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TWO–SCALE CONVERGENCE IN THIN DOMAINS WITH LOCALLY PERIODIC RAPIDLY OSCILLATING BOUNDARY

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Abstract. The aim of this paper is to adapt the notion of two-scale convergence in L^p to the case of a measure converging to a singular one. We present a specific case when a thin cylinder with locally periodic rapidly oscillating boundary shrinks to a segment, and the corresponding measure charging the cylinder converges to a one-dimensional Lebegues measure of an interval. The method is then applied to the asymptotic analysis of linear elliptic operators with locally periodic coefficients and a p-Laplacian stated in thin cylinders with locally periodic rapidly varying thickness.

1. Introduction

The goal of this paper is twofold. First, we want to adapt the classical two-scale convergence (see [25], [1], [33], [9]) to the case of a asymptotically thin domain. We consider a specific case when the domain has locally periodic rapidly oscillating boundary and shrinks to a segment. Second, we will apply the introduced definition to the asymptotic analysis of a linear and quasilinear elliptic operators in thin cylinders with oscillating thickness.

Boundary value and spectral problems in thin domains are usually treated using the analysis of resolvents ([16]), the method of asymptotic expansions ([13], [26], [8], [21], [24], [29]), two-scale convergence ([14], [20], [27], [28]), Γ-convergence ([22], [3], [12], [18], [11], [10]), compensated compactness agrument ([19]), and the unfolding method ([7], [5], [6]). The presented list of works devoted to the homogenization in thin structures is far from being complete, but our primary focus is the case of thin domains with locally periodic rapidly varying thickness, and the works treating the linear case closely related to our study are [21], [5], [16], [8], and [23]. We describe them briefly below.

The case of periodic rapidly oscillating boundary was considered in [21], where the authors studied the asymptotic behaviour of second-order self-adjoint elliptic operators with periodic coefficients, for different boundary conditions. In [5] the case of a locally periodic rapidly oscillating boundary was addressed, and the authors studied the Neumann boundary value problem for the Laplace operator in a two-dimensional thin domain by means of the unfolding method. Spectral asymptotics of the Laplace operator in thin domains with slowly varying thickness were considered in [16], [8],

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[23], where under the Dirichlet boundary conditions the localization of eigenfunctions occur.

The contribution of the present paper is an adapted notion of the two-scale convergence that covers both thin domains with slowly varying, periodic rapidly oscillating and locally periodic rapidly oscillating boundary. We do not make any restrictions on the dimension of the thin domains in the transverse direction. The method presented can be applied to both boundary value and spectral problems (exactly like the classical two-scale convergence), linear and nonlinear. In the present note we use it for the homogenization of a linear elliptic operator with locally periodic coefficients and a p-Laplacian operator stated in thin domains with locally periodic rapidly oscillating boundary. These two academic examples are given to illustrate the method. The results, even though new, can be predicted based on the existing literature. More advanced examples, like nonlinear convection-diffusion-reaction problems in thin domains with oscillating thickness, fluid flow through a thin pipe, or indefinite spectral problems possibly describing properties of metamaterials, are to be considered elsewhere.

The two-scale convergence is a powerful tool that allows us to characterize the leading term of the asymptotics without using asymptotic expansions, that reduces the amount of computations. It can be applied both to linear and nonlinear problems, which makes this method so popular for asymptotic analysis. In [20] the authors introduced the notion of the two-scale convergence for thin domains, but their definition does not catch the oscillations in the longitudinal variable. As a consequence, it works for operators with coefficients which are constant in the longitudinal variable. Our approach is based on the two-scale convergence in spaces with measure introduced in [9], [33]. It was introduced for the case of a scaled periodic measure, while in the present work we focus on a measure converging to a singular one. The proofs of the basic facts about the properties of the L^p-spaces and the two-scale convergence itself follow the lines of those in [33].

The study of the p-Laplacian operator $\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$ attracts a lot of attention in many different contexts because of its numerous applications. For small p it appears in total variation denoising [31], for $p > 1$ it models non-newtonian fluids, and for large p it is used to model growth and collapse of sandpiles [4].

One of the goals of this note is to analyse the asymptotic behavior of a boundary value problem for a p-Laplacian operator in a thin cylinder with locally periodic rapidly varying thickness, when the cylinder shrinks to a one-dimensional segment. We show that the limit equation is a one-dimensional p-Laplacian with an effective coefficient describing the varying thickness. Note that in general, when homogenizing a nonlinear operator, the limit operator does not have the same form. In the present case, however, the p-Laplacian is preserved due to the dimension reduction.

The homogenization of quasilinear operators in combination with oscillating boundary and dimension reduction was considered in many works. In [12] and [3] the authors study nonlinearly elastic thin films, including the case of non-convex energies, with a fast-oscillating profile and apply the Γ-convergence to find the homogenized functional. In such a general situation the dependence of the limit functional on the original one is not clear and should be studied for each particular situation. In [7] a class of monotone nonlinear Neumann problems in a thin plate with a “forest” of periodically distributed cylinders on the upper part of the plate. The work [2] exploits the method of local characteristics and study a quasilinear elliptic operator in very general non-periodic thin domains. The authors provide a periodic and locally periodic examples where the characteristics can be computed in terms of some auxiliary cell problems. It is shown that the limit operator is of the same form both in the case of thin domains with constant thickness and slowly varying thickness. The works [32] and [30] study a p-Laplacian operator in thin domains with slowly varying thickness. In the limit the authors obtain again a p-Laplacian with an effective coefficient describing the varying thickness of the domain.

The paper is organized as follows. In Section 2 we define the domain and introduce the corresponding spaces with measure charging this domain. In Section 3 we introduce the adapted two-scale convergence and discuss its properties. Section 4 concerns the application of the method to the asymptotic analysis of a linear elliptic operator with locally periodic coefficients (see Theorem 4.1). In Section 5 we study a p-Laplacian operator, and the main results of that part is given in Theorem 5.1.

2. Variable spaces with singular measure in a cylinder with locally periodic rapidly oscillating boundary

We are going to adapt the notion of the two-scale convergence to the case when a thin domain has a rapidly oscillating boundary modulated by some (slowly) varying function.

In what follows the points in $\mathbf{R}^d$ are denoted by $x=(x_1,x')$, and $I=(-1,1)$. We denote

$$Q(x_1,y_1)=\{y'\in\mathbf{R}^{d-1}: F(x_1,y_1,y')>0\},$$

where $F(x_1,y_1,y')$ is such that

(H1) $F(x_1,y_1,y')\in C^{1,\alpha}(\bar{I}\times\mathbf{T}^1\times\mathbf{R}^{d-1})$, where $\mathbf{T}^1$ is the one-dimensional torus.

(H2) $Q(x_1,y_1)$ is non-empty, bounded, and simply connected.

