

A PROBLEM INVOLVING THE p -LAPLACIAN OPERATOR

RATAN KR. GIRI AND D. CHOUDHURI

(Communicated by Jesús Ildefonso Díaz)

Abstract. Using a variational technique we guarantee the existence of a solution to the resonant Lane-Emden problem $-\Delta_p u = \lambda |u|^{q-2} u$, $u|_{\partial\Omega} = 0$ if and only if a solution to $-\Delta_p u = \lambda |u|^{q-2} u + f$, $u|_{\partial\Omega} = 0$, $f \in L^{p'}(\Omega)$ (p' being the conjugate of p), exists for $q \in (p, p^*)$ under certain condition on λ , where p^* is the Sobolev conjugate of p .

1. Introduction

The study of partial differential equations involving a p -Laplacian differential operator has become a major case of study in the recent times although it is still far from being completely understood, especially when $p = 1$ or ∞ . A few evidences of the limiting case can be found in [18], [20]. In fact, existence of a positive eigenvector to the eigenvalue problem can be found in [7]. When $p = 2$, the usual Laplacian is obtained for which a vast literature exists ([10], [11] and the references therein). For $p \neq 2$ the p -Laplacian operator has physical applications in the study of non-Newtonian fluids (dilatant fluids when $p > 2$) [15]. In practical life most of the problems are non linear by nature for which a numerical solution is seeked for, however, unearthing the existence of solution leads to a rich theory hidden behind the partial differential equations. The problems we are going to address in this article are the following. Let Ω be a bounded subset of \mathbb{R}^n , $n \geq 3$ with a Lipschitz boundary $\partial\Omega$. Given $1 < p < \infty$ and $q \in (p, p^*)$, where $p^* = \frac{np}{n-p}$ if $1 < p < n$ and $p^* = \infty$ if $p \geq n$, we consider the following problems.

1. $-\Delta_p u = \lambda |u|^{q-2} u$, $u|_{\partial\Omega} = 0$. This problem is also known as the resonant Lane-Emden problem.
2. $-\Delta_p u = \lambda |u|^{q-2} u + f$, $f \in L^{p'}(\Omega)$, $u|_{\partial\Omega} = 0$.

where λ is a real number, $\Delta_p = \nabla \cdot (|\cdot|^{p-2} \nabla \cdot)$. Throughout this paper we shall refer the problems in 1 and 2 as the first and the second problem respectively.

We call the first problem to be of sub-linear type if $1 < q < p < p^*$ and of super-linear type when $1 < p < q < p^*$. In this article, we restrict the first problem to be of

Mathematics subject classification (2010): 35A15, 35A01.

Keywords and phrases: p -Laplacian, elliptic PDE, Palais-Smale condition, Sobolev space.

One of the authors (R. Kr. Giri) thanks the financial assistantship received from the Ministry of Human Resource Development (M.H.R.D.), Govt. of India.

super-linear type. It is found in [5, 6] that a unique solution exists to the first problem for the sub-linear case whereas uniqueness is lost for the super-linear case. Readers interested in knowing more about the first problem can refer to examples found in [8], [16], where the domain is ring shaped for $q \sim p^*$ and the solution is non-unique. Kawohl [2] showed the same but the domain which was considered is of annulus type with the annulus being sufficiently small in size. Uniqueness of solution is also guaranteed in [9] for the sub-linear case whereas a subdifferential method has been used to prove existence in [13] for both sub and super linear cases. Grumiau and Parini [3] discussed the asymptotic behavior of the ground state solutions as $q \rightarrow p$. In recent times Véron et al [12] considered a similar problem but with a measure instead of the function f . They have characterized the ‘good’ measures for which the problem - $\Delta_p u + g(x, u) = \mu, u|_{\partial\Omega} = 0$ - where $g(\cdot, \cdot)$ is non-decreasing, Ω is a bounded domain in \mathbb{R}^n - has a solution. Interested readers can also refer to the work of Giri and Choudhuri [19] (and the references therein) who have used the notion of ‘reduced limit’ for problems with measure data.

In this paper we will use a well known variational technique to show the existence of a solution in $W_0^{1,p}(\Omega) = \{v \in L^p(\Omega) : \nabla v \in L^p(\Omega), v|_{\partial\Omega} = 0\}$. A Fredholm type alternative is also proposed thus showing a connection between the first and the second problem. We organize the paper into two sections. In Section 2 we give the Mathematical formulation. In Section 3 we discuss a few preliminary results and the main result.

2. Mathematical formulation

The following definitions and theorems will be used in the main result we prove.

