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NON HOMOGENEOUS DIRICHLET PROBLEM FOR THE KDVB EQUATION ON A SEGMENT

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Abstract. We study the Non homogeneous Dirichlet problem with large initial data for the KdVB equation on the interval $x \in (0,1)$

$$\begin{cases} u_t + u_x u - u_{xx} + u_{xxx} = 0, \; t > 0, \; x \in (0,1) \\ u(x,0) = u_0(x), \; x \in (0,1) \\ u(0,t) = u(1,t) = 0, \; t > 0 \\ u_x(1,t) = h(t), \; t > 0. \end{cases} \tag{1}$$

We prove that if the initial data $u_0 \in \mathbf{L}^2$ and boundary data $h(t) \in \mathbf{H}^1_\infty(0,\infty)$ then there exist a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^2\right) \cup \mathbf{C}\left((0,\infty);\mathbf{H}^1\right)$ of the initial-boundary value problem (1). We also obtain the large time asymptotic of solution uniformly with respect to $x \in (0,1)$ as $t \to \infty$.

1. Introduction

We study the global existence and large time asymptotic behavior of solutions to the initial-boundary value problem for the Korteweg–de Vries–Burgers (KdVB) equation on the interval $x \in (0,1)$

$$\begin{cases} u_t + u_x u - u_{xx} + u_{xxx} = 0, \; t > 0, \; x \in (0,1), \\ u(x,0) = u_0(x), \; x \in (0,1), \\ u(0,t) = u(1,t) = 0, \; t > 0, \\ u_x(1,t) = h(t), \; t > 0. \end{cases} \tag{2}$$

This equation is considered as one of the simplest partial differential equations (PDEs) that features dissipation, dispersion and nonlinearity. It is used to model many phenomena and has many applications in various fields of Physics, Biology and Electrical Engineering. A typical example in electrical engineering is a modified model of transmission line that has the feature of dispersion and dissipation, other examples are provided by the propagation of waves on an elastic tube filled with a viscous fluid [15], the flow of liquids containing gas bubbles [25] and turbulence

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[11]. KdVB equation has been explicitly derived by Shukla and Tagare in case of the ion-acoustic shock waves in multi-electron temperature collisional plasma using perturbation reduction technique [24].

In the case of travelling wave solutions of a KdVB equation with a non-local diffusion term were found in paper [1]. This model equation arises in the analysis of a shallow water flow by performing formal asymptotic expansions associated to the triple-deck regularization (which is an extension of classical boundary layer theory). The resulting non-local operator is a fractional derivative of order between 1 and 2.

In the case of the Cauchy problem, some estimations for the time decay rates of solutions to the KdVB type equations and the generalized KdVB equation were found in papers [4], [5], [6], [9], [26] and the large time asymptotic of solutions was obtained in [8], [10]. Recently, some researchers obtained many results about the Unique Continuation Property and Decay for the KdVB equation with localized damping (see, e.g. [21]).

In the case of the boundary value problem on half-line the large time asymptotics of solutions were studied in papers [2], [3], [7], [12], [14], [22], [17].

One of the most important developments in this area was the generalization of the Cauchy problem and problem on half-line to the case of the initial-boundary value problem on a segment. The boundary value problems on a segment are more natural for applications, however their mathematical investigations are more complicated. For example, it is necessary to answer the question of the well-posedness of the problem, in particular, how many boundary values should be given in the problem for its solvability and the uniqueness of the solution. And after that it is also interesting to study the influence of the boundary data on the qualitative properties of the solution.

As far as we know the non homogeneous initial-boundary value problem for the KdVB equation (2) on the interval was not considered previously. In this paper we study traditionally important problems of a theory of nonlinear partial differential equations, such as well-posedness and in time global existence of solutions to the initial-boundary value problem (2). Our main goal is to obtain the large time asymptotics of solutions. We consider (2) in the case of the initial data belonging to $\mathbf{L}^2$. Note that we do not assume the smallness condition on the data. In the case of large initial data it is more difficult than that small data to obtain exact representation of large time asymptotics of solutions and there are a few results (see, e.g. [23]). Another difficulty in the study of the boundary value problem for the KdVB equation (2) is that the linear operator $-\partial_x^2+\partial_x^3$ is not self-adjoint and we can not apply the Fourier method when we take the boundary value into account. To avoid this difficulty we apply the Laplace transformation with respect to space variable to derive the Green function of the resulting equation. We will show below that exactly three boundary values are necessary and sufficient in the problem (2) for its solvability and uniqueness. The Laplace transformation requires the boundary data $u(0,t)$, $u(1,t)$, $u_x(1,t)$ and so $u_x(0,t)$, $u_{xx}(0,t)$ and $u_{xx}(1,t)$ should be determined by the given data. To achieve this we need to solve the analytic condition of the function $\widehat{u}$. For obtaining $\mathbf{L}^p$ -estimates of the Green function we use the method of papers [12], [20], [18] and [19].

To state the results of the present paper precisely we give some notations.

Let us denote $\mathbf{H}^1=\left\{\varphi\in\mathbf{L}^2(0,1);\|\varphi\|_{\mathbf{H}^1}=\|\varphi\|_{\mathbf{L}^2}+\|\varphi_x\|_{\mathbf{L}^2}<\infty\right\}$. Direct

Laplace transformation $\mathscr{L}_{x\to p}$ is

$$\widehat{u}(p) \equiv \mathscr{L}u = \int_0^1 e^{-px}u(x)\,dx$$

and the inverse Laplace transformation $\mathscr{L}^{-1}_{p\to x}$ is defined by

$$u(x) \equiv \mathscr{L}^{-1}\widehat{u}(p) = (2\pi i)^{-1}\int_{-i\infty}^{i\infty} e^{px}\widehat{u}(p)\,dp.$$

By the same letter C we denote different positive constants if it does not make confusion.

We state the main result of this paper.

THEOREM 1. *Suppose that the initial data $u_0 \in \mathbf{L}^2(0,1)$, boundary data $h(t) \in \mathbf{H}^1(0,\infty)$, such as for some constant A the following asymptotics are valid*

$$h(t) = At^{-\beta} + O(t^{-\beta-\gamma}).$$

Then for $\beta > \frac{1}{2}$ there exists a unique solution of (2)

$$u \in \mathbf{C}\left([0,\infty);\mathbf{L}^2\right) \cup \mathbf{C}\left((0,\infty);\mathbf{H}^1\right).$$

Moreover there exists the function $\Psi(x) \in \mathbf{L}^\infty(0,1)$ such that the solution $u(x,t)$ has the following asymptotics

$$u(x,t) = h(t)\Psi(x) + O\left(t^{-\beta-\gamma}\right)$$

for $t \to \infty$ uniformly with respect to $x \in (0,1)$. The function $\Psi(x)$ is defined below in formula (43).

We organize our paper as follows. In Section 2 we solve the linear initial-boundary value problem corresponding to (2) with conditions homogeneous. In Section 3 we prove the Global existence of solutions to (2) for the case of small initial data. Section 4 is devoted to the proof of main theorem of solutions to (2) for the case of any initial data by using the time decay estimates of solutions obtained in Section 3.

