to |\nabla u(x)| \). So \( |\nabla u_n(x)|^{p-1} \to |\nabla u(x)|^{p-1} \) and hence \( |\nabla u_n|^{p-1} \to |\nabla u|^{p-1} \) in \( L^{p'}(\Omega) \). Thus we have \( \lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, \forall v \in W_0^{1,p}(\Omega) \).

Case 2: When \( p < 2 \).

This implies that \( p' \), the conjugate of \( p \), should be greater than 2, i.e., \( p < 2 < p' \).

Look at the map \( F : W_0^{1,p}(\Omega) \to L^{p'}(\Omega) \) defined by \( u \mapsto |\nabla u|^{p-1} \). Consider the range of \( F \), i.e., \( R(F) = \{|\nabla u|^{p-1} : u \in W_0^{1,p}(\Omega)\} \).

Observe that the map \( F \) is bounded in the sense that bounded sets are mapped to bounded sets. Hence if \( u_n \to u \) in \( W_0^{1,p}(\Omega) \) implies that \( \{u_n\} \) is bounded in \( W_0^{1,p}(\Omega) \). Hence \( \{F(u_n)\} = \{|\nabla u_n|^{p-1}\} \) is bounded in \( L^{p'}(\Omega) \). Since \( L^{p'}(\Omega) \) is reflexive, hence there exists a subsequence of \( \{|\nabla u_n|^{p-1}\} \) which weakly converges to, say, \( w \) in \( L^{p'}(\Omega) \).

We have the following: \( u_n \to u \) in \( W_0^{1,p}(\Omega) \) so \( |\nabla u_n|^{p-1} \to w \) in \( L^{p'}(\Omega) \). This implies that

\[ \langle |\nabla u_n|^{p-1} - w, v \rangle \to 0, \forall v \in L^{p'}(\Omega) \]
Since \( p < 2 < p' \) hence \(|\nabla u_n|^{p-1} - w \in L^p(\Omega)\). Thus \(|||\nabla u_n|^{p-1} - w||_2 \to 0\) and hence \(|||\nabla u_n|^{p-1} - w||_p \to 0\). Therefore we have a subsequence of \(\{|||\nabla u_n|^{p-1}\}\) such that \(|\nabla u_n|^{p-1} \to w\) pointwise a.e. (implying \(|\nabla u_n| \to w^{\frac{1}{p-1}}\) pointwise a.e.) and so \(|\nabla u_n| \to w^{\frac{1}{p-1}}\) in \(L^p(\Omega)\). Hence \(w = |\nabla u|^{p-1}\).

Thus in all the above cases we found the following.

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v \, dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx, \quad \forall v \in W^{1,p}_0(\Omega). \tag{4.2}
\]

Hence by the compact embedding due to Rellich-Kondrachov it can be concluded \(u_n \to u\) in \(L^q(\Omega)\). Thus we also have

\[
\lim_{n \to \infty} \int_{\Omega} |u_n|^{q-2} u_n v \, dx = \int_{\Omega} |u|^{q-2} u v \, dx, \quad \forall v \in W^{1,p}_0(\Omega). \tag{4.3}
\]

5. Conclusions

The resonant Lane-Emden problem has been studied. An existence result has been established to the non-homogeneous Lane-Emden problem for the super-linear case - \(1 < p < q < p^*\) for \(\lambda \in (0, \lambda'] - \lambda'\) being sufficiently large - if it is assumed that a solution exists to the homogeneous Lane-Emden problem for the super-linear case - \(1 < p < q < p^*\). We further proved the ‘converse’ that if the non-homogeneous problem has a solution then a solution to the homogeneous problem exists for the super linear case. We also established an ‘equivalence’ of eigenvalue problem and the non homogeneous Lane-Emden problem.

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