2.1 Definition: Let X be a Banach space and $H : X \rightarrow \mathbb{R}$ a C^1 functional. It is said to satisfy the *Palais-Smale condition* (PS) if the following holds.

Whenever $\{u_n\}$ is a sequence in X such that $\{H(u_n)\}$ is bounded and $H'(u_n) \rightarrow 0$ strongly in X' (the dual space), then $\{u_n\}$ has a strongly convergent subsequence in X .

The (PS) condition is a strong condition as very “well-behaved” function do not satisfy it (*Example:* $f(x) = c, x \in \mathbb{R}, c$ a real constant).

We now state the following important theorem due to Ambrosetti and Rabinowitz [1] which is a common tool used in the theory of modern PDEs.

Mountain-pass theorem: Let $H : X \rightarrow \mathbb{R}$ be a C^1 functional satisfying (PS). Let $u_0, u_1 \in X, c_0 \in \mathbb{R}$ and $r > 0$ such that

1. $\|u_1 - u_0\| > r$
2. $H(u_0), H(u_1) < c_0 \leq H(v), \forall v$ such that $\|v - u_0\| = r$. Then H has a critical value $c \geq c_0$ defined by

$$c = \inf_{\Gamma \in \mathcal{P}} \max_{t \in [0,1]} H(\Gamma(t)), \tag{2.1}$$

where \mathcal{P} is the collection of all continuous paths $\Gamma : [0, 1] \rightarrow X$ such that $\Gamma(0) = u_0, \Gamma(1) = u_1$.

2.2 Weak formulation of the problem: We now give the weak formulation of the first problem. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the first problem if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx - \lambda \int_{\Omega} |u|^{q-2} u v \, dx = 0 \tag{2.2}$$

for every $v \in W_0^{1,p}(\Omega)$.

The weak solutions of the Lane-Emden problem are the critical points of the energy function defined by

$$J_q(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx. \tag{2.3}$$

The following compact embedding theorems, due to Rellich-Kondrachov will be used in our work.

1. if $p < n$, $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leq q < p^*$,
2. if $p = n$, $W_0^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leq q < \infty$,
3. if $p > n$, $W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$.

We consider the non-homogeneous counterpart of the first problem - which is the second problem - and is as follows.

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2} u + f, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{2.4}$$

where $f \in L^{p'}(\Omega)$, p' being the conjugate of p , which is equal to $\frac{p}{p-1}$. Let the corresponding energy functional be denoted by J which is defined as follows.

$$J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx - \int_{\Omega} f u \, dx. \tag{2.5}$$

The Fréchet derivative of J , which is in $W_0^{-1,p'}(\Omega)$ where $p' = \frac{p}{p-1}$, is

$$\langle J'(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx - \lambda \int_{\Omega} |u|^{q-2} u v \, dx - \int_{\Omega} f v \, dx, \tag{2.6}$$

$\forall v \in W_0^{1,p}(\Omega)$. Thus $u \in W_0^{1,p}(\Omega)$ is a weak solution of the second problem if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx - \lambda \int_{\Omega} |u|^{q-2} u v \, dx - \int_{\Omega} f v \, dx = 0.$$

3. Few preliminary results and the main theorem

The main result of this paper, stated informally, is as follows. The problem $-\Delta_p u = \lambda |u|^{q-2} u$, $u|_{\partial\Omega} = 0$ has a weak solution if and only if the problem $-\Delta_p u = \lambda |u|^{q-2} u +$

$f, u|_{\partial\Omega} = 0$, where $f \in L^{p/p-1}(\Omega)$, has a weak solution. We prove the result for $p < n$. The case of $p \geq n$ follows the same proof as in the case $p < n$ which is based on the results on compact embedding stated after equation (2.3). But first we present a few technical lemmas on which the proof of this result will rely upon.

We first assume that a nontrivial solution exists to the problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2} u, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{3.1}$$

THEOREM 1. *The mapping J defined in (2.5) is a C^1 -functional over $W_0^{1,p}(\Omega)$.*

Proof. The functional J is differentiable which can be seen by extending the arguments in [17], Theorem 5.3.1. Thus it is enough to show that J' is continuous. Now from (2.6), we have

$$\begin{aligned} | \langle J'(u), v \rangle | &\leq \left| \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx \right| + |\lambda| \int_{\Omega} |u|^{q-1} |v| dx + \int_{\Omega} |f| |v| dx \\ &\leq \|\nabla u\|_{\frac{p}{p-1}} \|\nabla v\|_p + |\lambda| \|u\|_{\frac{q}{q-1}} \|v\|_q + \|f\|_{\frac{p}{p-1}} \|v\|_p \\ &\leq \left[\|\nabla u\|_{\frac{p}{p-1}} + C_1 |\lambda| \|u\|_{\frac{q}{q-1}} + C_2 \|f\|_{\frac{p}{p-1}} \right] \|\nabla v\|_p, \end{aligned} \tag{3.2}$$

$\forall v \in W_0^{1,p}(\Omega)$, where C_1, C_2 are the constants due to the embedding of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$ for $q \in [1, p^*]$. From (3.2) one can see that J is a C^1 functional over $W_0^{1,p}(\Omega)$.