To ensure that the conditions (H1), (H2) are fulfilled, we can take, for example, F satisfying the assumptions

(F1) For each x_1 and y_1, $F(x_1,y_1,0)>0$ and $F(x_1,y_1,y')<0$ for $|y'|\geqslant R$, for some $R>0$. This guarantees that $Q(x_1,y_1)$ is not empty and bounded.

(F2) $F(x_1,y_1,\cdot)$ does not have a nonpositive local maximum/minimum. This guarantees that $Q(x_1,y_1)$ is simply connected.

Now let $\varepsilon>0$ be a small parameter. We are going to work in a thin cylinder

$$\Omega_\varepsilon=\Big\{x=(x_1,x'): x_1\in I, x'\in\varepsilon Q\Big(x_1,\frac{x_1}{\varepsilon}\Big)\Big\}.$$

Here $Q(x_1,\frac{x_1}{\varepsilon})$ describes the locally periodic varying cross section of the cylinder (periodicity with respect to the second variable is inherited from F). When $F=F(y_1,y')$

we have a periodic oscillating boundary, when $F = F(x_1, y')$ we are in the case of slowly varying thickness, and finally, when $F = F(y')$ the cylinder is straight.

An example of Ω_ε is presented in Figure 1 for three different values of ε.

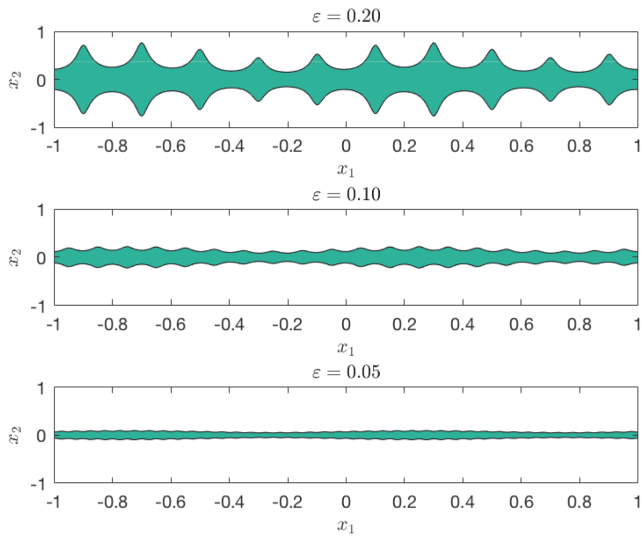

Figure 1: *A thin cylinder in* $\mathbf{R}^2$ *generated by*
$$F(x_1, y_1, y_2) = 2 + \sin(2\pi x_1) - y_2^2 \cdot (1 + 4\varepsilon \cos(2\pi y_1)).$$

The boundary of Ω_ε consists of the lateral boundary of the cylinder

$$\Sigma_\varepsilon = \left\{ x = (x_1, x') : x_1 \in I, F\left(x_1, \frac{x_1}{\varepsilon}, \frac{x'}{\varepsilon}\right) = 0 \right\},$$

and the bases $\Gamma_\varepsilon^\pm = \{\pm 1\} \times \varepsilon Q(\pm 1, \pm 1/\varepsilon)$.

The periodicity cell depending on x_1 is

$$\square(x_1) = \{ y = (y_1, y') : y_1 \in \mathbf{T}^1, y' \in Q(x_1, y_1) \},$$

where $\mathbf{T}^1$ is a one-dimensional torus.

Since $F(x_1, y_1, y')$ is periodic in y_1, the boundary of $\square(x_1)$ is $\partial\square(x_1) = \{ y = (y_1, y') : y_1 \in \mathbf{T}^1, F(x_1, y_1, y') = 0 \}$.

REMARK 1. In the two-dimensional case the definition of the thin cylinder with locally periodic oscillating thickness becomes easier (and more transparent). Namely, given two smooth functions $G_-(x_1, y_1)$ and $G_+(x_1, y_1)$, 1-periodic with respect to the second variable, we set

$$\Omega_\varepsilon = \left\{ x = (x_1, x') \in \mathbf{R}^2 : x_1 \in I, -\varepsilon G_-\left(x_1, \frac{x_1}{\varepsilon}\right) < x' < \varepsilon G_+\left(x_1, \frac{x_1}{\varepsilon}\right) \right\}.$$

In the d-dimensional case, however, an alternative way to describe the varying thickness would be to apply a family of diffeomorphisms depending on x_1 and ε (posing a local periodicity assumption) to a constant cross-section. An advantage of the definition given in the present paper is that it is easier to work with when defining the lateral boundary and the bases of the cylinder, and the periodicity cell, as well as to program when using numerical methods.

We define a Radon measure on $\mathbf{R}^d$ by

$$d\mu_\varepsilon = \varepsilon^{-(d-1)}\chi_{\Omega_\varepsilon}(x)\,dx, \tag{1}$$

where $\chi_{\Omega_\varepsilon}(x)$ is the characteristic function of the thin cylinder Ω_ε; dx is the d-dimensional Lebesgue measure.

The factor $\varepsilon^{-(d-1)}$ in (1) makes the measure of the cylinder Ω_ε of order 1.

LEMMA 2.1. *The measure μ_ε defined by (1) converges weak* in the space of Radon measures $\mathscr{M}(\mathbf{R}^d)$, as $\varepsilon \to 0$, to the measure μ_* defined by*

$$d\mu_* = |\square(x_1)|\chi_I(x_1)dx_1 \times \delta(x').$$

Proof. Let $\varphi \in C_0(\mathbf{R}^d)$. Then

$$\int_{\mathbf{R}^d} \varphi(x)\,d\mu_\varepsilon(x) = \int_I \varepsilon^{-(d-1)} \int_{\varepsilon Q(x_1,x_1/\varepsilon)} \varphi(x)dx'dx_1.$$

Rescaling $y' = x'/\varepsilon$ gives

$$\int_{\mathbf{R}^d} \varphi(x)\,d\mu_\varepsilon(x) = \int_I \int_{Q(x_1,x_1/\varepsilon)} \varphi(x_1,\varepsilon y')dy'dx_1.$$

Let us divide the interval I into small subintervals (translated periods) $I_j^\varepsilon = \varepsilon[0,1) + \varepsilon j$, $j \in \mathbf{Z}$. The two intervals intersecting the bases of the cylinder (if Ω_ε cannot be covered by an integer number of intervals) give an error of order ε.