2. Linear problem

We consider the following linear initial-boundary value problem

$$\begin{cases} v_t - v_{xx} + v_{xxx} = f(x,t),\ t>0,\ x\in(0,1), \\ v(x,0) = v_0(x),\ x\in(0,1), \\ v(0,t) = v(1,t) = 0, v_x(1,t) = h(t),\ t>0. \end{cases} \tag{3}$$

We define for $x \in (0,1)$

$$\psi(\xi,x) = \sum_{j=1}^{3} e^{-\varphi_j x}\varphi_j'(\xi). \tag{4}$$

Here $\phi_l(\xi)$ are the roots of the characteristic equation $-p^2+p^3+\xi=0$, such that $\operatorname{Re}\phi_l(\xi)>0, l=1,2$ and $\operatorname{Re}\phi_3(\xi)<0$, for all $\xi\in D_0$.
Where $D_0=\left\{\xi\in C: Re\xi\geqslant 0,\xi\notin\left[0,\frac{4}{27}\right]\right\}$. Note that the functions $\phi_l(\xi)$ are analytic in the domain $\left\{\xi\in C:\xi\notin\left[-\infty,\frac{4}{27}\right]\right\}$.

Denote by

$$\begin{aligned}\mathscr{G}\varphi &= \theta(x)\int_0^1 G(x,y,t)\varphi(y)dy, \\ \mathscr{H}h &= \int_0^t d\tau h(\tau)H(t-\tau).\end{aligned} \tag{5}$$

Here and below $\theta(x)=1$ for $x\in(0,1)$ and $\theta(x)=0$ for $x\notin(0,1)$,

$$G(x,y,t)=\begin{cases} F_1(x,y,t)=-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{\xi t}\frac{\psi(\xi,1-x)\psi(\xi,y)}{\psi(\xi,1)}d\xi, & \text{for } y<x, \\ F_2(x,y,t)=-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{\xi t}\frac{\psi(\xi,1-x)\psi(\xi,y)-\psi(\xi,y-x)\psi(\xi,1)}{\psi(\xi,1)}d\xi, & \text{for } x<y\end{cases} \tag{6}$$

and

$$H(x,t)=G_y(x,y,t)\big|_{y=1}. \tag{7}$$

PROPOSITION 1. *Let the initial data* $v_0\in\mathbf{L}^1(\mathbf{0},\mathbf{1})$ *and* $f\in\mathbf{C}\left([0,\infty);\mathbf{L}^1\right)$ *and* $h\in\mathbf{L}^1(0,\infty)$ *Then there exist a unique solution* $v(x,t)$ *of the initial-boundary value problem (3), which has integral representation*

$$v(x,t)=\mathscr{G}v_0+\int_0^t\mathscr{G}(t-\tau)f(\tau)d\tau+\mathscr{H}h, \tag{8}$$

where operators $\mathscr{G}$ *and* $\mathscr{H}$ *defined by (5)*

Proof. To derive an integral representation for the solutions of the problem (3) we suppose that there exists a solution $v(x,t)$, which is continued by zero outside of $x\in(0,1)$

$$\begin{aligned} v(x,t) &= 0 \ \text{ for all } x\notin[0,1], \\ \partial_x^j v(0,t) &= \lim_{x\to 0+}\partial_x^j v(x,t), j=0,1,2, \\ \partial_x^j v(1,t) &= \lim_{x\to 1-}\partial_x^j v(x,t), j=0,1,2. \end{aligned}$$

We define the operator

$$\mathbb{P}\left\{\widehat{\phi}(p,t)\right\}=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{(q-p)}-1}{q-p}\widehat{\phi}(q,t)dq.$$

It is readily observed that $\mathbb{P}\{\phi(p)\}$ constitutes a function analytic in the complex plane $p\in\mathbb{C}$.

Since $\mathscr{L}\{v\}$ is analytic for all $p\in\mathbb{C}$ we have

$$\widehat{v}(p,t)=\mathbb{P}\{\widehat{v}(p,t)\}. \tag{9}$$

Applying the method developed in [16] we obtain

$$\begin{cases} \mathbb{P}\left\{\widehat{v}_t + K(p)\widehat{v}(p,t) + B_1(p,t) - e^{-p}B_2(p,t) - \widehat{f}(p,t)\right\} = 0,\ t>0,\ x>0, \\ \widehat{v}(p,0) = \widehat{v}_0(p), \end{cases} \tag{10}$$

where $K(p) = p^3 - p^2$ and

$$\begin{aligned} B_1(p,t) &= p^2 \sum_{j=1}^{2} \frac{\partial_x^{j-1} v(0,t)}{p^j} - p^3 \sum_{j=1}^{3} \frac{\partial_x^{j-1} v(0,t)}{p^j}, \\ B_2(p,t) &= p^2 \sum_{j=1}^{2} \frac{\partial_x^{j-1} v(1,t)}{p^j} - p^3 \sum_{j=1}^{3} \frac{\partial_x^{j-1} v(1,t)}{p^j}. \end{aligned} \tag{11}$$

We rewrite (10) in the form

$$\widehat{v}_t + K(p)\widehat{v}(p,t) + B_1(p,t) - e^{-p}B_2(p,t) - \widehat{f}(p,t) = \Phi(p,t), \tag{12}$$

where some function $\Phi(p,t)$ is analytic for all $p \in \mathbb{C}$,

$$|\Phi(p,t)| \leqslant C\frac{1+|e^{-p}|}{|p|}, |p|>1 \tag{13}$$

and

$$\mathbb{P}\{\Phi(p,t)\} = 0. \tag{14}$$

Now we prove that under these conditions $\Phi(p,t) \equiv 0$.

We introduce functions of the Cauchy type

$$\begin{aligned} \Omega_1(z,t) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\Phi(q,t)dq, \\ \Omega_2(z,t) &= \frac{e^{-z}}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^q}{q-z}\Phi(q,t)dq. \end{aligned}$$

Since $\Phi(p,t)$ satisfies Hölder condition the functions $\Omega_1(z,\xi)$, $\Omega_2(z,\xi)$ are analytic in $\operatorname{Re} z \neq 0$. Denote by $\Omega_{1,2}^+(p,t) = \lim_{z\to p, \operatorname{Re} z<0} \Omega_{1,2}(z,t)$ and $\Omega_{1,2}^-(p,t) = \lim_{z\to p, \operatorname{Re} z>0} \Omega_{1,2}(z,t)$ for $\operatorname{Re} p = 0$. Since function $\Phi(p,t)$ is analytic for all $p \in \mathbb{C}$ from estimate (13) we have

$$\Omega_2^-(p,\xi) = \Omega_1^+(p,\xi) = 0.$$

In another hand by Sokhotsky-Plemelj formula we get

$$\begin{aligned} \Omega_2^-(p,\xi) &= \frac{e^{-p}}{2\pi i}\fint_{-i\infty}^{i\infty} \frac{1}{q-p}\Phi(p,t)dq - \frac{1}{2}\Phi(p,t) \\ \Omega_1^+(p,\xi) &= \frac{1}{2\pi i}\fint_{-i\infty}^{i\infty} \frac{1}{q-p}\Phi(p,t)dq + \frac{1}{2}\Phi(p,t). \end{aligned}$$

and therefore for $\operatorname{Re} p = 0$

$$\Omega_2^-(p,\xi) - \Omega_1^+(p,\xi) = \mathbb{P}\{\Phi(p,t)\} - \Phi(p,t) = 0.$$

Thus for $\operatorname{Re} p = 0$

$$\Phi(p,t) = \mathbb{P}\{\Phi(p,t)\} = 0$$

and therefore due to analyticity $\Phi(p,t) \equiv 0$ for all $p \in \mathbb{C}$.

Applying the Laplace transformation with respect to time variable to problem (12) we find $\mathscr{L}_{t\to\xi}\{\widehat{v}(p,t)\} = \widehat{\widehat{v}}(p,\xi)$ as

$$\widehat{\widehat{v}}(p,\xi) = \frac{1}{K(p)+\xi}\left(\widehat{v}_0(p) + \widehat{\widehat{f}}(p,\xi) - \widehat{B}_1(p,\xi) + e^{-p}\widehat{B}_2(p,\xi)\right) \tag{15}$$

for $p \in \mathbb{C}$.