THEOREM 2. *There exists $u_0, u_1 \in W_0^{1,p}(\Omega)$ and a positive real number c_0 such that $J(u_0), J(u_1) < c_0$ and $J(v) \geq c_0$, for every v satisfying $\|v - u_0\|_{1,p} = r$.*

Proof. Let $u_0 = 0$. Clearly u_0 is a solution of (3.1) and $J(0) = 0$. Now let $w \in B(0, 1) = \{u \in W_0^{1,p}(\Omega) : \|u\|_{1,p} = 1\}$ and consider $v = u_0 + rw$ for $r > 0$ and hence $\|v - u_0\|_{1,p} = r$. We first show the existence of r_0, c_0 such that for each v we have $\|v - u_0\|_{1,p} = r_0$ and for which $J(v) \geq c_0$, where $c_0 > 0$. Since $p < q < p^*$, we have

$$\begin{aligned} J(u_0 + rw) - J(u_0) &= \frac{r^p}{p} \int_{\Omega} |\nabla w|^p dx - \frac{r^q \lambda}{q} \int_{\Omega} |w|^q dx - r \int_{\Omega} f w dx, \\ &= \frac{r^p}{p} - \frac{r^q \lambda}{q} \int_{\Omega} |w|^q dx - r \int_{\Omega} f w dx. \end{aligned} \tag{3.3}$$

Further, $\|w\|_{1,p} = 1$ and hence $\int_{\Omega} w^p dx \leq \int_{\Omega} |w|^p dx = \|w\|_p^p \leq c_1 \|w\|_{1,p}^p = c_1$. Similarly, $|\int_{\Omega} w^q dx| \leq c_2$. Using these arguments leads to

$$J(u_0 + rw) - J(u_0) \geq r \left[\frac{r^{p-1}}{p} - \frac{r^{q-1} \lambda}{q} c_2 - c_1^{1/p} |f|_{p'} \right]. \tag{3.4}$$

We first analyze the term

$$\left[\frac{r^{p-1}}{p} - \frac{r^{q-1}\lambda}{q}c_2 - c_1^{1/p}\|f\|_{p'} \right] = F(r)$$

(say). Clearly $F(0) < 0$ and for $r_0 = \left(\frac{q(p-1)}{p(q-1)}\frac{1}{\lambda c_2}\right)^{\frac{1}{q-p}}$ we see that $F'(r_0) = 0$. A bit of calculus guarantees that $F''(r_0) < 0$ and hence r_0 is a maximizer of F . Note that, if

$$0 < \lambda < \lambda_1 = \frac{q(p-1)}{c_2 p(q-1)} \cdot \left(\frac{q-p}{p(q-1)} \cdot \frac{1}{c_1^{1/p}\|f\|_{p'}} \right)^{\frac{q-p}{p-1}},$$

then $F(r_0) > 0$. As $r \rightarrow \infty$ we have $F(r) \rightarrow -\infty$. Hence there exists $r_1, r_2 > 0, r_0 > 0$ (this r_0 could be different from the above one) such that $r_1 F(r_1) = r_2 F(r_2) = c'$ (say), $r_1 < r_0 < r_2$ and $rF(r) > 0$ for each $r \in (r_1, r_2)$. Thus for v such that $\|v - u_0\|_{1,p} = r_0$ we have $J(v) \geq c' > 0$ for each $v \in B(0, r_0) \subset W_0^{1,p}(\Omega)$.

Choice of u_1 : Let w_q be a nontrivial solution to the equation $-\Delta_p w_q = \lambda|w_q|^{q-2}w_q$ in Ω , $w_q = 0$ on $\partial\Omega$. Consider the function $g = kw_q, k \in \mathbb{R}$, where we have normalized w_q with respect to the Sobolev norm on $W_0^{1,p}(\Omega)$ without changing its notation. Note that,

$$J(g) = \left(\frac{k^p}{p} - \frac{\lambda k^q \int_{\Omega} |w_q|^q dx}{q} \right) - kC,$$

where $C = \int_{\Omega} f w_q dx$. Since $p < q < p^*$, we choose k_0 to be sufficiently large so that $\frac{k_0^p}{p} - \frac{\lambda k_0^q \int_{\Omega} |w_q|^q dx}{q} - k_0 C < 0$. Then $J(k_0 w_q) < 0$ and hence $J(k_0 w_q) < J(u_0)$. Thus we can choose $u_1 = k_0 w_q$, where $k_0 > r_0$, due to which $\|u_1 - u_0\|_{1,p} > r_0$. Hence the result.