On each interval we use the mean-value theorem choosing a point ξ_j and get

$$\sum_j \int_{I_j^\varepsilon} \int_{Q(x_1,x_1/\varepsilon)} \varphi(x_1,\varepsilon y')dy'dx_1 = \sum_j \int_{I_j^\varepsilon} \int_{Q(\xi_j,x_1/\varepsilon)} \varphi(\xi_j,\varepsilon y')dy'dx_1.$$

Since $Q(x_1,y_1)$ is periodic with respect to y_1, rescaling $y_1 = x_1\varepsilon$ yields

$$\sum_j \int_{\mathbf{T}^1} \int_{Q(\xi_1,y_1)} \varphi(\xi_j,\varepsilon y')dy'dy_1 = \sum_j \varepsilon \int_{\square(\xi_j)} \varphi(\xi_j,\varepsilon y')dy.$$

The last sum is a Riemann sum converging, as $\varepsilon \to 0$, to the following integral

$$\begin{aligned}\sum_j \varepsilon \int_{\square(\xi_j)} \varphi(\xi_j,\varepsilon y')dy &\to \int_I \int_{\square(x_1)} \varphi(x_1,0)\,dydx_1 \\ &= \int_I |\square(x_1)|\varphi(x_1,0)\,dx_1 = \int_{\mathbf{R}^d} \varphi(x)\,d\mu_*.\end{aligned}$$

Note that, for any $x_1 \in I$, due to the continuity of F, $|x'| \leqslant C\varepsilon^{d-1}$. Given $\gamma > 0$, we can choose ε small enough such that $x' \in \varepsilon Q$ implies $|\varphi(x_1,0) - \varphi(x)| < \gamma$ using the uniform continuity of φ. □

REMARK 2. We assume that the cylinder is bounded, but Lemma 2.1 is also valid in the case when the cylinder grows in the x_1 direction, as $\varepsilon \to 0$. The arguments are valid if the measure of the cross section is positive and bounded from above by $C\varepsilon^{d-1}$ (the whole cylinder is contained in another cylinder of diameter of order ε). In the case of a cylinder growing in x_1, as $\varepsilon \to 0$, the limit measure is $d\mu_* = |\Box(x_1)| dx_1 \times \delta(x')$. An example of how the two-scale convergence in spaces with measure is used in such case can be found in [28].

REMARK 3. Note that the geometry of the boundary of the periodicity cell is of no importance in Lemma 2.1.

For any ε and $1 < p < \infty$, the space of Borel measurable functions $g : \mathbf{R}^d \to \mathbf{R}$ such that

$$\int_{\mathbf{R}^d} |g|^p \, d\mu_\varepsilon < \infty,$$

is denoted by $L^p(\mathbf{R}^d, \mu_\varepsilon)$. For vector functions $g : \mathbf{R}^d \to \mathbf{R}^d$ we denote the corresponding space by $L^p(\mathbf{R}^d, \mu_\varepsilon)^d$.

DEFINITION 2.2. A sequence u_ε is bounded in $L^p(\mathbf{R}^d, \mu_\varepsilon)$ if

$$\limsup_{\varepsilon \to 0} \int_{\mathbf{R}^d} |u_\varepsilon|^p d\mu_\varepsilon < \infty.$$

A bounded sequence $u_\varepsilon \in L^p(\mathbf{R}^d, \mu_\varepsilon)$ is said to converge weakly in $L^p(\mathbf{R}^d, \mu_\varepsilon)$ to $u \in L^p(\mathbf{R}^d, \mu_*)$ if

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^d} u_\varepsilon \varphi \, d\mu_\varepsilon = \int_{\mathbf{R}^d} u \varphi \, d\mu_*, \qquad \varphi \in C_0^\infty(\mathbf{R}^d).$$

We say that $u_\varepsilon \in L^p(\mathbf{R}^d, \mu_\varepsilon)$ converges strongly to $u \in L^p(\mathbf{R}^d, \mu_*)$ if for any $v_\varepsilon \in L^{p'}(\mathbf{R}^d, \mu_\varepsilon)$ weakly converging to $v \in L^{p'}(\mathbf{R}^d, \mu_*)$, $1/p + 1/p' = 1$, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^d} u_\varepsilon v_\varepsilon \, d\mu_\varepsilon = \int_{\mathbf{R}^d} u v \, d\mu_*.$$

In the case of strong convergence we write $u_\varepsilon \to u$, $\varepsilon \to 0$.

Proofs of the following facts valid for a sequence of measures μ_ε weakly convergent to μ_* (no specific assumptions on the structure of μ_ε), can be found in [34].

- The property of weak compactness of a bounded sequence in a separable Hilbert space remains valid with respect to the convergence in variable spaces. Any bounded sequence in $L^p(\mathbf{R}^d, \mu_\varepsilon)$ contains a weakly convergent subsequence.

- For $u_\varepsilon \in L^p(\mathbf{R}^d, \mu_\varepsilon)$ weakly converging to $u \in L^p(\mathbf{R}^d, \mu_*)$ the lower semicontinuity property holds:

$$\liminf_{\varepsilon \to 0} \int_{\mathbf{R}^d} |u_\varepsilon|^p d\mu_\varepsilon \geqslant \int_{\mathbf{R}^d} |u|^p d\mu_*.$$

- A sequence $u_\varepsilon \in L^p(\mathbf{R}^d, \mu_\varepsilon)$ converges strongly to $u \in L^p(\mathbf{R}^d, \mu_*)$ if and only if u_ε converges to u weakly and

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^d} |u_\varepsilon|^p d\mu_\varepsilon = \int_{\mathbf{R}^d} |u|^p d\mu_*.$$

Let us also recall the definition of the Sobolev space with measure.

DEFINITION 2.3. A function $g \in L^p(\mathbf{R}^d, \mu_\varepsilon)$ is said to belong to the space $W^{1,p}(\mathbf{R}^d, \mu_\varepsilon)$ if there exists a vector function $z \in L^p(\mathbf{R}^d, \mu_\varepsilon)^d$ and a sequence $\varphi_k \in C_0^\infty(\mathbf{R}^d)$ such that

$$\varphi_k \to g \quad \text{in } L^p(\mathbf{R}^d, \mu_\varepsilon), \quad k \to \infty,$$
$$\nabla \varphi_k \to z \quad \text{in } L^p(\mathbf{R}^d, \mu_\varepsilon)^d, \quad k \to \infty.$$

In this case z is called a gradient of g and is denoted by $\nabla^{\mu_\varepsilon} g$.

Since in our case the measure μ_ε is a weighted Lebesgue measure, we have $\nabla^{\mu_\varepsilon} g = \nabla g$ and the space $W^{1,p}(\mathbf{R}^d, \mu_\varepsilon)$ is identical to the usual Sobolev space $W^{1,p}(\Omega_\varepsilon)$, in contrast to the scaled periodic singular measure considered in [33] when the gradient is not unique and is defined up to a gradient of zero.

The spaces $L^2(\mathbf{R}^d, \mu_*)$ and $W^{1,p}(\mathbf{R}^d, \mu_*)$ are defined in a similar way, however the μ_*-gradient is not unique and is defined up to a gradient of zero. A zero function might have a nontrivial gradient as it is demostrated by Example 1 in Ch. 3, [33]. Following the proof in the last example, one can see that for $p = 2$ the subspace of vectors of the form $(0, \psi_2(z_1), \ldots, \psi_d(z_1))$, $\psi_j \in L^2(\mathbf{R})$ is the subspace of gradients of zero. Any μ_*-gradient of $v \in W^{1,2}(\mathbf{R}^d, \mu_*)$ takes the form

$$\nabla^{\mu_*} v(z) = (v'(z_1, 0), \psi_2(z_1), \ldots, \psi_d(z_1)), \quad \psi_j \in L^2(\mathbf{R}),$$

where $v'(z_1, 0)$ is the derivative of the restriction of $v(z)$ to $\mathbf{R} \times \{0\}$.