Here functions $\widehat{B}_1(p,\xi)$ and $\widehat{B}_2(p,\xi)$ are the Laplace transforms of $B_1(p,t)$ and $B_2(p,t)$ with respect to time.

In order to get the integral formula for solution, we need to know the functions $\widehat{B}_1(p,\xi)$ and $\widehat{B}_2(p,\xi)$. We will find its using the analytic condition (9) of function $\widehat{\widehat{v}}$ for $p \in \mathbb{C}$ and $\operatorname{Re}\ \xi > 0$. Via (15) we rewrite (9) in the form

$$\begin{aligned} &\frac{1}{K(p)+\xi}\left(\widehat{v}_0(p) + \widehat{\widehat{f}}(p,\xi) - \widehat{B}_1(p,\xi) + e^{-p}\widehat{B}_2(p,\xi)\right) \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{(q-p)}-1}{q-p}\frac{1}{K(q)+\xi}\left(\widehat{v}_0(q) + \widehat{\widehat{f}}(q,\xi) - \widehat{B}_1(q,\xi) + e^{-q}\widehat{B}_2(q,\xi)\right)dq. \end{aligned} \tag{16}$$

By Cauchy Theorem we have for all $p \in \mathbb{C}$

$$\begin{aligned} &\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{(q-p)}-1}{q-p}\frac{1}{K(q)+\xi}\left(\widehat{v}_0(q) + \widehat{\widehat{f}}(q,\xi) - \widehat{B}_1(q,\xi) + e^{-q}\widehat{B}_2(q,\xi)\right)dq \\ &= \frac{1}{K(p)+\xi}\left(\widehat{v}_0(p) + \widehat{\widehat{f}}(p,\xi) - \widehat{B}_1(p,\xi) + e^{-p}\widehat{B}_2(p,\xi)\right) \\ &\quad + \frac{e^{(\varphi_3(\xi)-p)}}{\varphi_3 - p}\varphi_3'(\xi)\left(-\widehat{v}_0(\varphi_3) - \widehat{\widehat{f}}(\varphi_3,\xi) + \widehat{B}_1(\varphi_3,\xi) - e^{-\varphi_3}\widehat{B}_2(\varphi_3,\xi)\right) \\ &\quad + \sum_{j=1}^{2}\frac{1}{\varphi_j - p}\varphi_j'(\xi)\left(-\widehat{v}_0(\varphi_j) - \widehat{\widehat{f}}(\varphi_j,\xi) + \widehat{B}_1(\varphi_j,\xi) - e^{-\varphi_j}\widehat{B}_2(\varphi_j,\xi)\right). \end{aligned}$$

By the analytic condition of function $\widehat{\widehat{v}}$ for $p \in \mathbb{C}$ we get

$$\begin{cases} \widehat{B}_2(\varphi_3,\xi) = e^{\varphi_3}\left(-\widehat{v}_0(\varphi_3) - \widehat{\widehat{f}}(\varphi_3,\xi) + \widehat{B}_1(\varphi_3,\xi)\right) \\ \widehat{B}_1(\varphi_j,\xi) = \widehat{v}_0(\varphi_j) + \widehat{\widehat{f}}(\varphi_j,\xi) + e^{-\varphi_j}\widehat{B}_2(\varphi_j,\xi),\ j = 1,2. \end{cases} \tag{17}$$

So we need to put in the initial-boundary value problem one boundary data in the point $x = 0$ and two boundary data in the point $x = 1$. Let for example

$$v(0,t) = v(1,t) = 0, v_x(1,t) = h(t). \tag{18}$$

Applying this conditions and using (11) we get the following system

$$\begin{cases} \quad e^{-\varphi_j}\partial_{xx}\widehat{v}(1,\xi)+(1-\varphi_j)\partial_x\widehat{v}(0,\xi)-\partial_{xx}\widehat{v}(0,\xi) \\ =\widehat{v}_0(\varphi_j)+\widehat{\widehat{f}}(\varphi_j,\xi)+e^{-\varphi_j}(1-\phi_j)\widehat{h}(\xi),\ j=1,2,3 \end{cases} \tag{19}$$

Denote the determinant of this system by $\triangle(\varphi_1,\varphi_2,\varphi_3)$, then it has a form

$$\triangle(\varphi_1,\varphi_2,\varphi_3)=e^{-\varphi_1}(\varphi_2-\varphi_3)+e^{-\varphi_2}(\varphi_3-\varphi_1)+e^{-\varphi_3}(\varphi_1-\varphi_2). \tag{20}$$

in the domain $\xi\in D_0$. Since $\sum_{j=1}^{3}\varphi_j=1$ and

$$\begin{aligned} \varphi_1'(\xi) &= -\frac{1}{(\varphi_1-\varphi_2)(\varphi_1-\varphi_3)};\ \varphi_2'(\xi)=-\frac{1}{(\varphi_2-\varphi_1)(\varphi_2-\varphi_3)}; \\ \varphi_3'(\xi) &= -\frac{1}{(\varphi_3-\varphi_1)(\varphi_3-\varphi_2)} \end{aligned} \tag{21}$$

we can rewrite $\triangle(\varphi_1,\varphi_2,\varphi_3)$ as

$$\triangle(\varphi_1,\varphi_2,\varphi_3)=V(\xi)\sum_{j=1}^{3}e_j^{-\varphi_j(\xi)}\varphi\prime(\xi). \tag{22}$$

where $V(\xi)=(\varphi_1-\varphi_2)(\varphi_2-\varphi_3)(\varphi_3-\varphi_1)$. Since $V(\xi)\neq 0$ and $\operatorname{Re}\varphi_l(\xi)>0$, $l=1,2$, $\operatorname{Re}\varphi_3(\xi)<0$ in domain $\xi\in D_0$ we easily get for $|\xi|\gg 1$, $\xi\in D_0$ and by numeric computations we can check that $\triangle(\varphi_1,\varphi_2,\varphi_3)\neq 0$ for all $|\xi|\leqslant C$, $\xi\in D_0=\left\{\xi\in\mathbf{C}:\operatorname{Re}\xi\geqslant 0,\xi\notin\left[0,\frac{4}{27}\right]\right\}$. Therefore there exists a unique solution of the system (19) which can be written as follows

$$\begin{aligned} \begin{pmatrix}\partial_{xx}\widehat{v}(1,\xi)\\ \partial_x\widehat{v}(0,\xi)\\ \partial_{xx}\widehat{v}(0,\xi)\end{pmatrix} &= \int_0^1 dy\left[v_0(y)+\widehat{f}(y,\xi)\right]\begin{pmatrix}e^{-\varphi_1} & 1-\varphi_1 & -1\\ e^{-\varphi_2} & 1-\varphi_2 & -1\\ e^{-\varphi_3} & 1-\varphi_3 & -1\end{pmatrix}^{-1}\begin{pmatrix}e^{-\varphi_1 y}\\ e^{-\varphi_2 y}\\ e^{-\varphi_3 y}\end{pmatrix} \\ &\quad +\widehat{h}(\xi)\begin{pmatrix}e^{-\varphi_1} & 1-\varphi_1 & -1\\ e^{-\varphi_2} & 1-\varphi_2 & -1\\ e^{-\varphi_3} & 1-\varphi_3 & -1\end{pmatrix}^{-1}\begin{pmatrix}e^{-\varphi_1}(1-\phi_1)\\ e^{-\varphi_2}(1-\phi_2)\\ e^{-\varphi_3}(1-\phi_3)\end{pmatrix}. \end{aligned} \tag{23}$$