THEOREM 3. *The functional J satisfies the Palais-Smale condition.*

Proof. Let u_n be a sequence in $W_0^{1,p}(\Omega)$ such that $|J(u_n)| \leq M$ and $J'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ in $W_0^{-1,p'}(\Omega)$, p' being the conjugate of p . Now

$$J(u_n) = \frac{1}{p} \int_{\Omega} |\nabla u_n|^p dx - \frac{\lambda}{q} \int_{\Omega} |u_n|^q dx - \int_{\Omega} f u_n dx, \tag{3.5}$$

$$\langle J'(u_n), v \rangle = \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx - \lambda \int_{\Omega} |u_n|^{q-2} u_n v dx - \int_{\Omega} f v dx, \tag{3.6}$$

for all $v \in W_0^{1,p}(\Omega)$. Consider the following.

$$\langle J'(u_n), u_n \rangle = \int_{\Omega} |\nabla u_n|^p dx - \lambda \int_{\Omega} |u_n|^q dx - \int_{\Omega} f u_n dx, \tag{3.7}$$

$$\begin{aligned} J(u_n) &= \frac{1}{p} \int_{\Omega} |\nabla u_n|^p dx - \frac{\lambda}{q} \int_{\Omega} |u_n|^q dx - \int_{\Omega} f u_n dx, \\ &= \frac{1}{p} |u_n|_{1,p}^p - \frac{\lambda}{q} \int_{\Omega} |u_n|^q dx - \int_{\Omega} f u_n dx \\ \lambda \int_{\Omega} |u_n|^q dx &= \frac{q}{p} |u_n|_{1,p}^p - qJ(u_n) - q \int_{\Omega} f u_n dx, \\ \frac{p-q}{p} |u_n|_{1,p}^p &= \langle J'(u_n), u_n \rangle - qJ(u_n) - (q-1) \int_{\Omega} f u_n dx. \end{aligned} \tag{3.8}$$

From (3.8) $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ and hence by Eberlein-Šmulian’s theorem (refer Dunford-Schwartz [1; p. 430] [14]) it has a weakly convergent subsequence, say $\{u_{n_k}\}$, in $W_0^{1,p}(\Omega)$.

Claim. The subsequence $\{u_{n_k}\}$ is strongly convergent in $W_0^{1,p}(\Omega)$.

Proof. Applying limit $n_k \rightarrow \infty$ to (3.6) (refer Appendix) and using the strong convergence of (u_{n_k}) in $L^q(\Omega)$ due to compact embedding we obtain

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \lambda \int_{\Omega} |u|^{q-2} u v dx + \int_{\Omega} f v dx, \tag{3.9}$$

We then pass on the limit $n_k \rightarrow \infty$ to (3.7) to get

$$\lim_{n_k \rightarrow \infty} |u_{n_k}|_{1,p}^p = \lambda \int_{\Omega} |u|^q dx + \int_{\Omega} f u dx = |u|_{1,p}^p. \tag{3.10}$$

Since, a weakly convergent sequence which is convergent in norm is strongly convergent, hence $u_{n_k} \rightarrow u$ in $W_0^{1,p}(\Omega)$ as $n_k \rightarrow \infty$.

So, by the Mountain-pass theorem an extreme point for J exists in $W_0^{1,p}(\Omega)$. We summarize the results proved in Theorems 1, 2 and 3 in the form of a unified theorem as follows.

THEOREM 4. *Suppose $-\Delta_p u = \lambda |u|^{q-2} u$, $u|_{\partial\Omega} = 0$ has a nontrivial solution for some $\lambda > 0$, where $q \in (p, p^*)$. Then the problem $-\Delta_p u = \lambda |u|^{q-2} u + f$, $f \in L^{p'}(\Omega)$, $u|_{\partial\Omega} = 0$ has a nontrivial solution whenever $\lambda \in (0, \lambda']$ where $\lambda' < \lambda_1$ and $\lambda_1 = \frac{q(p-1)}{c_2 p(q-1)} \cdot \left(\frac{q-p}{p(q-1)} \cdot \frac{1}{c_1^{\frac{1}{p}} \|f\|_{p'}} \right)^{\frac{q-p}{p-1}}$, $p' = \frac{p}{p-1}$, $c_1^{1/p}$, $c_2^{1/q}$ are the Sobolev constants corresponding to the embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$, $L^q(\Omega)$ respectively.*

Arguing on similar lines, as in Theorems 1, 2 and 3, we conclude the following result.

