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A PROBLEM INVOLVING THE p–LAPLACIAN OPERATOR

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Abstract. Using a variational technique we guarantee the existence of a solution to the *resonant Lane-Emden* problem $-\Delta_p u = \lambda|u|^{q-2}u,\ u|_{\partial\Omega} = 0$ if and only if a solution to $-\Delta_p u = \lambda|u|^{q-2}u + f,\ u|_{\partial\Omega} = 0,\ f \in L^{p'}(\Omega)$ (p' being the conjugate of p), exists for $q \in (p, p^*)$ under certain condition on λ, where p^* is the Sobolev conjugate of p.

1. Introduction

The study of partial differential equations involving a p-Laplacian differential operator has become a major case of study in the recent times although it is still far from being completely understood, especially when $p = 1$ or ∞. A few evidences of the limiting case can be found in [18], [20]. In fact, existence of a positive eigenvector to the eigenvalue problem can be found in [7]. When $p = 2$, the usual Laplacian is obtained for which a vast literature exists ([10], [11] and the references therein). For $p \neq 2$ the p-Laplacian operator has physical applications in the study of non-Newtonian fluids (dilatant fluids when $p > 2$) [15]. In practical life most of the problems are non linear by nature for which a numerical solution is seeked for, however, unearthing the existence of solution leads to a rich theory hidden behind the partial differential equations. The problems we are going to address in this article are the following. Let Ω be a bounded subset of $\mathbb{R}^n$, $n \geqslant 3$ with a Lipschitz boundary $\partial\Omega$. Given $1 < p < \infty$ and $q \in (p, p^*)$, where $p^* = \frac{np}{n-p}$ if $1 < p < n$ and $p^* = \infty$ if $p \geqslant n$, we consider the following problems.

1. $-\Delta_p u = \lambda|u|^{q-2}u,\ u|_{\partial\Omega} = 0$. This problem is also known as the resonant Lane-Emden problem.
2. $-\Delta_p u = \lambda|u|^{q-2}u + f,\ f \in L^{p'}(\Omega),\ u|_{\partial\Omega} = 0$.

where λ is a real number, $\Delta_p = \nabla\cdot(|\cdot|^{p-2}\nabla\cdot)$. Throughout this paper we shall refer the problems in 1 and 2 as the first and the second problem respectively.

We call the first problem to be of sub-liear type if $1 < q < p < p^*$ and of super-linear type when $1 < p < q < p^*$. In this article, we restrict the first problem to be of

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super-linear type. It is found in [5, 6] that a unique solution exists to the first problem for the sub-linear case whereas uniqueness is lost for the super-linear case. Readers interested in knowing more about the first problem can refer to examples found in [8], [16], where the domain is ring shaped for $q \sim p^*$ and the solution is non-unique. Kawohl [2] showed the same but the domain which was considered is of annulus type with the annulus being sufficiently small in size. Uniqueness of solution is also guaranteed in [9] for the sub-linear case whereas a subdifferential method has been used to prove existence in [13] for both sub and super linear cases. Grumiau and Parini [3] discussed the asymptotic behavior of the ground state solutions as $q \to p$. In recent times Véron et al [12] considered a similar problem but with a measure instead of the function f. They have characterized the 'good' measures for which the problem - $\Delta_p u + g(x,u) = \mu$, $u|_{\partial\Omega} = 0$ - where $g(.,.)$ is non-decreasing, Ω is a bounded domain in $\mathbb{R}^n$ - has a solution. Interested readers can also refer to the work of Giri and Choudhuri [19] (and the references therein) who have used the notion of 'reduced limit' for problems with measure data.

In this paper we will use a well known variational technique to show the existence of a solution in $W_0^{1,p}(\Omega) = \{v \in L^p(\Omega) : \nabla v \in L^p(\Omega), v|_{\partial\Omega} = 0\}$. A Fredholm type alternative is also proposed thus showing a connection between the first and the second problem. We organize the paper into two sections. In Section 2 we give the Mathematical formulation. In Section 3 we discuss a few preliminary results and the main result.