By (11) and (15) we have

$$\begin{aligned} \widehat{\widehat{v}}(p,\xi)=\frac{1}{K(p)+\xi}\Big[&\widehat{v}_0(p)+\widehat{\widehat{f}}(p,\xi)+(p-1)\widehat{v}_x(0,\xi) \\ &+\widehat{v}_{xx}(0,\xi)-e^{-p}\widehat{v}_{xx}(1,\xi)+e^{-p}(1-p)h(t)\Big]. \end{aligned}$$

Taking inverse Laplace transform with respect to space and time variables we get

$$v(x,t)=\mathscr{G}v_0+\int_0^t\mathscr{G}(t-\tau)f(\tau)d\tau+\mathscr{H}h,$$

where

$$\mathscr{G}\phi = \int_0^1 G(x,y,t)\phi(y)dy, \mathscr{H}h = \int_0^t h(\tau)H(x,t-\tau)d\tau$$

$$\begin{aligned} G(x,y,t) &= \left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{+i\infty} e^{\xi t} d\xi \int_{-i\infty}^{+i\infty} dp e^{px} \frac{1}{K(p)+\xi} \\ &\quad \times \left[e^{-py} + (p-1)\widehat{v}_x^0(0,\xi) + \widehat{v}_{xx}^0(0,\xi) - e^{-p}\widehat{v}_{xx}^0(1,\xi)\right] \end{aligned}$$

and

$$\begin{aligned} H(x,t) &= \left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{+i\infty} e^{\xi t} \int_{-i\infty}^{+i\infty} dp e^{px} \frac{1}{K(p)+\xi} \\ &\quad \times \left[e^{-p}(1-p) + (p-1)\widehat{v}_x^1(0,\xi) + \widehat{v}_{xx}^1(0,\xi) - e^{-p}\widehat{v}_{xx}^1(1,\xi)\right]. \end{aligned} \tag{24}$$

Here

$$\begin{pmatrix} \partial_{xx}\widehat{v}^0(1,\xi) \\ \partial_x\widehat{v}^0(0,\xi) \\ \partial_{xx}\widehat{v}^0(0,\xi) \end{pmatrix} = \begin{pmatrix} e^{-\varphi_1} & 1-\varphi_1 & -1 \\ e^{-\varphi_2} & 1-\varphi_2 & -1 \\ e^{-\varphi_3} & 1-\varphi_3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} e^{-\varphi_1 y} \\ e^{-\varphi_2 y} \\ e^{-\varphi_3 y} \end{pmatrix}$$

and

$$\begin{pmatrix} \partial_{xx}\widehat{v}^1(1,\xi) \\ \partial_x\widehat{v}^1(0,\xi) \\ \partial_{xx}\widehat{v}^1(0,\xi) \end{pmatrix} = \begin{pmatrix} e^{-\varphi_1} & 1-\varphi_1 & -1 \\ e^{-\varphi_2} & 1-\varphi_2 & -1 \\ e^{-\varphi_3} & 1-\varphi_3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} e^{-\varphi_1}(1-\phi_1) \\ e^{-\varphi_2}(1-\phi_2) \\ e^{-\varphi_3}(1-\phi_3) \end{pmatrix}.$$

Firstly we consider $G(x,y,t)$.

We apply Cauchy theorem (see [16]) to get

$$G(x,y,t) = \begin{cases} F_1(x,y,t), \text{for } y > x \\ F_2(x,y,t), \text{ for } x > y \end{cases},$$

where

$$\begin{aligned} F_1(x,y,t) &= -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{\xi t} \frac{\psi(\xi,1-x)\psi(\xi,y)}{\psi(\xi,1)} d\xi, \\ F_2(x,y,t) &= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{\xi t} \frac{\psi(\xi,y-x)\psi(\xi,1) - \psi(\xi,1-x)\psi(\xi,y)}{\psi(\xi,1)} d\xi. \end{aligned} \tag{25}$$

Now we consider $H(x,t)$. By direct calculation via definition (24) we have

$$H(x,t) = \left(G(x,y,t) + G_y(x,y,t)\right)\big|_{y=1}$$

and as consequence via (25) we have

$$H(x,t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{\xi t} \frac{\psi_y(\xi,1-x)\psi(\xi,1) - \psi(\xi,1-x)\psi_y(\xi,1)}{\psi(\xi,1)} d\xi.$$

Proposition is proved.

LEMMA 1. *We have the asymptotics for large time*

$$F_j(x,y,t) = -e^{-\xi_0 t}\Lambda(x)\,\psi(-\xi_0,y) + O\left(e^{-(\xi_0+\delta)t}\right) \tag{26}$$

and estimates

$$\left|\partial_x^n F_j(x,y,t)\right| \leqslant Ce^{-\xi_0 t}\{t\}^{-\alpha}|x-y|^{2\alpha-1-n} \tag{27}$$

for $x,y \in (0,1)$, $x \neq y$, $t > 0$, *where* $\alpha \in \left[0,\frac{n+1}{2}\right]$, $n = 0,1$, $j = 1,2$, $\xi_0 > 0$, $\psi(-\xi_0,1) = 0$.

Proof. We consider a curve in the complex left-half plane $\operatorname{Re}\xi < 0$ such that $\operatorname{Re}\varphi_1(\xi) = 0$, it is defined by the equation $(iy)^2 - (iy)^3 = \xi$ with $y = \operatorname{Im}\varphi_1(\xi)$. Therefore there exists a contour

$$\mathscr{C}_0 = \left\{\xi \in \mathbf{C}, \operatorname{Re}\xi < 0 : \operatorname{Re}\xi = O\left(|\xi+\xi_0+\delta|^{\frac{2}{3}}\right)\right\}$$

such that

$$\operatorname{Re}\varphi_l(\xi) > 0, l = 1,2, \operatorname{Re}\varphi_3(\xi) < 0 \text{ for all } \xi \in \mathscr{C}_0.$$

We also consider a contour

$$\mathscr{C}_1 = (-\xi_0-\delta-i0,-i0)\cup(i0,-\xi_0-\delta+i0)$$

We now define a contour $\mathscr{C} = \mathscr{C}_0 \cup \mathscr{C}_1$. We represent $p^2 = \frac{\xi}{1-p}$ for $|p| < 1$ and $p^3 = \frac{-\xi}{1-\frac{1}{p}}$ for $|p| > 1$, hence we get the asymptotics

$$\varphi_1(\xi) = \begin{cases} \sqrt{\xi} + O(|\xi|), \xi \to 0, \operatorname{Im}\xi > 0,\ 1 + O(|\xi|), \xi \to 0, \operatorname{Im}\xi < 0, \\ e^{i\frac{\pi}{3}}\sqrt[3]{\xi} + O(1), |\xi| \to \infty, \end{cases} \tag{28}$$

$$\varphi_2(\xi) = \begin{cases} 1 + O(|\xi|), \xi \to 0, \operatorname{Im}\xi > 0,\ \sqrt{\xi} + O(|\xi|), \xi \to 0, \operatorname{Im}\xi < 0, \\ e^{-i\frac{\pi}{3}}\sqrt[3]{\xi} + O(1), |\xi| \to \infty, \end{cases} \tag{29}$$

and

$$\varphi_3(\xi) = \begin{cases} -\sqrt{\xi} + O(|\xi|), |\xi| \to 0, \\ -\sqrt[3]{\xi} + O(1), |\xi| \to \infty, \end{cases} \tag{30}$$

for all $\xi \in \mathbf{C} : \xi \notin \left(-\infty, \frac{4}{27}\right]$ (by $\sqrt{\xi}$ and $\sqrt[3]{\xi}$ we denote the main value of the analytic function, i.e. $\sqrt{1} = \sqrt[3]{1} = 1$).