THEOREM 5. *If the eigenvalue problem*

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u \text{ in } \Omega, \\ u|_{\partial\Omega} &= 0 \text{ on } \partial\Omega, \end{aligned}$$

has a nontrivial solution, then the non homogeneous Lane-Emden problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2} u + f, \quad f \in L^{p'}(\Omega), \\ u|_{\partial\Omega} &= 0, \end{aligned}$$

has a nontrivial solution for $q \in (p, p^)$ whenever $\lambda \in (0, \lambda']$ where $\lambda' < \lambda_1$ and*

$$\lambda_1 = \frac{q(p-1)}{c_2 p(q-1)} \cdot \left(\frac{q-p}{p(q-1)} \cdot \frac{1}{c_1^{\frac{p}{q-1}} \|f\|_{p'}} \right)^{\frac{q-p}{p-1}}, \quad p' = \frac{p}{p-1}, \quad c_1^{1/p}, \quad c_2^{1/q} \text{ are the Sobolev constants}$$

corresponding to the embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$, $L^q(\Omega)$ respectively.

Conversely, suppose to each $f \in L^{p'}(\Omega)$ the problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2} u + f, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{3.11}$$

has a nontrivial solution on the set $\mathfrak{M} = \{u \in W_0^{1,p}(\Omega) : \|u\|_q = 1\}$ for some $\lambda > 0$, where $q \in [p, p^*)$. Existence of such solution can be assumed from the weak lower semi continuity and coercivity of the corresponding energy functional J on the subset \mathfrak{M} of $W_0^{1,p}(\Omega)$ (refer [17]). In order to prove the existence of nontrivial solution of the first problem for $q \in [p, p^*)$, we let $\{f_n\} \subset L^{p'}(\Omega)$ be a sequence such that $f_n \rightarrow 0$ in $L^{p'}(\Omega)$. Then for each f_n , there exists a solution, say u_n .

We have

$$\begin{aligned} B[u, v] &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} u v dx, \\ &= \int_{\Omega} f v dx, \quad \forall v \in W_0^{1,p}(\Omega), \end{aligned} \tag{3.12}$$

where B is a ‘non linear form’ in two variables u and v . It is easy to check that $B(.,.)$ is the Fréchet derivative of the C^1 functional $\frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\lambda}{q} \int_{\Omega} |u|^q$ and hence is continuous.

Clearly, for each $v \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned} B[u_n, v] &= \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx - \lambda \int_{\Omega} |u_n|^{q-2} u_n v dx, \\ &= \int_{\Omega} f_n v dx, \\ &\leq \|f_n\|_{p'} \|v\|_p \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \tag{3.13}$$

Hence $\int_{\Omega} f_n v dx \rightarrow 0$ as $n \rightarrow \infty$. Consider $T_n(v) = \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx$. Then T_n ’s are bounded linear over $W_0^{1,p}(\Omega)$ and $\|T_n\| = \|\nabla u_n\|_{p'}^{p-1}$ for $n \geq 1$. From the above

definition of T_n , for a fixed $v \in W_0^{1,p}(\Omega)$ we have the sequence $\{T_n(v)\}$ to be bounded which implies that $\{T_n(v)\}$ is pointwise bounded. Thus by the uniform boundedness principle $\{\|T_n\|\}$ is bounded. Thus $\{\|\nabla u_n\|_p\}$ is bounded. Hence, there exists a subsequence $\{u_{n_k}\}$ which weakly converges to u_∞ with respect to the norm $\|\cdot\|_{1,p}$ in $W_0^{1,p}(\Omega)$. Hence we have

$$\begin{aligned} \lim_{n_k \rightarrow \infty} \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla u_{n_k} dx &= \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla u_\infty dx, \forall v \in W_0^{1,p}(\Omega). \\ \Rightarrow \lim_{n_k \rightarrow \infty} \int_{\Omega} |\nabla u_{n_l}|^{p-2} \nabla u_{n_l} \cdot \nabla u_{n_k} dx &= \int_{\Omega} |\nabla u_{n_l}|^{p-2} \nabla u_{n_l} \cdot \nabla u_\infty dx, \end{aligned} \tag{3.14}$$