3. Two-scale convergence in spaces with measure converging to a singular one

In what follows μ_ε denotes the measure given by

$$d\mu_\varepsilon = \chi_{\Omega_\varepsilon}(x) \varepsilon^{-(d-1)} dx,$$

and its weak limit is

$$d\mu_* = |\square(x_1)| \chi_I(x_1)\, dx_1 \times \delta(x').$$

For each $x_1 \in I$, we introduce $C^k(\square(x_1))$, $L^p(\square(x_1))$ and $W^{1,p}(\square(x_1))$ in a usual way. Functions belonging to these spaces are 1-periodic with respect to y_1.

In the present context two-scale convergence is described as follows.

DEFINITION 3.1. We say that $g^\varepsilon \in L^p(\mathbf{R}^d,\mu_\varepsilon)$, $1<p<\infty$, converges two-scale weakly, as $\varepsilon\to 0$, in $L^p(\mathbf{R}^d,\mu_\varepsilon)$ if

(i) $\limsup_{\varepsilon\to 0}\|g^\varepsilon\|_{L^p(\mathbf{R}^d,\mu_\varepsilon)}<\infty$,

(ii) there exists a function $g(x_1,y)\in L^p(I;L^p(\square(x_1))$ 1-periodic in y_1 such that the following limit relation holds:

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^d} g^\varepsilon(x)\,\varphi(x)\,\psi(\frac{x}{\varepsilon})d\mu_\varepsilon(x)=\int_{\mathbf{R}^d}\frac{1}{|\square(x_1)|}\int_{\square(x_1)} g(x_1,y)\,\varphi(x)\,\psi(y)\,dy\,d\mu_*(x)$$
$$=\int_I\int_{\square(x_1)} g(x_1,y)\,\varphi(x_1,0)\,\psi(y)\,dy\,dx_1,$$

for any $\varphi\in C_0^\infty(\mathbf{R}^d)$ and $\psi(y)\in C^\infty(\overline{\square(x_1)})$ periodic in y_1.

We write $g^\varepsilon \xrightarrow{2} g(x_1,y)$ if g^ε converges two-scale weakly to $g(x_1,y)$ in $L^p(\mathbf{R}^d,\mu_\varepsilon)$.

The definition of the two-scale convergence holds for more general classes of test functions. Following the lines of the proof of Lemma 2.1 one can see that for $\psi(y)\in L^1(\square(x_1))$ we have the mean-value property

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^d}\varphi(x)\,\psi\Big(\frac{x}{\varepsilon}\Big)d\mu_\varepsilon(x)=\int_{\mathbf{R}^d}\frac{1}{|\square(x_1)|}\int_{\square(x_1)}\varphi(x)\,\psi(y)\,dy\,d\mu_*(x)$$
$$=\int_I\varphi(x_1,0)\Big(\int_{\square(x_1)}\psi(y)\,dy\Big)\,dx_1.$$

For example, as it is shown in Lemma 3.1 in [34], one can take a Caratheodory function $\Phi(x,y)$ such that

$$|\Phi(x,y)|\leqslant\Phi_0(y),\qquad \Phi_0\in L^1(\square(x_1)).$$

Such test functions are called admissible, and the mean-value property holds

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^d}\Phi\Big(x,\frac{x}{\varepsilon}\Big)d\mu_\varepsilon=\int_{\mathbf{R}^d}\frac{1}{|\square(x_1)|}\int_{\square(x_1)}\Phi(x,y)\,dy\,d\mu_*$$
$$=\int_I\int_{\square(x_1)}\Phi(x_1,0,y)\,dy\,dx_1.$$

The proof of the mean-value property follows the lines of the proof of Lemma 3.1 in [34]. As it was shown in [1], the property of continuity with respect to one of the arguments can not be dropped.

The following compactness result can be proved in the same way as Theorem 4.2 in [34].

LEMMA 3.2. (Compactness) *Suppose that g^ε satisfies the estimate*

$$\limsup_{\varepsilon\to 0}\|g^\varepsilon\|_{L^p(\mathbf{R}^d,\mu_\varepsilon)}<\infty.$$

Then g^ε, up to a subsequence, converges two-scale weakly in $L^p(\mathbf{R}^d,\mu_\varepsilon)$ to some function $g(x_1,y)\in L^p(\mathbf{R}^d\times\square(x_1),\mu_*\times dy)$.

DEFINITION 3.3. A sequence g^ε is said to converge two-scale strongly to a function $g(x_1,y) \in L^p(\mathbf{R}^d \times \Box(x_1), \mu_* \times dy)$ if

(i) g^ε converges two-scale weakly to $g(x_1,y)$,

(ii) the following limit relation holds:

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^d} |g^\varepsilon(x)|^p d\mu_\varepsilon(x) = \int_{\mathbf{R}^d} \frac{1}{|\Box(x_1)|}\int_{\Box(x_1)} |g(x_1,y)|^p\, dy\, d\mu_*(x).$$

We write $g^\varepsilon \xrightarrow{2} g(x_1,y)$ if g^ε converges two-scale strongly to the function $g(x_1,y)$ in $L^p(\mathbf{R}^d,\mu_\varepsilon)$.

The following properties of the weak two-scale limit hold (see [34] for the proof in spaces with measure):

- If $u_\varepsilon \overset{2}{\rightharpoonup} u(x_1,y)$ in $L^p(\mathbf{R}^d,\mu_\varepsilon)$, then u_ε converges weakly in $L^p(\mathbf{R}^d,\mu_\varepsilon)$ to the local average of the two-scale limit:

$$u_\varepsilon \rightharpoonup \frac{1}{|\Box(x_1)|}\int_{\Box(x_1)} u(x_1,y)dy.$$

To see this it is suffices to take a test function independent of y in the definition of the two-scale convergence.