2. Mathematical formulation

The following definitions and theorems will be used in the main result we prove.

2.1 Definition: Let X be a Banach space and $H : X \to \mathbb{R}$ a C^1 functional. It is said to satisfy the *Palais-Smale condition* (PS) if the following holds.

Whenever $\{u_n\}$ is a sequence in X such that $\{H(u_n)\}$ is bounded and $H^{'}(u_n) \to 0$ strongly in $X^{'}$ (the dual space), then $\{u_n\}$ has a strongly convergent subsequence in X.

The (PS) condition is a strong condition as very "well-behaved" function do not satisfy it (*Example*: $f(x) = c$, $x \in \mathbb{R}$, c a real constant).

We now state the following important theorem due to Ambrosetti and Rabinowitz [1] which is a common tool used in the theory of modern PDEs.

Mountain-pass theorem: Let $H : X \to \mathbb{R}$ be a C^1 functional satisfying (PS). Let u_0, $u_1 \in X$, $c_0 \in \mathbb{R}$ and $r > 0$ such that

1. $||u_1 - u_0|| > r$

2. $H(u_0)$, $H(u_1) < c_0 \leqslant H(v)$, $\forall v$ such that $||v - u_0|| = r$. Then H has a critical value $c \geqslant c_0$ defined by

$$c = \inf_{\Gamma \in \wp} \max_{t \in [0,1]} H(\Gamma(t)), \tag{2.1}$$

where $\wp$ is the collection of all continuous paths $\Gamma : [0,1] \to X$ such that $\Gamma(0) = u_0$, $\Gamma(1) = u_1$.

2.2 Weak formulation of the problem: We now give the weak formulation of the first problem. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the first problem if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx - \lambda \int_\Omega |u|^{q-2} u v dx = 0 \tag{2.2}$$

for every $v \in W_0^{1,p}(\Omega)$.

The weak solutions of the Lane-Emden problem are the critical points of the energy function defined by

$$J_q(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{q}\int_\Omega |u|^q dx. \tag{2.3}$$

The following compact embedding theorems, due to Rellich-Kondrachov will be used in our work.

1. if $p < n$, $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leqslant q < p^*$,
2. if $p = n$, $W_0^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leqslant q < \infty$,
3. if $p > n$, $W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$.

We consider the non-homogeneous counterpart of the first problem - which is the second problem - and is as follows.

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2} u + f, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{2.4}$$

where $f \in L^{p'}(\Omega)$, p' being the conjugate of p, which is equal to $\frac{p}{p-1}$. Let the corresponding energy functional be denoted by J which is defined as follows.

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{q}\int_\Omega |u|^q dx - \int_\Omega f u dx. \tag{2.5}$$

The Fréchet derivative of J, which is in $W_0^{-1,p'}(\Omega)$ where $p' = \frac{p}{p-1}$, is

$$< J'(u), v > = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx - \lambda \int_\Omega |u|^{q-2} u v dx - \int_\Omega f v dx, \tag{2.6}$$

$\forall v \in W_0^{1,p}(\Omega)$. Thus $u \in W_0^{1,p}(\Omega)$ is a weak solution of the second problem if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx - \lambda \int_\Omega |u|^{q-2} u v dx - \int_\Omega f v dx = 0.$$

3. Few preliminary results and the main theorem

The main result of this paper, stated informally, is as follows. The problem $-\Delta_p u = \lambda |u|^{q-2} u$, $u|_{\partial\Omega} = 0$ has a weak solution if and only if the problem $-\Delta_p u = \lambda |u|^{q-2} u +$

f, $u|_{\partial\Omega}=0$, where $f\in L^{p/p-1}(\Omega)$, has a weak solution. We prove the result for $p<n$. The case of $p\geqslant n$ follows the same proof as in the case $p<n$ which is based on the results on compact embedding stated after equation (2.3). But first we present a few technical lemmas on which the proof of this result will rely upon.