for a fixed l . Therefore, since $u_{n_k} \rightharpoonup u_\infty$ in $W_0^{1,p}(\Omega)$ implies that $|\nabla u_{n_k}|^{p-1} \rightharpoonup |\nabla u_\infty|^{p-1}$ (for a subsequence) in $L^{p'}(\Omega)$ (Refer Appendix). But $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ and hence

$$\begin{aligned} \lim_{n_l \rightarrow \infty} \int_{\Omega} |\nabla u_{n_l}|^{p-2} \nabla u_{n_l} \cdot \nabla v dx &= \int_{\Omega} |\nabla u_\infty|^{p-2} \nabla u_\infty \cdot \nabla v dx, \forall v \in W_0^{1,p}(\Omega), \\ \Rightarrow \lim_{n_l \rightarrow \infty} \int_{\Omega} |\nabla u_{n_l}|^{p-2} \nabla u_{n_l} \cdot \nabla u_\infty dx &= \int_{\Omega} |\nabla u_\infty|^p dx. \end{aligned} \tag{3.15}$$

Therefore, $\lim_{n_k \rightarrow \infty} \int_{\Omega} |\nabla u_{n_k}|^p dx = \int_{\Omega} |\nabla u_\infty|^p dx$. It immediately can be concluded that there exists a u_∞ such that $u_{n_k} \rightarrow u_\infty$ in $W_0^{1,p}(\Omega)$. Hence using the continuity of $B[.,.]$ in (3.12) we have

$$\begin{aligned} \lim_{n_k \rightarrow \infty} B[u_{n_k}, v] &= \lim_{n_k \rightarrow \infty} \int_{\Omega} |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \cdot \nabla v dx - \lim_{n_k \rightarrow \infty} \lambda \int_{\Omega} |u_{n_k}|^{q-2} u_{n_k} v dx, \\ &= \lim_{n_k \rightarrow \infty} \int_{\Omega} f_{n_k} v dx, \\ \Rightarrow B[u_\infty, v] &= 0, \forall v \in W_0^{1,p}(\Omega). \end{aligned}$$

In other words,

$$\int_{\Omega} |\nabla u_\infty|^{p-2} \nabla u_\infty \cdot \nabla v dx - \lambda \int_{\Omega} |u_\infty|^{q-2} u_\infty v dx = 0, \forall v \in W_0^{1,p}(\Omega). \tag{3.16}$$

Assume that $\lambda \in \left(0, \inf_{u \neq 0 \in W_0^{1,p}(\Omega)} \left\{ \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} |u|^q} \right\} \right)$ (infimum exists and is strictly greater than zero which follows from the embedding result for $p < N$). Since $\|u_{n_k}\|_q = 1$ and $u_{n_k} \rightarrow u_\infty$ in $W_0^{1,p}(\Omega)$, hence we have

$$\begin{aligned} 0 < \lambda &\leq \liminf \frac{\int_{\Omega} |\nabla u_{n_k}|^p}{\int_{\Omega} |u_{n_k}|^q} \\ &= \liminf \int_{\Omega} |\nabla u_{n_k}|^p \\ &= \liminf \|\nabla u_{n_k}\|_p^p = \|\nabla u_\infty\|_p^p = \|u_\infty\|_{1,p}^p \end{aligned}$$

This implies that u_∞ is a nontrivial solution of the first problem. Thus we summarize the result proved as follows.

THEOREM 6. Suppose to each $f \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$, the problem $-\Delta_p u = \lambda |u|^{q-2}u + f$, $u|_{\partial\Omega} = 0$ has a solution in $\mathfrak{M} \subset W_0^{1,p}(\Omega)$ for some $\lambda > 0$, then the problem $-\Delta_p u = \lambda |u|^{q-2}u$, $u|_{\partial\Omega} = 0$, has a nontrivial solution in $W_0^{1,p}(\Omega)$ for $q \in [p, p^*)$, whenever $\lambda \in \left(0, \inf_{u \neq 0 \in W_0^{1,p}(\Omega)} \left\{ \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} |u|^q} \right\} \right)$.

We end this section with a small observation from Theorem 6 that if to each $f \in L^{p'}(\Omega)$, the problem $-\Delta_p u = \lambda |u|^{q-2}u + f$, $u|_{\partial\Omega} = 0$ has a solution in $\mathfrak{M} \subset W_0^{1,p}(\Omega)$ for some $\lambda > 0$ and $q \in [p, p^*)$ then the eigenvalue problem $-\Delta_p u = \lambda |u|^{p-2}u$, $u|_{\partial\Omega} = 0$, has a nontrivial solution in $W_0^{1,p}(\Omega)$, whenever $\lambda \in \left(0, \inf_{u \neq 0 \in W_0^{1,p}(\Omega)} \left\{ \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} |u|^q} \right\} \right)$.