- If $u_\varepsilon \overset{2}{\rightharpoonup} u(x_1,y)$ in $L^p(\mathbf{R}^d,\mu_\varepsilon)$, then the lower semicontinuity property holds

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_{\mathbf{R}^d}|u_\varepsilon|^p d\mu_\varepsilon &\geqslant \int_{\mathbf{R}^d}\frac{1}{|\Box(x_1)|}\int_{\Box(x_1)}|u(x_1,y)|^p dy d\mu_* \\ &= \int_I\int_{\Box(x_1)}|u(x_1,y)|^p dy dx_1.\end{aligned}$$

A proof is based on the Young inequality

$$a\cdot b \leqslant \frac{1}{p}|a|^p + \frac{1}{p'}|b|^{p'}, \quad \frac{1}{p}+\frac{1}{p'} = 1.$$

For any $\varphi(x_1,y) \in C_0^\infty(\mathbf{R}; C^\infty(\Box(x_1)))$

$$\frac{1}{p}\int_{\mathbf{R}^d}|u_\varepsilon|^p d\mu_\varepsilon \geqslant \int_{\mathbf{R}^d} u_\varepsilon \varphi\Big(x_1,\frac{x}{\varepsilon}\Big)dy d\mu_\varepsilon - \frac{1}{p'}\int_{\mathbf{R}^d}\Big|\varphi\Big(x_1,\frac{x}{\varepsilon}\Big)\Big|^{p'} d\mu_\varepsilon.$$

Passing to the limit yields

$$\begin{aligned}\frac{1}{p}\liminf_{\varepsilon\to 0}\int_{\mathbf{R}^d}|u_\varepsilon|^p d\mu_\varepsilon \geqslant &\int_{\mathbf{R}^d}\frac{1}{|\Box(x_1)|}\int_{\Box(x_1)} u(x_1,y)\varphi(x_1,y)dy d\mu_* \\ &-\frac{1}{p'}\int_{\mathbf{R}^d}\frac{1}{|\Box(x_1)|}\int_{\Box(x_1)}|\varphi(x_1,y)|^{p'}dy d\mu_*.\end{aligned}$$

By density of smooth functions in $L^p(\mathbf{R}^d,\mu_\varepsilon)$, we can take

$$\varphi(x_1,y) = |u(x_1,y)|^{p-2}u(x_1,y),$$

which completes the proof.

The next proposition provides additional information about the two-scale limit in the case when it is possible to estimate the derivatives. The original statement is given for a fixed domain Ω and the Lebesgue measure in [1] (Proposition 1.14). A more general case of a periodic scaled measure μ_ε is considered in [17] (Theorem 10.3). The proof is essentially the same in all these cases and is therefore omitted.

LEMMA 3.4. *Assume that $u_\varepsilon(x)$ is bounded in $W^{1,p}(\mathbf{R}^d,\mu_\varepsilon)$, $1 \leqslant p < \infty$. Then there exists $u(x_1) \in W^{1,p}(\mathbf{R}^d,\mu_*)$ and $u_1(x_1,y) \in L^p(\mathbf{R};W^{1,p}(\square(x_1)))$ periodic in y_1 such that, as $\varepsilon \to 0$,*

(i) u_ε converges strongly in $L^p(\mathbf{R}^d,\mu_\varepsilon)$ and strongly two-scale in $L^p(\mathbf{R}^d,\mu_\varepsilon)$ to $u(x_1) \in L^p(\mathbf{R}^d,\mu_*)$.

(ii) ∇u_ε, along a subsequence, weakly two-scale converges to $\nabla^{\mu_*}u(x_1)+\nabla_y u_1(x_1,y)$ in $L^p(\mathbf{R}^d,\mu_\varepsilon)$. Here $\nabla^{\mu_*}u(x_1)$ is one of the gradients of u with respect to μ_* (which are defined up to a gradient of zero).

4. Homogenization of a linear elliptic operator with locally periodic coefficients

Let us illustrate how one can apply the adapted notion of the two-scale convergence to the asymptotic analysis of a linear second-order elliptic operator with locally periodic coefficients stated in a thin domain with locally periodic rapidly oscillating boundary. One can think about it as a model problem for steady state thermal conduction in a thin rod made of a composite material, where the conductivity varies “almost” periodically (as a rapidly oscillating function modulated by a function slowly varying along the rod). One can assume that the lateral boundary of the rod is insulated, so a Neumann boundary condition is to be imposed, and keep a given temperature at the ends of the rod, which leads to Dirichlet boundary conditions at the bases. Other applications include modelling of electrostatic problems and reaction-diffusion processes.

Note that the method developed here can be applied to higher order linear elliptic operators with locally periodic coefficients.

We consider the following boundary value problem:

$$\begin{aligned} -\operatorname{div}\big(a^\varepsilon \nabla u_\varepsilon\big) + c^\varepsilon u_\varepsilon = f, &\quad \Omega_\varepsilon, \\ a^\varepsilon \nabla u_\varepsilon \cdot n = 0, &\quad \Sigma_\varepsilon, \\ u_\varepsilon = 0, &\quad \Gamma_\varepsilon^{\pm}. \end{aligned} \tag{2}$$

Our main assumptions are

(H1) The coefficients have the form $a^\varepsilon(x) = a(x_1,\frac{x}{\varepsilon})$, $c^\varepsilon(x) = c(x_1,\frac{x}{\varepsilon})$, where $c(x_1,y)$, $a_{ij}(x_1,y) \in C^{1,\alpha}(\overline{I};C^\alpha(\overline{\square}(x_1)))$ are 1-periodic in y_1, $0<\alpha<1$.

(H2) The matrix a is symmetric and satisfies the uniform ellipticity condition: There exists $\Lambda_0 > 0$ such that for all $x_1 \in I$ and $y \in \square(x_1)$,

$$a_{ij}(x_1,y)\xi_i\xi_j \geqslant \Lambda_0|\xi|^2, \quad \xi \in \mathbf{R}^d.$$

(H3) $f(x_1) \in L^2(I)$.

We study the asymptotic behavior of the solution u_ε of (2) as $\varepsilon \to 0$.

Problem (2) being stated in a bulk domain is classical and can be homogenized by any method of asymptotic analysis. We present the convergence result in the case when the domain is thin and has a locally periodic rapidly varying thickness using singular measures approach. Corrector terms, as well as the estimates for the rate of convergence can be obtained for example by using the asymptotic expansion method.

THEOREM 4.1. *Let u^ε be a solution of problem* (2). *Under the assumptions (H1)–(H3), the following convergence result holds:*

(i) u^ε converges two-scale, as $\varepsilon \to 0$, in $L^2(\mathbf{R}^d, \mu_\varepsilon)$ to the solution u of the one-dimensional problem

$$-(a^{\text{eff}}(x_1)u')' + \bar{c}(x_1)u = |\Box(x_1)|f(x_1), \quad x \in (-L, L), \qquad (3)$$
$$u(\pm L) = 0.$$

The effective diffusion coefficient a^{eff} and the potential $\bar{c}$ are given by the formulae

$$a^{\text{eff}}(x_1) = \int_{\Box(x_1)} a_{1j}(x_1, y)(\delta_{1j} + \partial_{y_j} N_1(x_1, y))\, dy,$$
$$\bar{c}(x_1) = \int_{\Box(x_1)} c(x_1, y)\, dy.$$

The auxiliary function $N_1(x_1, y)$ solves the following cell problem:

$$\begin{cases} -\operatorname{div}_y(a(x_1, y)\nabla_y N_1(x_1, y)) = \partial_{y_i} a_{i1}(x_1, y), & y \in \Box(x_1), \\ a(x_1, y)\nabla_y N_1(x_1, y) \cdot n = -a_{i1}(x_1, y)\, n_i, & y \in \partial\Box(x_1). \end{cases}$$