Using (4), the asymptotics formulas (28)-(30) and since $\varphi_l' = O\left(|\xi|^{-\frac{1}{2}}\right)$, $l = 1$, 2, 3 for $|\xi| < 1$, $\xi \in D_0$ we have

$$\frac{\psi(\xi,1-x)\psi(\xi,y)}{\psi(\xi,1)} = O\left(|\xi|^{-\frac{1}{2}}\right) \tag{31}$$

and

$$\psi(\xi, y-x) = O\left(|\xi|^{-\frac{1}{2}}\right). \tag{32}$$

Due to the fact that $\operatorname{Re}\varphi_l(\xi) > 0,\ l = 1,\ 2,\ \operatorname{Re}\varphi_3(\xi) < 0$ for $|\xi| > 1,\ \xi \in D_0$, we obtain for $|\xi| > 1$

$$\frac{\psi(\xi,1-x)\psi(\xi,y)}{\psi(\xi,1)} = e^{\varphi_3(x-y)}\varphi_3'\left(1+\sum_{i=1}^{2} O\left(e^{\left(-\varphi_j+\varphi_3\right)y}\right)+\sum_{i=1}^{2} O\left(e^{\left(-\varphi_j+\varphi_3\right)(1-x)}\right)\right). \tag{33}$$

Therefore taking the asymptotics (28)-(30) into account we find that

$$\widehat{F}_1(x,y,\xi) = O\left(\xi^{-\frac{2}{3}} e^{-C\sqrt[3]{|\xi|}(x-y)}\right). \tag{34}$$

for $\xi \in D_0,\ |\xi| > 1,\ y < x.$ Also from (33) we get

$$\begin{aligned}
\widehat{F}_2(x,y,\xi) &= \sum_{j=1}^{2}\left(\varphi_j' e^{-\varphi_j(y-x)}\right. \\
&\qquad \left. + O\left(\varphi_3' e^{-\operatorname{Re}\varphi_j y+\operatorname{Re}\varphi_3 x}\right)+O\left(\varphi_3' e^{-\operatorname{Re}\varphi_j(1-x)+\operatorname{Re}\varphi_3(1-y)}\right)\right) \\
&= O\left(\xi^{-\frac{2}{3}} e^{-C\sqrt[3]{|\xi|}(y-x)}\right)
\end{aligned} \tag{35}$$

for $\xi \in D_0,\ |\xi| > 1,\ x < y.$

In view of them we have (31)-(35) for $\xi \in \mathscr{C}$. Note that the asymptotics formulas (28)-(30) are valid on the contour $\mathscr{C}$. Therefore changing the contour of integration to $\mathscr{C}$ we obtain for $x > y$

$$\begin{aligned}
F_1(x,y,t) &= -\frac{1}{2\pi i}\int_{\xi\in\mathscr{C}_1} e^{\xi t}\frac{1}{\psi(\xi,1)}\psi(\xi,1-x)\psi(\xi,y)d\xi \\
&\quad -\frac{1}{2\pi i}\int_{\xi\in\mathscr{C}_0} e^{\xi t}\frac{1}{\psi(\xi,1)}\psi(\xi,1-x)\psi(\xi,y)d\xi.
\end{aligned} \tag{36}$$

Via $\psi(x+i0,q) = \psi(x-i0,q)$ by Cauchy Theorem taking residue in the point $\xi = \xi_0 > 0, (\psi(\xi_0,1) = 0)$, we obtain

$$\begin{aligned}
&-\frac{1}{2\pi i}\int_{\xi\in\mathscr{C}_1} e^{\xi t}\frac{1}{\psi(\xi,1)}\psi(\xi,1-x)\psi(\xi,y)d\xi \\
&= -\frac{1}{2\pi i}\int_{-\xi_0-\delta-i0}^{-i0} e^{\xi t}\frac{\psi(\xi,1-x)\psi(\xi,y)}{\psi(\xi,1)}d\xi \\
&\quad -\frac{1}{2\pi i}\int_{+i0}^{-\xi_0-\delta+i0} e^{\xi t}\frac{\psi(\xi,1-x)\psi(\xi,y)}{\psi(\xi,1)}d\xi \\
&= -e^{-\xi_0 t}\frac{\psi(-\xi_0,1-x)\psi(-\xi_0,y)}{\psi'(-\xi_0,1)}.
\end{aligned} \tag{37}$$

Taking into account (34) we get the following estimate for second term of (36)

$$\begin{aligned}
&\left|\int_{\xi\in\mathscr{C}_1} e^{\xi t}\frac{\psi(\xi,1-x)\psi(\xi,y)}{\psi(\xi,1)}d\xi\right| \\
&\leqslant Ce^{-(\xi_0+\delta)t}\int_{\xi\in\mathscr{C}_0} e^{-Ct|\xi|^{\frac{2}{3}}+t(\xi_0+\delta)-C|x-y||\xi|^{\frac{1}{3}}}|\xi|^{-\frac{2}{3}}d\xi \\
&\leqslant Ce^{-(\xi_0+\delta)t}t^{-\alpha}|x-y|^{2\alpha-1} \qquad (38)
\end{aligned}$$

since $C|\xi|^{\frac{2}{3}}-\xi_0-\delta\geqslant 0$ for $\xi\in\mathscr{C}_1,$ where $\alpha\in\left[0,\frac{1}{2}\right].$ Therefore by (37)-(38) we have from (36)

$$F_1(x,y,t)=-e^{-\xi_0 t}\Lambda(x)\psi(-\xi_0,y)+O\left(e^{-(\xi_0+\delta)t}\right).$$

where $\Lambda(x)=\frac{\psi(-\xi_0,1-x)}{\psi'(-\xi_0,1)}$ for $x,y>0,t\geqslant 1,$ and moreover

$$|F_1(x,y,t)|\leqslant Ce^{-\xi_0 t}\left(1+\{t\}^{-\alpha}|x-y|^{2\alpha-1}\right)$$

for all $x,\ y\in(0,1),\ x\neq y,\ t>0,$ where $\alpha\in\left[0,\frac{1}{2}\right].$ Thus the result of the lemma is true for the case $n=0.$

Consider the case $n=1.$ In view of the asymptotics formulas (28)-(30) we get

$$\begin{aligned}
&\frac{\partial_x\psi(\xi,1-x)\psi(\xi,y)}{\psi(\xi,1)} \\
&=\frac{\left(\sum_{j=1}^{3}e^{-\varphi_j(\xi)(1-x)}\left(\varphi_j'\varphi_j\right)\right)\left(\sum_{j=1}^{3}e^{-\varphi_j(\xi)y}\varphi_j'\right)}{\sum_{j=1}^{3}e^{-\varphi_j(\xi)}\varphi_j'}=O(1) \qquad (39)
\end{aligned}$$

and

$$\partial_x\psi(\xi,y-x)=O(1) \qquad (40)$$

for $|\xi|<1,\ \xi\in\mathscr{C}$ and in the same argument as in the proof of the estimate (31) we get

$$\begin{aligned}
&\frac{\partial_x\psi(\xi,1-x)\psi(\xi,y)}{\psi(\xi,1)} \\
&=e^{-\varphi_3(y-x)}\varphi_3\varphi_3'\left(1+O\left(e^{-C\sqrt[3]{|\xi|}y}\right)+O\left(e^{-C\sqrt[3]{|\xi|}(1-x)}\right)\right) \qquad (41)
\end{aligned}$$