4. Appendix

We show that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v \, dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx, \quad \forall v \in W_0^{1,p}(\Omega). \quad (4.1)$$

We divide the explanation into two cases:

Case 1: When $p > 2$.

This implies that p' , the conjugate of p , should be lesser than 2, i.e., $1 < p' < 2 < p$. Thus we have $W_0^{1,p}(\Omega) \hookrightarrow_{compact} L^{p'}(\Omega)$ (since $W_0^{1,p}(\Omega) \hookrightarrow_{compact} L^q(\Omega)$ for $q \in [1, p^*)$). Since ∇u_n converges weakly to, say ∇u , in $L^p(\Omega)$, hence $\langle |\nabla u_n| - |\nabla u|, v \rangle \rightarrow 0$ for each $v \in L^{p'}(\Omega)$. Thus $\langle |\nabla u_n| - |\nabla u|, |\nabla u_n| - |\nabla u| \rangle \rightarrow 0$, i.e., $\|\nabla u_n\|_2 \rightarrow \|\nabla u\|_2$. Hence $\|\nabla u_n\|_{p'} \rightarrow \|\nabla u\|_{p'}$ because $p' < 2 < p$. By the Riesz-Fischer theorem [4], there exists a subsequence of $\{\nabla u_n\}$ which converges pointwise a.e., i.e., $|\nabla u_n(x)| \rightarrow |\nabla u(x)|$. So $|\nabla u_n(x)|^{p-1} \rightarrow |\nabla u(x)|^{p-1}$ and hence $|\nabla u_n|^{p-1} \rightharpoonup |\nabla u|^{p-1}$ in $L^{p'}(\Omega)$. Thus we have $\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v \, dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx, \forall v \in W_0^{1,p}(\Omega)$.

Case 2: When $p < 2$.

This implies that p' , the conjugate of p , should be greater than 2, i.e., $p < 2 < p'$.

Look at the map $F : W_0^{1,p}(\Omega) \rightarrow L^{p'}(\Omega)$ defined by $u \mapsto |\nabla u|^{p-1}$. Consider the range of F , i.e., $R(F) = \{|\nabla u|^{p-1} : u \in W_0^{1,p}(\Omega)\}$.

Observe that the map F is bounded in the sense that bounded sets are mapped to bounded sets. Hence if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ implies that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence $\{F(u_n)\} = \{|\nabla u_n|^{p-1}\}$ is bounded in $L^{p'}(\Omega)$. Since $L^{p'}(\Omega)$ is reflexive, hence there exists a subsequence of $\{|\nabla u_n|^{p-1}\}$ which weakly converges to, say, w in $L^{p'}(\Omega)$.

We have the following: $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ so $|\nabla u_n|^{p-1} \rightharpoonup w$ in $L^{p'}(\Omega)$. This implies that

$$\langle |\nabla u_n|^{p-1} - w, v \rangle \rightarrow 0, \forall v \in L^{p'}(\Omega)$$

Since $p < 2 < p'$ hence $|\nabla u_n|^{p-1} - w \in L^p(\Omega)$. Thus $\| |\nabla u_n|^{p-1} - w \|_2 \rightarrow 0$ and hence $\| |\nabla u_n|^{p-1} - w \|_p \rightarrow 0$. Therefore we have a subsequence of $\{ |\nabla u_n|^{p-1} \}$ such that $|\nabla u_n|^{p-1} \rightarrow w$ pointwise a.e. (implying $|\nabla u_n| \rightarrow w^{\frac{1}{p-1}}$ pointwise a.e.) and so $|\nabla u_n| \rightarrow w^{\frac{1}{p-1}}$ in $L^p(\Omega)$. Hence $w = |\nabla u|^{p-1}$.

Thus in all the above cases we found the following.

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, \quad \forall v \in W_0^{1,p}(\Omega). \quad (4.2)$$

Hence by the compact embedding due to Rellich-Kondrachov it can be concluded $u_n \rightarrow u$ in $L^q(\Omega)$. Thus we also have

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{q-2} u_n v dx = \int_{\Omega} |u|^{q-2} u v dx, \quad \forall v \in W_0^{1,p}(\Omega). \quad (4.3)$$

5. Conclusions

The resonant Lane-Emden problem has been studied. An existence result has been established to the non-homogeneous Lane-Emden problem for the super-linear case - $1 < p < q < p^*$ for $\lambda \in (0, \lambda']$ - λ' being sufficiently large - if it is assumed that a solution exists to the homogeneous Lane-Emden problem for the super-linear case - $1 < p < q < p^*$. We further proved the 'converse' that if the non-homogeneous problem has a solution then a solution to the homogeneous problem exists for the super linear case. We also established an 'equivalence' of eigenvalue problem and the non homogeneous Lane-Emden problem.