We first assume that a nontrivial solution exists to the problem

$$\begin{aligned} -\Delta_p u &= \lambda|u|^{q-2}u, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{3.1}$$

THEOREM 1. *The mapping J defined in (2.5) is a C^1-functional over $W_0^{1,p}(\Omega)$.*

Proof. The functional J is differentiable which can be seen by extending the arguments in [17], Theorem 5.3.1. Thus it is enough to show that J' is continuous. Now from (2.6), we have

$$\begin{aligned} |<J^{'}(u),v>| &\leqslant |\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx|+|\lambda|\int_\Omega |u|^{q-1}|v|dx+\int_\Omega |f||v|dx \\ &\leqslant ||\nabla u||_{\frac{p}{p-1}}||\nabla v||_p+|\lambda|||u||_{\frac{q}{q-1}}||v||_q+||f||_{\frac{p}{p-1}}||v||_p \\ &\leqslant \left[||\nabla u||_{\frac{p}{p-1}}+C_1|\lambda|||u||_{\frac{q}{q-1}}+C_2||f||_{\frac{p}{p-1}}\right]||\nabla v||_p, \end{aligned} \tag{3.2}$$

$\forall v\in W_0^{1,p}(\Omega)$, where C_1, C_2 are the constants due to the embedding of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$ for $q\in[1,p^*]$. From (3.2) one can see that J is a C^1 functional over $W_0^{1,p}(\Omega)$.

THEOREM 2. *There exists $u_0,u_1\in W_0^{1,p}(\Omega)$ and a positive real number c_0 such that $J(u_0),J(u_1)<c_0$ and $J(v)\geqslant c_0$, for every v satisfying $||v-u_0||_{1,p}=r$.*

Proof. Let $u_0=0$. Clearly u_0 is a solution of (3.1) and $J(0)=0$. Now let $w\in B(0,1)=\{u\in W_0^{1,p}(\Omega):||u||_{1,p}=1\}$ and consider $v=u_0+rw$ for $r>0$ and hence $||v-u_0||_{1,p}=r$. We first show the existence of r_0, c_0 such that for each v we have $||v-u_0||_{1,p}=r_0$ and for which $J(v)\geqslant c_0$, where $c_0>0$.
Since $p<q<p^*$, we have

$$\begin{aligned} J(u_0+rw)-J(u_0) &= \frac{r^p}{p}\int_\Omega|\nabla w|^p dx-\frac{r^q\lambda}{q}\int_\Omega|w|^q dx-r\int_\Omega fwdx, \\ &= \frac{r^p}{p}-\frac{r^q\lambda}{q}\int_\Omega|w|^q dx-r\int_\Omega fwdx. \end{aligned} \tag{3.3}$$

Further, $|w|_{1,p}=1$ and hence $|\int_\Omega w^p dx|\leqslant\int_\Omega|w^p|dx=||w||_p^p\leqslant c_1|w|_{1,p}^p=c_1$. Similarly, $|\int_\Omega w^q dx|\leqslant c_2$. Using these arguments leads to

$$J(u_0+rw)-J(u_0)\geqslant r\left[\frac{r^{p-1}}{p}-\frac{r^{q-1}\lambda}{q}c_2-c_1^{1/p}||f||_{p'}\right]. \tag{3.4}$$

We first analyze the term

$$\left[\frac{r^{p-1}}{p} - \frac{r^{q-1}\lambda}{q}c_2 - c_1^{1/p}||f||_{p'}\right] = F(r)$$