(ii) $\displaystyle \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{d-1}} \int_{\Omega_\varepsilon} |u^\varepsilon(x) - u(x_1)|^2\, dx = 0.$

(iii) As $\varepsilon \to 0$, the corresponding fluxes converge two-scale in $L^2(\mathbf{R}^d, \mu_\varepsilon)$:

$$a^\varepsilon(x)\nabla u^\varepsilon \xrightharpoonup{2} a^{\text{eff}}(x_1)u'(x_1)e_1 + \nabla_y N(x_1, y)u'(x_1), \quad e_1 = (1, 0, \cdots, 0) \in \mathbf{R}^d.$$

Proof. The weak formulation of (2) in terms of the measure μ_ε reads

$$\int_{\mathbf{R}^d} a^\varepsilon \nabla u_\varepsilon \cdot \nabla\Phi\, d\mu_\varepsilon + \int_{\mathbf{R}^d} c^\varepsilon u_\varepsilon \Phi\, d\mu_\varepsilon = \int_{\mathbf{R}^d} f\Phi\, d\mu_\varepsilon, \qquad (4)$$

where $\Phi \in H^1(\Omega_\varepsilon)$, $\Phi\big|_{\Gamma_\varepsilon^\pm} = 0$. Taking u_ε as a test function we obtain the following a priori estimate:

$$\|u_\varepsilon\|_{L^2(\mathbf{R}^d, \mu_\varepsilon)} + \|\nabla u_\varepsilon\|_{L^2(\mathbf{R}^d, \mu_\varepsilon)} \leqslant C. \qquad (5)$$

Thus, up to a subsequence, u_ε converges two-scale weakly in $L^2(\mathbf{R}^d,\mu_\varepsilon)$ to some $u(x_1)\in L^2(\mathbf{R}^d,\mu_*)$, and due to Lemma 3.4, there exists $u_1(x_1,y)\in L^2(\mathbf{R};H^1(\square(x_1)))$ periodic in y_1 such that ∇u_ε converges two-scale in $L^2(\mathbf{R}^d,\mu_\varepsilon)$ to $\nabla^{\mu_*}u(x_1)+\nabla_y u_1(x_1,y)$.

We proceed in two steps. First we choose an oscillating test function to determine the structure of $u_1(x_1,y)$. Then we use a smooth test function of a slow argument to obtain the limit problem for u.

Let us take

$$\Phi_\varepsilon(x)=\varepsilon\,\varphi(x)\,\psi\Big(\frac{x}{\varepsilon}\Big),\quad \varphi\in C_0^\infty(\mathbf{R}^d),\ \psi\in C^\infty(\mathbf{T}^1\times\mathbf{R}^{d-1}),$$

as a test function in (4).

The gradient of Φ_ε takes the form

$$\nabla\Phi_\varepsilon(x)=\varepsilon\,\psi\Big(\frac{x}{\varepsilon}\Big)\nabla_x\varphi(x)+\varphi(x)\,\nabla_y\psi(y)\big|_{\zeta=x/\varepsilon}.$$

In the first term on the left hand side in (4) we can regard a^ε as a part of the test function. Passing to the limit we get

$$\int_{\mathbf{R}^d}\Big(\frac{1}{|\square(x_1)|}\int_{\square(x_1)}a(x_1,y)\nabla_y\psi(y)\,dy\Big)\cdot\nabla^{\mu_*}u(x_1,0)\,\varphi(x_1,0)\,d\mu_*$$
$$+\int_{\mathbf{R}^d}\Big(\frac{1}{|\square(x_1)|}\int_{\square(x_1)}a(x_1,y)\nabla_y\psi(y)\cdot\nabla_y u_1(x_1,y)\,dy\Big)\,\varphi(x_1,0)\,d\mu_*=0.$$

Looking for u_1 in the form

$$u_1(x_1,y)=N(x_1,y)\cdot\nabla^{\mu_*}u(x_1,0)\tag{6}$$

gives the following relation for the components of $N(y)$:

$$\int_{\mathbf{R}^d}\Big(\frac{1}{|\square(x_1)|}\int_{\square(x_1)}a(x_1,y)\nabla_y N_k(y)\cdot\nabla\psi(y)\,dy\Big)\varphi(x_1,0)\,d\mu_*$$
$$=-\int_{\mathbf{R}^d}\Big(\frac{1}{|\square(x_1)|}\int_{\square(x_1)}a_{kj}(x_1,y)\partial_{y_j}\psi(y)\,dy\Big)\varphi(x_1,0)\,d\mu_*,$$

for any $\varphi\in C_0^\infty(\mathbf{R}^d)$, $\psi\in C^\infty(\mathbf{T}^1\times\mathbf{R}^{d-1})$. The last integral identity is a variational formulation associated to

$$\begin{cases}-\mathrm{div}_y(a(x_1,y)\nabla_y N_k(x_1,y))=\partial_{y_i}a_{ik}(x_1,y), & y\in\square(x_1),\\ a(x_1,y)\nabla_y N_k(y)\cdot n=-a_{ik}(x_1,y)\,n_i,\quad y\in\partial\square(x_1), & k=1,2,\dots.\end{cases}\tag{7}$$

For each $x_1\in I$, there exists a unique solution $N_k(x_1,\cdot)\in C^{1,\alpha}(\bar I;C^{1,\alpha}(\overline{\square(x_1)})/\mathbf{R}$ to (7).

In this way

$$\nabla u_\varepsilon\xrightarrow{2}(\nabla^{\mu_*}u(x_1,0)+\nabla_y N(x_1,y)\cdot\nabla^{\mu_*}u(x_1,0)),\quad \varepsilon\to 0.$$

Now the structure of the function $v^1(z_1,\zeta)$ is known, and we can proceed by deriving the problem for u.