and

$$\partial_x\psi(\xi,y-x)=e^{-\varphi_3(y-x)}\varphi_3\varphi_3'\left(1+O\left(e^{-C\sqrt[3]{|\xi|}y}\right)+O\left(e^{-C\sqrt[3]{|\xi|}(1-x)}\right)\right) \qquad (42)$$

for all $|\xi|>1,\ \xi\in\mathscr{C}_0.$ Hence by the similar way to (37)-(38) we get

$$
\begin{aligned}
|\partial_x F_1(x,y,t)| &\leqslant e^{-\xi_0 t}\left|\frac{\psi(-\xi_0,y)\partial_x\psi(-\xi_0,1-x)}{\psi'(-\xi_0,1)}\right| + Ce^{-\xi_0 t} \\
&\quad + Ce^{-(\xi_0+\delta)t}\int_{\xi\in\mathscr{C}_0} e^{-Ct|\xi|^{\frac{2}{3}}-C|x-y||\xi|^{\frac{1}{3}}}|\xi|^{-\frac{1}{3}}d\xi \\
&\leqslant e^{-\xi_0 t}\left(C + C\{t\}^{-\alpha}|x-y|^{2\alpha-2}\right)
\end{aligned}
$$

for all $x,\ y\in(0,1),\ x\neq y,\ t>0,$ where $\alpha\in[0,1]$. The function $F_2(x,y,t)$ is considered in the same way for $y>x$. Lemma 1 is proved.

LEMMA 2. *The following estimates are valid*

$$
\|\mathscr{G}\varphi\|_{\mathbf{H}^1} \leqslant Ce^{-\xi_0 t}t^{-\frac{1}{2}}\|\phi\|_{\mathbf{L}^2},\ \|\mathscr{H}h\|_{\mathbf{H}^1} \leqslant C\langle t\rangle^{-\beta}\|h\|_{\mathbf{H}^1},
$$

and

$$
\begin{aligned}
\mathscr{G}\varphi &= e^{-\xi_0 t}B\Lambda(x) + e^{-\xi_0(t+\delta)}\|\phi\|_{\mathbf{L}^2}, \\
\mathscr{H}h &= h(t)\Psi(x,-\xi_0) + O(t^{-\beta-\gamma})\|h\|_{\mathbf{H}^1}
\end{aligned}
$$

where

$$
\begin{aligned}
B &= \int_0^1 \varphi(y)\psi(-\xi_0,y)dy, \qquad (43)\\
\Psi &= res(\widehat{H}(x,\xi),-\xi_0)\left(1+(-\xi_0+i)\frac{1}{2\pi i}\int_0^\infty e^{-\xi}\xi^{-1+\beta}d\xi\right), \\
\widehat{H}(x,\xi) &= \mathscr{L}_t H.
\end{aligned}
$$

Proof. Since

$$
G(x,y,t) = \begin{cases} F_1(x,y,t), & \text{for } y<x, \\ F_2(x,y,t), & \text{for } x<y \end{cases}.
$$

using the Young inequality and Lemma 1 we have

$$
\|\mathscr{G}\varphi\|_{\mathbf{H}^1} \leqslant C(\|G\|_{\mathbf{L}^1} + \|G_x\|_{\mathbf{L}^1})\|\phi\|_{\mathbf{L}^2} \leqslant Ce^{-\xi_0 t}\{t\}^{-\gamma}\|\phi\|_{\mathbf{H}^1}.
$$

Also we have after integrating by part

$$
\begin{aligned}
\mathscr{H}h &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{\xi+i}\frac{\psi_y(\xi,1-x)\psi(\xi,1)-\psi(\xi,1-x)\psi_y(\xi,1)}{\psi(\xi,1)}d\xi \qquad (44)\\
&\quad \left(h(t)-e^{\xi t}h(0)-\int_0^t e^{\xi(t-\tau)}(1+i\partial_\tau)h(\tau)\right)d\tau
\end{aligned}
$$

Using estimates (41) and (42) we get

$$
\partial_x^n\frac{\psi_y(\xi,1-x)\psi(\xi,1)-\psi(\xi,1-x)\psi_y(\xi,1)}{\psi(\xi,1)} = e^{-\sqrt{\xi}(1-x)}\langle\xi\rangle^{\frac{n}{3}}\langle\xi\rangle^{-\frac{1}{3}}.
$$

Therefore for $t < 1$

$$\|\mathscr{H}h\|_{\mathbf{H}^1} \leqslant C\|h\|_{\mathbf{H}^1}.$$

Also via Lemma 1 we get

$$\mathscr{G}\varphi = -e^{-\xi_0 t}\Lambda(x)\int_0^1 \psi(-\xi_0,y)\phi(y)dy + O\left(e^{-(\xi_0+\delta)t}\right)\|\phi\|_{\mathbf{L}^2}.$$

To get asymptotics of $\mathscr{H}$ we use (44). We have

$$\mathscr{H}h = h(t)\Psi(x,0) + \int_0^t \Psi(x,t-\tau)(1+\partial_\tau)h(\tau)d\tau + R, \tag{45}$$

where

$$\begin{aligned}\Psi(x,t) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}\frac{1}{\xi+i}\frac{\psi_y(\xi,1-x)\psi(\xi,1)-\psi(\xi,1-x)\psi_y(\xi,1)}{\psi(\xi,1)}d\xi,\\ R &= e^{-\xi_0 t}|h(0)|.\end{aligned}$$

By Cauchy theorem since integrand function is analytic in $\operatorname{Re}\xi > 0$ we get

$$\Psi(x,0) = 0.$$

By the same way as (41) and (42) we get

$$\Psi(x,t) = e^{-\xi_0 t}\Phi(x,-\xi_0) + e^{-\xi_0(t+\delta)}\langle t\rangle^{-\alpha}, \alpha > 1, \delta > 0,$$

where

$$\Phi(x,-\xi_0) = \frac{1}{-\xi_0+i}\frac{\psi_y(-\xi_0,1-x)\psi(-\xi_0,1)-\psi(-\xi_0,1-x)\psi_y(-\xi_0,1)}{\psi'(-\xi_0,1)}.$$

Therefore

$$\int_0^t \Psi(x,t-\tau)(1+\partial_\tau)h(\tau)d\tau = \Phi(x,-\xi_0)\int_0^t e^{-\xi_0(t-\tau)}(1-i\partial_\tau)h(\tau)d\tau + R,$$

where

$$R = \int_0^t e^{-\xi_0(t-\tau)}(1-i\partial_\tau)h(\tau)O(e^{-\xi_0(t+\delta)}\langle\tau\rangle^{-\alpha})d\tau = O(t^{-\beta-\gamma})\|h\|_{\mathbf{H}^1}$$

We have

$$\int_0^t e^{-\xi_0(t-\tau)}\partial_\tau h(\tau)d\tau = h(t) - h(0)e^{-\xi_0 t} - i\xi_0\int_0^t e^{-\xi_0(t-\tau)}h(\tau)d\tau$$

and as consequence

$$\mathscr{H}h = h(t)\Phi(x,-\xi_0) + (-\xi_0+i)\Phi(x,-\xi_0)\int_0^t e^{-\xi_0(t-\tau)}h(\tau)d\tau + O(t^{-\beta-\gamma})\|h\|_{\mathbf{H}^1}.$$

Also we have

$$\int_0^t e^{-\xi_0(t-\tau)}h(\tau)d\tau = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}\widehat{h}(\xi)\frac{1}{\xi+\xi_0}d\xi = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi}\frac{\widehat{h}(\xi t^{-1})}{\xi+\xi_0 t}d\xi.$$