(say). Clearly $F(0) < 0$ and for $r_0 = \left(\frac{q(p-1)}{p(q-1)}\frac{1}{\lambda c_2}\right)^{\frac{1}{q-p}}$ we see that $F'(r_0) = 0$. A bit of calculus guarantees that $F''(r_0) < 0$ and hence r_0 is a maximizer of F. Note that, if

$$0 < \lambda < \lambda_1 = \frac{q(p-1)}{c_2 p(q-1)}.\left(\frac{q-p}{p(q-1)}.\frac{1}{c_1^{\frac{1}{p}}||f||_{p'}}\right)^{\frac{q-p}{p-1}},$$

then $F(r_0) > 0$. As $r \to \infty$ we have $F(r) \to -\infty$. Hence there exists $r_1, r_2 > 0$, $r_0 > 0$ (this r_0 could be different from the above one) such that $r_1F(r_1) = r_2F(r_2) = c'$ (say), $r_1 < r_0 < r_2$ and $rF(r) > 0$ for each $r \in (r_1, r_2)$. Thus for v such that $||v - u_0||_{1,p} = r_0$ we have $J(v) \geqslant c' > 0$ for each $v \in B(0, r_0) \subset W_0^{1,p}(\Omega)$.

Choice of u_1: Let w_q be a nontrivial solution to the equation $-\Delta_p w_q = \lambda |w_q|^{q-2}w_q$ in Ω, $w_q = 0$ on $\partial\Omega$. Consider the function $g = kw_q$, $k \in \mathbb{R}$, where we have normalized w_q with respect to the Sobolev norm on $W_0^{1,p}(\Omega)$ without changing its notation. Note that,

$$J(g) = \left(\frac{k^p}{p} - \frac{\lambda k^q \int_\Omega |w_q|^q dx}{q}\right) - kC,$$

where $C = \int_\Omega f w_q dx$. Since $p < q < p^*$, we choose k_0 to be sufficiently large so that $\frac{k_0^p}{p} - \frac{\lambda k_0^q \int_\Omega |w_q|^q dx}{q} - k_0 C < 0$. Then $J(k_0 w_q) < 0$ and hence $J(k_0 w_q) < J(u_0)$. Thus we can choose $u_1 = k_0 w_q$, where $k_0 > r_0$, due to which $||u_1 - u_0||_{1,p} > r_0$. Hence the result.

THEOREM 3. *The functional J satisfies the Palais-Smale condition.*

Proof. Let u_n be a sequence in $W_0^{1,p}(\Omega)$ such that $|J(u_n)| \leqslant M$ and $J'(u_n) \to 0$ as $n \to \infty$ in $W_0^{-1,p'}(\Omega)$, p' being the conjugate of p. Now

$$J(u_n) = \frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \frac{\lambda}{q}\int_\Omega |u_n|^q dx - \int_\Omega f u_n dx, \tag{3.5}$$

$$< J'(u_n), v > = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n.\nabla v dx - \lambda \int_\Omega |u_n|^{q-2}u_n v dx - \int_\Omega f v dx, \tag{3.6}$$

for all $v \in W_0^{1,p}(\Omega)$. Consider the following.

$$
\begin{aligned}
< J^{'}(u_n), u_n > &= \int_\Omega |\nabla u_n|^p dx - \lambda \int_\Omega |u_n|^q dx - \int_\Omega f u_n dx, \qquad (3.7)\\
J(u_n) &= \frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \frac{\lambda}{q}\int_\Omega |u_n|^q dx - \int_\Omega f u_n dx,\\
&= \frac{1}{p}|u_n|_{1,p}^p - \frac{\lambda}{q}\int_\Omega |u_n|^q dx - \int_\Omega f u_n dx\\
\lambda \int_\Omega |u_n|^q dx &= \frac{q}{p}|u_n|_{1,p}^p - qJ(u_n) - q\int_\Omega f u_n dx,\\
\frac{p-q}{p}|u_n|_{1,p}^p &= < J^{'}(u_n), u_n > - qJ(u_n) - (q-1)\int_\Omega f u_n dx. \qquad (3.8)
\end{aligned}
$$

From (3.8) $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ and hence by Eberlein-Šmulian's theorem (refer Dunford-Schwartz [1; p. 430] [14]) it has a *weakly* convergent subsequence, say $\{u_{n_k}\}$, in $W_0^{1,p}(\Omega)$.

Claim. The subsequence $\{u_{n_k}\}$ is strongly convergent in $W_0^{1,p}(\Omega)$.