Acknowledgement. We would like to thank the anonymous referee(s) for carefully reading this paper and making many useful comments.

REFERENCES

- [1] A. AMBROSETTI AND P. H. RABINOWITZ, *Dual Variational Methods in Critical Point Theory and Applications*, J. Funct. Anal., **14** (1973), 349–381.
- [2] B. KAWOHL, *Symmetry results for functions yielding best constants in Sobolev-type inequalities*, Discrete Contin. Dynam. Systems, **6** (2000), 683–690.
- [3] C. GRUMIAU AND E. PARINI, *On the asymptotics of solutions of the Lane-Emden problem for the p -Laplacian*, Archiv der Mathematik, **91**, 4 (2008), 354–365.
- [4] G. BACHMAN AND L. NARICI, *Functional Analysis*, Dover Publications, Mineola, New York, 1966.
- [5] GREY ERCOLE, *On the resonant Lane-Emden problem for the p -Laplacian*, Communications in Contemporary Mathematics, **16**, 4 (2014), 1350033-1-22.
- [6] GREY ERCOLE, *On the resonant Lane-Emden problem for the p -Laplacian*, Communications in Contemporary Mathematics, **16**, 4 (2014), 1492001-1.
- [7] IDRIS LY, *The first eigenvalue for the p -laplacian operator*, Journal of inequalities of pure and applied Math., **6**, 3 (2005), 1–28.
- [8] J. GARCÍA AZORERO AND I. PERAL ALONSO, *On limits of solutions of elliptic problems with nearly critical exponent*, Comm. Partial Differential Equations, **17** (1992), 2113–2126.
- [9] J. I. DÍAZ AND J.E. SAA, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilineaires*, C. R. Acad. Sci. Paris, **305** (1987), 521–524.
- [10] S. KESAVAN, *Functional Analysis and applications*, New age international pvt. Ltd., 2003.

- [11] L.C. EVANS, *Partial Differential Equations*, Amer. Math. Soc., 2009.
- [12] M. F. BIDAUT-VÉRON, N. Q. HUNG AND LAURENT VÉRON, *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math. Pures Appl., **102**, 2 (2014), 315–337.
- [13] M. ÔTANI, *Existence and nonexistence of nontrivial solutions of some nonlinear degenerate elliptic equations*, J. Funct. Anal., **76** (1988), 140–159.
- [14] N. DUNFORD AND J. SCHWARTZ, *Linear Operators. Part 2: Spectral theory*, Wiley (Interscience), New York, 1958.
- [15] NIKOS E. MASTORAKIS, HASSAN FATHABADI, *On the solution of p -Laplacian for non-newtonian fluid flow*, W.S.E.A.S. transactions on Math., **8**, 6 (2009), 238–245.
- [16] P. DRÁBEK, *A note on the nonuniqueness for some quasilinear eigenvalue problem*, Appl. Math. Lett., **13** (2000), 39–41.
- [17] PHILIPPE G. CIARLET, *The finite element method for elliptic problems*, Studies in Mathematics and its Applications, Vol. 4, North-Holland Publishing Co., Amsterdam, 1978.
- [18] P. JUNTINEN, P. LINDQVIST, *On the higher eigenvalues for the ∞ -eigenvalue problem*, Calc. Var. partial differential equations, **23** (2005), 169–192.
- [19] RATAN KR. GIRI AND D. CHOUDHURI, *Reduced limit approach to semilinear elliptic PDEs with measure data* (arXiv:1605.00870 [math.AP]).
- [20] T. BHATTACHARYA, E. D. BENEDETTO AND J. MANFREDI, *Limits as $p \rightarrow \infty$ of $\Delta_p u = f$ and related extremal problems*, Rend. Sem. Mat. Univ. Politec. Torino, special issue (1991), 15–68.

(Received March 28, 2016)

Ratan Kr. Giri
 Department of Mathematics
 National Institute of Technology Rourkela
 Rourkela - 769008, India
 e-mail: giri90ratan@gmail.com

D. Choudhuri
 Department of Mathematics
 National Institute of Technology Rourkela
 Rourkela - 769008, India
 e-mail: dc.iit12@gmail.com