Using $h(t) = At^{-\beta} + O(t^{-\beta-\gamma})$ we get $\widehat{h}(\xi) = \xi^{-1+\beta} + O(\xi^{-1+\beta+\gamma})$. Therefore

$$\begin{aligned}\int_0^t e^{-\xi_0(t-\tau)}h(\tau)d\tau &= At^{1-\beta}\frac{1}{2\pi i}\int e^{\xi}\frac{\xi^{-\beta+1}+O(\xi^{1-\beta-\gamma})t^{-\gamma}}{\xi+\xi_0 t}d\xi \\ &= At^{-\beta}\frac{1}{2\pi i}\int_0^{\infty} e^{-\xi}\xi^{-1+\beta}d\xi + O(t^{-\beta-\gamma}).\end{aligned}$$

Finally we get

$$\mathscr{H}h = h(t)B(x,-\xi_0) + At^{-\beta}(-\xi_0+i)B(x,-\xi_0)\frac{1}{2\pi i}\int_0^{\infty} e^{-\xi}\xi^{-1+\beta}d\xi + O(t^{-\beta-\gamma}).$$

By the same way we can prove

$$\|\partial_x \mathscr{H}h\|_{\mathbf{L}^\infty} \leqslant C\langle t\rangle^{-\beta}\|h\|_{\mathbf{H}^1}.$$

Lemma is proved.

Using results of Lemma 1 by standard contraction mapping principle we obtain the following local existence result.

THEOREM 2. *Let the initial data $u_0(x) \in \mathbf{L}^2, h(t) \in \mathbf{H}^1$ Then there exists $T > 0$ and a unique solution $u(x,t) \in \mathbf{C}([0,T];\mathbf{L}^\infty) \cup \mathbf{C}\left((0,T];\mathbf{H}^1\right)$ of the nonlinear initial boundary value problem (2) where $T > 0$ depends on $\|u_0\|_{\mathbf{L}^2}$.*

3. Global existence in the case of small initial data

THEOREM 3. *Suppose that the initial data $u_0 \in \mathbf{L}^2$ and $\|u_0(x)\| \leqslant \varepsilon$ where $\varepsilon > 0$ is sufficiently small. Boundary data $h(t) \in \mathbf{H}^1(0,\infty)$, such as for some constant A the following asymptotics are valid*

$$h(t) = At^{-\beta} + O(t^{-\beta-\gamma}). \tag{46}$$

Then for $\beta > \frac{1}{2}$ there exists a unique solution of (2)

$$u \in \mathbf{C}\left([0,\infty);\mathbf{L}^2\right) \cup \mathbf{C}\left((0,\infty);\mathbf{H}^1\right).$$

Moreover the solution has the following asymptotics

$$u(x,t) = h(t)\Psi(x,-\xi_0) + O\left(t^{-\beta-\gamma}\right) \tag{47}$$

for $t \to \infty$ uniformly with respect to $x \in (0,1)$, where $\gamma > 0$, the function $\Psi(x,-\xi_0)$ is defined by (43)

Proof. We prove the global existence of the solution of the problem (2) using the contraction mapping principle.

Let

$$\mathbf{X} = \left\{ \varphi(t,x) \in \mathbf{H}^1 \right\}, \tag{48}$$

$$\mathbf{Y} = \left\{ h(t) \in \mathbf{H}^1(0,\infty)), \langle t \rangle^{\beta} \|h\|_{\mathbf{H}^1} < \varepsilon \right\}, \tag{49}$$

a complete metric space $\mathbf{Z}$, for $\beta > \frac{1}{2}$

$$\mathbf{Z} = \left\{ \varphi(t,x) \in \mathbf{C}((0,\infty;\mathbf{H}^1(0,1)), \|\varphi\|_{\mathbf{X}} = \sup_{t>0} \langle t \rangle^{\beta} \|\varphi(t)\|_{\mathbf{H}^1} \leqslant \varepsilon_1 \right\}. \tag{50}$$

By Proposition 1 we define a mapping $\mathscr{A}$ for $u(x,t)$ given by

$$\mathscr{A}(u) = \mathscr{G}u_0 - \int_0^t \mathscr{G}(t-\tau)\mathbb{N}(u)\,d\tau + \mathscr{H}h. \tag{51}$$

We suppose that $u \in \mathbf{Z}$ with $\|u\|_{\mathbf{Z}} \leqslant \varepsilon_1$ and using (51) we probe that

$$\|\mathscr{A}(u)\|_{\mathbf{Z}} \leqslant \varepsilon_1.$$

Where $\varepsilon_1 < \varepsilon$. Applying the definition (51) we write

$$\begin{aligned}\|\mathscr{A}(u)\|_{\mathbf{Z}} &\leqslant \|\mathscr{G}u_0\|_{\mathbf{Z}} + \left\| \int_0^t \mathscr{G}(t-\tau)\mathbb{N}(u)\,d\tau \right\|_{\mathbf{Z}} + \|\mathscr{H}h\|_{\mathbf{Z}} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

We consider each integral separately, using definition of the operator $\mathscr{G}$ and (50) we obtain

$$I_1 \leqslant \sup_{t>0} \langle t \rangle^{\beta} \left\| \int_0^x u_0(y) F_1(x,y,t)dy \right\|_{\mathbf{L}^2} + \sup_{t>0} \langle t \rangle^{\beta} \left\| \int_x^1 u_0(y) F_2(x,y,t)dy \right\|_{\mathbf{L}^2}.$$

We can extend the domain of definition of $x \in (-\infty,\infty)$ whereas $u_0(x) = 0$ for $x \notin (0,1)$ and $F_1(x,y,t) \leqslant F_1(x-y,t)$, so we can write the following using the Young inequality

$$\begin{aligned}I_1 &\leqslant \sup_{t>0} \langle t \rangle^{\beta} \int_{-\infty}^{\infty} \|F_1(x-y,t)\|_{\mathbf{L}^1} \|u_0(y)\|_{\mathbf{L}^2}\,dy \\ &\quad + \sup_{t>0} \langle t \rangle^{\beta} \int_{-\infty}^{\infty} \|F_2(x-y,t)\|_{\mathbf{L}^1} \|u_0(y)\|_{\mathbf{L}^2}\,dy.\end{aligned}$$

Using the estimation by Lemma 1 and simplifying we obtain

$$\begin{aligned}I_1 &\leqslant C\|u_0(y)\|_{\mathbf{L}^2} \sup_{t>0} \langle t \rangle^{\beta} e^{-\xi_0 t} \{t\}^{-\alpha} \int_0^1 |x-y|^{2\alpha-1}\,dy \\ &\leqslant C\|u_0(y)\|_{\mathbf{L}^2}\end{aligned} \tag{52}$$

Similarly for I_2 we obtain using the Young inequality and Lemma 1

$$
\begin{aligned}
I_2 &\leqslant C\sup_{t>0}\langle t\rangle^{\beta}\int_0^t e^{-\xi_0(t-\tau)}\{t-\tau\}^{-\alpha}d\tau\int_0^x |x-y|^{2\alpha-1}\|\mathbb{N}(v)\|_{\mathbf{L}^2}dy \\
&\quad +C\sup_{t>0}\langle t\rangle^{\beta}\int_0^t e^{-\xi_0(t-\tau)}\{t-\tau\}^{-\alpha}d\tau\int_x^1 |x-y|^{2\alpha-1}\|\mathbb{N}(v)\|_{\mathbf{L}^2}dy.
\end{aligned}
\tag{53}
$$

since $v\in\mathbf{Z}$ then $\|v\|_{\mathbf{Z}}\leqslant\varepsilon$ therefore

$$
\begin{aligned}
\|\mathbb{N}(v)\|_{\mathbf{L^2}} &= \left(\int_0^1 |v|^2|v_x|^2dx\right)^{\frac12}\leqslant C\|v\|_{\mathbf{L}^\infty}\|v_x\|_{\mathbf{L}^2} \\
&\leqslant t^{-2\beta}\|v\|_{\mathbf{Z}}^2.
\end{aligned}
\tag{54}
$$