Proof. Applying limit $n_k \to \infty$ to (3.6) (refer Appendix) and using the strong convergence of (u_{n_k}) in $L^q(\Omega)$ due to compact embedding we obtain

$$
\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx = \lambda \int_\Omega |u|^{q-2} u v dx + \int_\Omega f v dx, \qquad (3.9)
$$

We then pass on the limit $n_k \to \infty$ to (3.7) to get

$$
\lim_{n_k \to \infty} |u_{n_k}|_{1,p}^p = \lambda \int_\Omega |u|^q dx + \int_\Omega f u dx = |u|_{1,p}^p. \qquad (3.10)
$$

Since, a weakly convergent sequence which is convergent in norm is strongly convergent, hence $u_{n_k} \to u$ in $W_0^{1,p}(\Omega)$ as $n_k \to \infty$.

So, by the Mountain-pass theorem an extreme point for J exists in $W_0^{1,p}(\Omega)$. We summarize the results proved in Theorems 1, 2 and 3 in the form of a unified theorem as follows.

THEOREM 4. *Suppose* $-\Delta_p u = \lambda |u|^{q-2}u,\ u|_{\partial\Omega} = 0$ *has a nontrivial solution for some* $\lambda > 0$, *where* $q \in (p, p^*)$. *Then the problem* $-\Delta_p u = \lambda |u|^{q-2}u + f,\ f \in L^{p'}(\Omega)$, $u|_{\partial\Omega} = 0$ *has a nontrivial solution whenever* $\lambda \in (0, \lambda']$ *where* $\lambda' < \lambda_1$ *and* $\lambda_1 = \frac{q(p-1)}{c_2 p(q-1)} \cdot \left(\frac{q-p}{p(q-1)} \cdot \frac{1}{c_1^{\frac{1}{p}} ||f||_{p'}} \right)^{\frac{q-p}{p-1}}$, $p' = \frac{p}{p-1}$, $c_1^{1/p}$, $c_2^{1/q}$ *are the Sobolev constants corresponding to the embedding of* $W_0^{1,p}(\Omega)$ *in* $L^p(\Omega)$, $L^q(\Omega)$ *respectively.*

Arguing on similar lines, as in Theorems 1, 2 and 3, we conclude the following result.

THEOREM 5. *If the eigenvalue problem*

$$-\Delta_p u = \lambda |u|^{p-2}u \text{ in } \Omega,$$
$$u|_{\partial\Omega} = 0 \text{ on } \partial\Omega,$$

has a nontrivial solution, then the non homogeneous Lane-Emden problem

$$-\Delta_p u = \lambda |u|^{q-2}u + f,\ f \in L^{p'}(\Omega),$$
$$u|_{\partial\Omega} = 0,$$

has a nontrivial solution for $q \in (p,p^*)$ *whenever* $\lambda \in (0,\lambda']$ *where* $\lambda' < \lambda_1$ *and* $\lambda_1 = \frac{q(p-1)}{c_2 p(q-1)} \cdot \left(\frac{q-p}{p(q-1)} \cdot \frac{1}{c_1^{\frac{1}{p}} ||f||_{p'}} \right)^{\frac{q-p}{p-1}}$, $p' = \frac{p}{p-1}$, $c_1^{1/p}$, $c_2^{1/q}$ *are the Sobolev constants corresponding to the embedding of* $W_0^{1,p}(\Omega)$ *in* $L^p(\Omega)$, $L^q(\Omega)$ *respectively.*

Conversely, suppose to each $f \in L^{p'}(\Omega)$ the problem

$$-\Delta_p u = \lambda |u|^{q-2}u + f,$$
$$u|_{\partial\Omega} = 0, \tag{3.11}$$

has a nontrivial solution on the set $\mathfrak{M} = \{u \in W_0^{1,p}(\Omega) : ||u||_q = 1\}$ for some $\lambda > 0$, where $q \in [p,p^*)$. Existence of such solution can be assumed from the weak lower semi continuity and coercivity of the corresponding energy functional J on the subset $\mathfrak{M}$ of $W_0^{1,p}(\Omega)$ (refer [17]). In order to prove the existence of nontrivial solution of the first problem for $q \in [p,p^*)$, we let $\{f_n\} \subset L^{p'}(\Omega)$ be a sequence such that $f_n \to 0$ in $L^{p'}(\Omega)$. Then for each f_n, there exists a solution, say u_n.