We pass to the limit in the integral identity (4) with $\varphi(x) \in C_0^\infty(\mathbf{R}^d)$:

$$\int_{\mathbf{R}^d} \Big(\frac{1}{|\square(x_1)|} \int_{\square(x_1)} a(x_1,y)(\mathrm{Id} + \nabla_y N(x_1,y))\,dy \Big) \nabla^{\mu_*} u(x_1,0) \cdot \nabla\varphi(x_1,0)\,d\mu_*$$
$$+ \int_{\mathbf{R}^d} \frac{1}{|\square(x_1)|} \int_{\square(x_1)} c(x_1,y) u(x_1,0) \varphi(x_1,0)\,dy d\mu_*$$
$$= \int_{\mathbf{R}^d} f(x_1,0)\,\varphi(x_1,0)\,d\mu_*.$$

Here $\nabla N = \{\partial_{\zeta_i} N_j(\zeta)\}_{ij=1}^d$, and $\mathrm{Id} = \{\delta_{ij}\}_{ij=1}^d$ is the unit matrix. Denote

$$A_{ij}^{\mathrm{eff}} = \int_{\square(x_1)} a_{ik}(x_1,y)(\delta_{kj} + \partial_{y_k} N_j(x_1,y))\,dy.$$

In this way the limit problem in the weak form reads

$$\int_{\mathbf{R}^d} \frac{1}{|\square(x_1)|} A^{\mathrm{eff}} \nabla^{\mu_*} u(x_1,0) \cdot \nabla\varphi(x_1,0)\,d\mu_* + \int_{\mathbf{R}^d} \frac{1}{|\square(x_1)|} \overline{c}(x_1) u(x_1,0)\varphi(x_1,0)\,d\mu_*$$
$$= \int_{\mathbf{R}^d} f(x_1,0)\,\varphi(x_1,0)\,d\mu_*. \tag{8}$$

The μ_*-gradient is not unique, but the flux $A^{\mathrm{eff}}\nabla^{\mu_*} u(x_1,0)$ is uniquely determined by the condition of orthogonality of the vector $A^{\mathrm{eff}}\nabla^{\mu_*} u$ to the subspace of the gradients of zero. This can be seen by taking in (8) any test function with zero trace $\varphi(x_1,0,\dots,0) = 0$ and non-zero μ_*-gradient, for example $\varphi(x) = \sum_{j\neq 1} x_j \psi_j(x_1)$ with arbitrary $\psi_j \in C_0^\infty(\mathbf{R}) \setminus \{0\}$. By the density of smooth functions, the subspace of vectors in the form $(0, \psi_2(x_1), \dots, \psi_d(x_1))$, $\psi_j \in L^2(\mathbf{R})$ is the subspace of the gradients of zero, and the condition of orthogonality to the gradients of zero gives that

$$A^{\mathrm{eff}}\nabla^{\mu_*} u = (A_{1j}^{\mathrm{eff}} \partial_{x_j}^{\mu_*} u(x_1,0), 0, \dots, 0).$$

If we define a solution of (8) as a function $u(x) \in H^1(\mathbf{R}^d, \mu_*)$ satisfying the integral identity, then this solution is unique. A solution $(u, A^{\mathrm{eff}}\nabla^{\mu_*} u)$, as a pair, is also unique due to the orthogonality to the gradients of zero. If one, however, defines a solution of (8) as a pair $(u, \nabla^{\mu_*} u)$, then a solution is not unique. This has to do with the fact that the matrix A^{eff} is not positive definite, and the uniqueness of the flux does not imply the uniqueness of the gradient.

Next step is to prove that $A_{1j}^{\mathrm{eff}} = 0$ for all $j \neq 1$. To this end we rewrite the problem for N_k in the following form:

$$\begin{cases} -\mathrm{div}_y (a(x_1,y)\nabla_y (N_k(x_1,y) + y_k) = 0, & y \in \square(x_1), \\ a(x_1,y)\nabla_y (N_k(x_1,y) + y_k) \cdot n = 0,\ k = 1,2,\dots, & y \in \partial\square(x_1). \end{cases} \tag{9}$$

We multiply (9) by y_m, $m \neq 1$, and integrate over $\square(x_1)$. For $m \neq 1$, the function y_m is periodic in y_1 and can be used as a test function. This gives

$$\int_{\square(x_1)} a(x_1,y)\nabla_y (y_k + N_k(x_1,y)) \cdot \nabla y_m\,dy = 0,$$

and since $\partial_{y_j} y_m = \delta_{jm}$, $A^{\text{eff}}_{km} = 0$ for any $k = 1,\dots,d$ and $m \neq 1$. Thus

$$A^{\text{eff}} \nabla^{\mu_*} u = (A^{\text{eff}}_{11} u'(x_1,0), 0, \dots, 0),$$

and (8) takes the form

$$\int_{\mathbf{R}} A^{\text{eff}}_{11} u'(x_1,0)\varphi'(x_1,0)\,dx_1 + \int_{\mathbf{R}} \overline{c}(x_1) u(x_1,0)\varphi(x_1,0)\,dx_1$$
$$= \int_{\mathbf{R}} f(x_1,0)\,|\square(x_1)|\varphi(x_1,0)\,dx_1.$$

Denoting $a^{\text{eff}} = A^{\text{eff}}_{11}$, $u(x_1) = u(x_1,0)$, we see that the last integral identity is the weak formulation of (3).

Using N_i as a test function in (9) gives

$$A^{\text{eff}}_{ik}(x_1) = \int_{\square(x_1)} a(x_1,\zeta)\nabla_y(y_i + N_i(x_1,y)) \cdot \nabla_y(y_k + N_k(x_1,y))\,dy,$$

which shows that A^{eff} is symmetric and positive semidefinite due to the corresponding properties of $a(x_1,y)$. If $e_1 = (1,0,\dots,0)$,

$$a^{\text{eff}} = A^{\text{eff}}_{11} = A^{\text{eff}} e_1 \cdot e_1 \geqslant \Lambda_0 \int_{\square(x_1)} |\nabla_y(y_1 + N_1(x_1,y))|^2\,dy \geqslant 0.$$

Assuming that $\partial_{y_i}(y_1 + N_1(x_1,y)) = 0$ for all i, leads to a contradiction since N_1 is periodic in y_1. Thus, the effective coefficient a^{eff} is strictly positive.

It is left to prove the strong convergence of u_ε in $L^2(\mathbf{R}^d,\mu_\varepsilon)$. To this end we consider the local average of u_ε

$$\overline{u}_\varepsilon(x_1) = \frac{1}{\varepsilon^{d-1}|Q(x_1,x_1/\varepsilon)|}\int_{\varepsilon Q(x_1,x_1/\varepsilon)} u_\varepsilon(x)dx'.$$

Applying the Poincaré inequality we obtain

$$\int_{\varepsilon Q(x_1,x_1/\varepsilon)} (u_\varepsilon - \overline{u}_\varepsilon)^2 dx' \leqslant C\varepsilon^2 \int_{\varepsilon Q(x_1,x_1/\varepsilon)} |\nabla(u_\varepsilon - \overline{u}_\varepsilon)|^2 dx'.$$

Integrating with respect to x_1, using (5) and the definition of $\overline{u}_\varepsilon$, we have

$$\int_{\Omega_\varepsilon} (u_\varepsilon - \overline{u}_\varepsilon)^2 dx \leqslant C\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - \overline{u}_\varepsilon)|^2 dx \leqslant C\varepsilon. \tag{10}$$

At the same time, since $\overline{u}_\varepsilon$ is bounded in $H^1(I)$, it converges strongly in $L^2(\mathbf{R}^d,\mu_*)$ (equivalently in $L^2(I)$) to some $\overline{u}(x_1)$, which together with (10) gives the strong convergence of u_ε in $L^2(\Omega_\varepsilon,\mu_\varepsilon)$ to $u(x_1) = \overline{u}(x_1)$. $\square$

5. Homogenization of quasilinear operators in a thin domain

We proceed with the nonlinear case and apply the adapted notion of the two-scale convergence to a model of nonlinear degenerate diffusion. Namely, we study the homogenization of a quasilinear elliptic operator stated in a thin domain with locally periodic rapidly oscillating boundary. Let again the domain be that described in Section 2. We consider the following boundary value problem for the p-Laplace operator, $1 < p < \infty$:

$$\begin{aligned} -\mathrm{div}\big(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\big) &= f, \quad \Omega_\varepsilon, \\ |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot n_\varepsilon &= 0, \quad \Sigma_\varepsilon, \\ u_\varepsilon &= 0, \quad \Gamma_\varepsilon^{\pm}, \end{aligned} \tag{11}$$

where $f \in L^{p'}(\Omega_\varepsilon)$, $(1/p)+(1/p')=1$.