Substituting (54) into (53) and making change of variable $\tau=tz$ we get

$$
\begin{aligned}
I_2 &\leqslant C\sup_{t>0}\langle t\rangle^{\beta}\|v\|_{\mathbf{Z}}^2\int_0^t e^{-\xi_0(t-\tau)}\{t-\tau\}^{-\alpha}\langle\tau\rangle^{-2\beta}d\tau \\
&\leqslant C\sup_{t>0}\langle t\rangle^{\beta}\|v\|_Z^2 t^{-2\beta+1-\alpha}\leqslant C\sup_{t>0}\|v\|_{\mathbf{Z}}^2 t^{-\beta+1-\alpha},
\end{aligned}
\tag{55}
$$

where $\beta>\frac12$. Via Lemma 2

$$
\|\mathscr{H}h\|_{\mathbf{Z}}\leqslant C\|h(t)\|_{\mathbf{Y}}.
$$

From (52) and (55) we get for $t>0$

$$
\begin{aligned}
\|\mathscr{A}(u)\|_{\mathbf{Z}} &\leqslant C(\|u_0\|_{\mathbf{L}^2}+\|h\|_{\mathbf{Y}})+T^{1-\gamma}\|v\|_Z^2 \\
&\leqslant C(\|u_0\|_{\mathbf{L}^2}+\|h\|_{\mathbf{Y}})+T^{1-\gamma}\varepsilon_1^2 \\
&\leqslant \varepsilon.
\end{aligned}
$$

Analogously we can estimate the difference $\|\mathscr{A}(v_1)-\mathscr{A}(v_2)\|_{\mathbf{Z}}\leqslant\varepsilon$ for $t>0$. Therefore the mapping $\mathscr{A}$ is a contraction mapping in $\mathbf{H}_\rho^1$ into itself and there exists a unique solution $v(x,t)\in\mathbf{C}\left((0,\infty);\mathbf{H}^1\right)$ of the initial-value problem (2).

Now we prove that the solution

$$
u(x,t)=\mathscr{G}u_0-\int_0^t\mathscr{G}(t-\tau)\mathbb{N}(u)d\tau+\mathscr{H}h \tag{56}
$$

has asymptotics (47) for $t\to\infty$ uniformly with respect to $x\in(0,1)$. Indeed, due to Lemma 2

$$
\begin{aligned}
\mathscr{G}\varphi &= e^{-\xi_0 t}B\Lambda(x)+e^{-\xi_0(t+\delta)}\|\phi\|_{\mathbf{L}^1}, \\
\mathscr{H}h &= h(t)\Psi(x,-\xi_0)+O(t^{-\beta-\gamma})\|h\|_{\mathbf{H}^1}.
\end{aligned}
$$

Therefore from (56) we write solution in the form

$$u(x,t) = h(t)\Psi(x,-\xi_0) + R(x,t),$$

where

$$|R(x,t)| \leqslant Ce^{-\xi_0 t}\|u_0\|_{\mathbf{L}^2} + C\left|\int_0^t d\tau \int_0^1 G(x,y,t-\tau)\mathbb{N}(v)\,dy\right|$$
$$= J_1 + J_2$$

Since $\|\mathbb{N}(v)\|_{\mathbf{L}^2} \leqslant Ct^{-2\beta}$ in the same way as (45) we obtain for J_2

$$J_2 = O\left(t^{-2\beta-\gamma}\right).$$

Therefore, we can finally express the asymptotic $u(x,t)$ as follows

$$u(x,t) = h(t)\Psi(x,-\xi_0) + O\left(t^{-\beta-\gamma}\right)$$

for all $t \geqslant 1$, where $\gamma > 0$. Theorem 3 is proved.

The following section is devoted to proof of main theorem with large initial data by using the time decay estimates of solutions obtained in Section 3.

4. Large initial data. Proof of Theorem

We consider the initial-boundary value problem (2) with any initial data $\|u_0\|_{\mathbf{L}^2} \leqslant C$. Multiplying equation (2) by u and integrating with respect to $x \in (0,1)$ we get

$$\frac{d}{dt}\|u\|_{\mathbf{L}^2}^2 + 2\int_0^1 \left(u^2u_x - uu_{xx} + uu_{xxx}\right)dx = 0. \tag{57}$$

In view of the boundary data $u(0,t) = u(1,t) = 0$ we have

$$\int_0^1 u^2u_x dx = \frac{1}{3}u^3\bigg|_0^1 = 0. \tag{58}$$

For

$$\int_0^1 uu_{xx}dx = uu_x|_0^1 - \int_0^1 u_x^2dx = -\int_0^1 u_x^2dx,$$

substituting $u(0,t) = u(1,t) = 0$ we obtain

$$\int_0^1 uu_{xx}dx = -\int_0^1 u_x^2dx. \tag{59}$$

Now

$$\int_0^1 uu_{xxx}dx = uu_{xx}|_0^1 - \frac{1}{2}\,u_x^2\big|_0^1 = -\frac{1}{2}\left[u_x^2(1,t) - u_x^2(0,t)\right]$$

substituting $u_x(1,t) = h(t)$ for $t > 0$ we get

$$\int_0^1 uu_{xxx}dx = -\frac{1}{2}\left[h^2(t) - u_x^2(0,t)\right]. \tag{60}$$

Substituting relations (58)-(60) into (57) we get

$$\frac{d}{dt}\|u\|_{\mathbf{L}^2}^2 + 2\|u_x(\tau)\|_{\mathbf{L}^2}^2 - h^2(t) + u_x^2(0,t) = 0.$$

Integration with respect to $t > 0$ yields

$$\|u(t)\|_{\mathbf{L}^2} + 2\int_0^t \|u_x(\tau)\|_{\mathbf{L}^2}^2 d\tau \leqslant \|u_0\|_{\mathbf{L}^2} + \int_0^t h^2(\tau)d\tau$$

for all $t \in (0,\infty)$. It follows that the norm $\|u(t)\|_{\mathbf{L}^2} \leqslant \|u_0\|_{\mathbf{L}^2} + \|h\|_{\mathbf{L}^2}$ for all $t \geqslant 0$. Since the existence time T depends only on $\|u_0\|_{\mathbf{L}^2} + \|h\|_{\mathbf{L}^2}$ by the standard continuation process via local existence Theorem 2 we obtain that there exists a unique global solution $u \in \mathbf{C}\left((0,\infty);\mathbf{H}^1\right)$. Moreover for any $\varepsilon > 0$ there exists a time $T > 0$ such that $\|u_x(T)\|_{\mathbf{L}^2}^2 < \varepsilon$. By the inequality $\left|u^2(x,T)\right| = 2\left|\int_0^x uu_y dy\right| \leqslant 2\|u\|_{\mathbf{L}^2}\|u_x\|_{\mathbf{L}^2}$ we obtain that the norm $\|u(T)\|_{\mathbf{L}^\infty}$ is small. Hence by the estimate $\|u(T)\|_{\mathbf{L}^2} \leqslant \|u(T)\|_{\mathbf{L}^\infty}$ the norm $\|u(T)\|_{\mathbf{L}^2}$, is also small. Then we consider the initial-boundary value problem (2) for $t \geqslant T$ and apply Theorem 3 we prove Theorem 1.

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