We have

$$B[u,v] = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx - \lambda \int_\Omega |u|^{q-2}uv dx,$$
$$= \int_\Omega f v dx,\ \forall v \in W_0^{1,p}(\Omega), \tag{3.12}$$

where B is a '*non linear form*' in two variables u and v. It is easy to check that $B(.,.)$ is the Fréchet derivative of the C^1 functional $\frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q$ and hence is continuous.

Clearly, for each $v \in W_0^{1,p}(\Omega)$ we have

$$B[u_n,v] = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx - \lambda \int_\Omega |u_n|^{q-2}u_n v dx,$$
$$= \int_\Omega f_n v dx,$$
$$\leqslant ||f_n||_{p'} ||v||_p \to 0 \text{ as } n \to \infty. \tag{3.13}$$

Hence $\int_\Omega f_n v dx \to 0$ as $n \to \infty$. Consider $T_n(v) = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx$. Then T_n's are bounded linear over $W_0^{1,p}(\Omega)$ and $||T_n|| = |||\nabla u_n|^{p-1}||_{p'}$ for $n \geqslant 1$. From the above definition of T_n, for a fixed $v \in W_0^{1,p}(\Omega)$ we have the sequence $\{T_n(v)\}$ to be bounded which implies that $\{T_n(v)\}$ is pointwise bounded. Thus by the uniform boundedness principle $\{||T_n||\}$ is bounded. Thus $\{||\nabla u_n||_p\}$ is bounded. Hence, there exists a subsequence $\{u_{n_k}\}$ which weakly converges to u_∞ with respect to the norm $||\cdot||_{1,p}$ in $W_0^{1,p}(\Omega)$. Hence we have

$$\lim_{n_k\to\infty}\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u_{n_k}dx = \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla u_\infty dx, \forall v \in W_0^{1,p}(\Omega).$$
$$\Rightarrow \lim_{n_k\to\infty}\int_\Omega |\nabla u_{n_l}|^{p-2}\nabla u_{n_l}\cdot\nabla u_{n_k}dx = \int_\Omega |\nabla u_{n_l}|^{p-2}\nabla u_{n_l}\cdot\nabla u_\infty dx, \tag{3.14}$$

for a fixed l. Therefore, since $u_{n_k} \rightharpoonup u_\infty$ in $W_0^{1,p}(\Omega)$ implies that $|\nabla u_{n_k}|^{p-1} \rightharpoonup |\nabla u_\infty|^{p-1}$ (for a subsequence) in $L^{p'}(\Omega)$ (Refer Appendix). But $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ and hence

$$\lim_{n_l\to\infty}\int_\Omega |\nabla u_{n_l}|^{p-2}\nabla u_{n_l}\cdot\nabla v dx = \int_\Omega |\nabla u_\infty|^{p-2}\nabla u_\infty\cdot\nabla v dx, \forall v \in W_0^{1,p}(\Omega),$$
$$\Rightarrow \lim_{n_l\to\infty}\int_\Omega |\nabla u_{n_l}|^{p-2}\nabla u_{n_l}\cdot\nabla u_\infty dx = \int_\Omega |\nabla u_\infty|^p dx. \tag{3.15}$$

Therefore, $\lim_{n_k\to\infty}\int_\Omega |\nabla u_{n_k}|^p dx = \int_\Omega |\nabla u_\infty|^p dx$. It immediately can be concluded that there exists a u_∞ such that $u_{n_k} \to u_\infty$ in $W_0^{1,p}(\Omega)$. Hence using the continuity of $B[.,.]$ in (3.12) we have

$$\begin{aligned}\lim_{n_k\to\infty} B[u_{n_k}, v] &= \lim_{n_k\to\infty}\int_\Omega |\nabla u_{n_k}|^{p-2}\nabla u_{n_k}\cdot\nabla v dx - \lim_{n_k\to\infty}\lambda\int_\Omega |u_{n_k}|^{q-2}u_{n_k}v dx,\\ &= \lim_{n_k\to\infty}\int_\Omega f_{n_k}v dx,\\ \Rightarrow B[u_\infty, v] &= 0, \forall v \in W_0^{1,p}(\Omega).\end{aligned}$$

In other words,

$$\int_\Omega |\nabla u_\infty|^{p-2}\nabla u_\infty\cdot\nabla v dx - \lambda\int_\Omega |u_\infty|^{q-2}u_\infty v dx = 0, \forall v \in W_0^{1,p}(\Omega). \tag{3.16}$$

Assume that $\lambda \in \left(0, \inf_{u\neq 0\in W_0^{1,p}(\Omega)}\left\{\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^q}\right\}\right]$ (infimum exists and is strictly greater than zero which follows from the embedding result for $p < N$). Since $||u_{n_k}||_q = 1$ and $u_{n_k} \to u_\infty$ in $W_0^{1,p}(\Omega)$, hence we have

$$\begin{aligned} 0 < \lambda &\leqslant \liminf \frac{\int_\Omega |\nabla u_{n_k}|^p}{\int_\Omega |u_{n_k}|^q}\\ &= \liminf \int_\Omega |\nabla u_{n_k}|^p\\ &= \liminf ||\nabla u_{n_k}||_p^p = ||\nabla u_\infty||_p^p = ||u_\infty||_{1,p}^p\end{aligned}$$

This implies that u_∞ is a nontrivial solution of the first problem. Thus we summarize the result proved as follows.

THEOREM 6. *Suppose to each* $f \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$, *the problem* $-\Delta_p u = \lambda |u|^{q-2}u + f$, $u|_{\partial\Omega} = 0$ *has a solution in* $\mathfrak{M} \subset W_0^{1,p}(\Omega)$ *for some* $\lambda > 0$, *then the problem* $-\Delta_p u = \lambda |u|^{q-2}u$, $u|_{\partial\Omega} = 0$, *has a nontrivial solution in* $W_0^{1,p}(\Omega)$ *for* $q \in [p, p^*)$, *whenever* $\lambda \in \left(0, \inf_{u \neq 0 \in W_0^{1,p}(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^q} \right\}\right]$.

We end this section with a small observation from Theorem 6 that if to each $f \in L^{p'}(\Omega)$, the problem $-\Delta_p u = \lambda |u|^{q-2}u + f$, $u|_{\partial\Omega} = 0$ has a solution in $\mathfrak{M} \subset W_0^{1,p}(\Omega)$ for some $\lambda > 0$ and $q \in [p, p^*)$ then the eigenvalue problem $-\Delta_p u = \lambda |u|^{p-2}u$, $u|_{\partial\Omega} = 0$, has a nontrivial solution in $W_0^{1,p}(\Omega)$, whenever $\lambda \in \left(0, \inf_{u \neq 0 \in W_0^{1,p}(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^q} \right\}\right]$.

4. Appendix

We show that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, \ \forall v \in W_0^{1,p}(\Omega). \tag{4.1}$$

We divide the explanation into two cases:

Case 1: When $p > 2$.

This implies that p', the conjugate of p, should be lesser than 2, i.e., $1 < p' < 2 < p$. Thus we have $W_0^{1,p}(\Omega) \hookrightarrow_{compact} L^{p'}(\Omega)$ (since $W_0^{1,p}(\Omega) \hookrightarrow_{compact} L^q(\Omega)$ for $q \in [1, p^*)$). Since ∇u_n converges weakly to, say ∇u, in $L^p(\Omega)$, hence $< |\nabla u_n| - |\nabla u|, v > \to 0$ for each $v \in L^{p'}(\Omega)$. Thus $< |\nabla u_n| - |\nabla u|, |\nabla u_n| - |\nabla u| > \to 0$, i.e., $||\nabla u_n||_2 \to ||\nabla u||_2$. Hence $||\nabla u_n||_{p'} \to ||\nabla u||_{p'}$ because $p' < 2 < p$. By the Riesz-Fischer theorem [4], there exists a subsequence of $\{\nabla u_n\}$ which converges pointwise a.e., i.e., $|\nabla u_n(x)| \to |\nabla u(x)|$. So $|\nabla u_n(x)|^{p-1} \to |\nabla u(x)|^{p-1}$ and hence $|\nabla u_n|^{p-1} \rightharpoonup |\nabla u|^{p-1}$ in $L^{p'}(\Omega)$. Thus we have $\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx$, $\forall v \in W_0^{1,p}(\Omega)$.

Case 2: When $p < 2$.

This implies that p', the conjugate of p, should be greater than 2, i.e., $p < 2 < p'$.

Look at the map $F : W_0^{1,p}(\Omega) \to L^{p'}(\Omega)$ defined by $u \mapsto |\nabla u|^{p-1}$. Consider the range of F, i.e., $R(F) = \{|\nabla u|^{p-1} : u \in W_0^{1,p}(\Omega)\}$.

Observe that the map F is bounded in the sense that bounded sets are mapped to bounded sets. Hence if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ implies that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence $\{F(u_n)\} = \{|\nabla u_n|^{p-1}\}$ is bounded in $L^{p'}(\Omega)$. Since $L^{p'}(\Omega)$ is reflexive, hence there exists a subsequence of $\{|\nabla u_n|^{p-1}\}$ which weakly converges to, say, w in $L^{p'}(\Omega)$.

We have the following: $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ so $|\nabla u_n|^{p-1} \rightharpoonup w$ in $L^{p'}(\Omega)$. This implies that

$$< |\nabla u_n|^{p-1} - w, v > \to 0, \forall v \in L^p(\Omega)$$

Since $p < 2 < p'$ hence $|\nabla u_n|^{p-1} - w \in L^p(\Omega)$. Thus $|||\nabla u_n|^{p-1} - w||_2 \to 0$ and hence $|||\nabla u_n|^{p-1} - w||_p \to 0$. Therefore we have a subsequence of $\{|\nabla u_n|^{p-1}\}$ such that $|\nabla u_n|^{p-1} \to w$ pointwise a.e. (implying $|\nabla u_n| \to w^{\frac{1}{p-1}}$ pointwise a.e.) and so $|\nabla u_n| \rightharpoonup w^{\frac{1}{p-1}}$ in $L^p(\Omega)$. Hence $w = |\nabla u|^{p-1}$.

Thus in all the above cases we found the following.

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \ \forall v \in W_0^{1,p}(\Omega). \tag{4.2}$$

Hence by the compact embedding due to Rellich-Kondrachov it can be concluded $u_n \to u$ in $L^q(\Omega)$. Thus we also have

$$\lim_{n\to\infty} \int_\Omega |u_n|^{q-2}u_n v dx = \int_\Omega |u|^{q-2}u v dx, \ \forall v \in W_0^{1,p}(\Omega). \tag{4.3}$$

5. Conclusions

The resonant Lane-Emden problem has been studied. An existence result has been established to the non-homogeneous Lane-Emden problem for the super-linear case - $1 < p < q < p^*$ for $\lambda \in (0, \lambda']$ - λ' being sufficiently large - if it is assumed that a solution exists to the homogeneous Lane-Emden problem for the super-linear case - $1 < p < q < p^*$. We further proved the '*converse*' that if the non-homogeneous problem has a solution then a solution to the homogeneous problem exists for the super linear case. We also established an '*equivalence*' of eigenvalue problem and the non homogeneous Lane-Emden problem.

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