Such equations appear, for example, while modeling non-Newtonian fluids, turbulent flows of gas in porous media, and glaciology. Moreover, the technique used in this section allows one to consider more general monotone operators stated in thin domains with oscillating thickness.

We study the asymptotic behaviour of the solution u_ε of (11) as $\varepsilon \to 0$.

Equivalently we can study the minimization problem for the functional

$$I_\varepsilon(v) = \int_{\Omega_\varepsilon} \Big(\frac{1}{p}|\nabla v|^p - fv\Big)\,dx, \tag{12}$$

where $v \in W^{1,p}(\Omega_\varepsilon)$, $v=0$ on $\Gamma_\varepsilon^{\pm}$. In general, an Euler-Largange equation might have other solutions that are not solving the corresponding minimization problem, but since the map $\xi \to |\xi|^p$ is convex, each weak solution of (11) is also a minimizer of I_ε (see [15], Ch. 8.2). In terms of the measure μ_ε:

$$J_\varepsilon(v) = \int_{\mathbf{R}^d} \Big(\frac{1}{p}|\nabla v|^p - fv\Big)\,d\mu_\varepsilon, \quad v \in W^{1,p}(\mathbf{R}^d,\mu_\varepsilon), \quad v\Big|_{\Gamma_\varepsilon^{\pm}} = 0. \tag{13}$$

Due to the convexity of $|\cdot|^p$, there exists a unique minimizer $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$, $u_\varepsilon\big|_{\Gamma_\varepsilon^{\pm}} = 0$ (see [15]).

We study the asymptotic behaviour of u_ε as $\varepsilon \to 0$.

The main result of this section is given in the following theorem.

THEOREM 5.1. *Let u_ε be a solution of (11) and the assumptions (H1)–(H3) are satisfied. Then, as $\varepsilon \to 0$,*

(i) u_ε converges strongly to $u(x_1)$ in $L^p(\mathbf{R}^d,\mu_\varepsilon)$:

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)} \int_{\Omega_\varepsilon} |u_\varepsilon|^p\,dx = \int_I |\Box(x_1)|\,|u(x_1)|^p\,dx_1,$$

where u solves the following one-dimensional equation

$$\begin{gathered} (|\Box(x_1)|\,|u'(x_1)|^{p-2}u'(x_1))' = |\Box(x_1)| f(x_1), \quad x_1 \in I, \\ u(\pm L) = 0. \end{gathered}$$

(ii) ∇u_ε *converges strongly in* $L^p(\mathbf{R}^d,\mu_\varepsilon)^d$ *to* $\{u'(x_1),0,\cdots,0\}$*:*

$$\lim_{\varepsilon\to 0}\varepsilon^{-(d-1)}\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^p\,dx=\int_I|\square(x_1)|\,|u'(x_1)|^p\,dx_1.$$

Proof. The proof is similar to one given in [1], but it should be adapted to the case when the domain is thin and the dimension reduction occur. The limit two-scale problem, in contrast to the one for a general quasilinear operator, can be decoupled. We will derive a cell problem, the expression for the effective diffusion, and a homogenized one-dimensional equation.

Weak formulation of (11) reads

$$\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla\phi\,dx=\int_{\Omega_\varepsilon}f\phi\,dx,\quad \phi\in W^{1,p}(\Omega_\varepsilon),\quad \phi\big|_{\Gamma_\varepsilon^\pm}=0.$$

In terms of measures:

$$\int_{\mathbf{R}^d}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla\phi\,d\mu_\varepsilon=\int_{\mathbf{R}^d}f\phi\,d\mu_\varepsilon,\quad \phi\in W^{1,p}(\mathbf{R}^d,\mu_\varepsilon),\quad \phi\big|_{\Gamma_\varepsilon^\pm}=0. \tag{14}$$

Taking u_ε as a test function in (14) and using Friedrichs' inequality, we obtain a priori estimates

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon,\mu_\varepsilon)^d}+\|u_\varepsilon\|_{L^p(\Omega_\varepsilon,\mu_\varepsilon)}\leqslant C. \tag{15}$$

By Lemma 3.4, up to a subsequence, u_ε converges two-scale in $L^p(\Omega_\varepsilon,\mu_\varepsilon)$ to $u(x_1)\in L^p(\mathbf{R}^d,\mu_*)$ (equivalently $L^p(I)$), and ∇u_ε converges two-scale to $\nabla^{\mu_*}u(x_1)+\nabla u_1(x_1,y)$ in $L^p(\Omega_\varepsilon,\mu_\varepsilon)^d$. Here $u_1(x_1,y)\in L^p(\mathbf{R};W^{1,p}(\square(x_1)))$ is periodic in y_1 and $\nabla^{\mu_*}u(x_1)$ is one of the gradients of u with respect to μ_*, defined up to a gradient of zero, that is

$$\nabla^{\mu_*}u(x_1)=\{u'(x_1),\psi_2(x_1),\cdots,\psi_d(x_1)\},\quad \phi_i\in L^p(\mathbf{R}).$$

We will derive lower and upper bounds for the functional (12), which will give us the homogenized functional.

By convexity,

$$|b|^p\geqslant|a|^p+p|a|^{p-2}a\cdot(b-a),\quad p\geqslant 1.$$

Thus

$$J_\varepsilon(u_\varepsilon)\geqslant\int_{\Omega_\varepsilon}\Big(\frac{1}{p}|\Phi_\varepsilon|^p+|\Phi_\varepsilon|^{p-2}\Phi_\varepsilon\cdot(\nabla u_\varepsilon-\Phi_\varepsilon)-fu_\varepsilon\Big)d\mu_\varepsilon,$$

for $\Phi_\varepsilon=\Phi(x_1,x/\varepsilon)$, with $\Phi(x_1,y)\in C_0^\infty(\mathbf{R};C^\infty(\square(x_1)))^d$.

Passing to the limit, as $\varepsilon\to 0$, we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0}J_\varepsilon(u_\varepsilon)\geqslant&\int_{\mathbf{R}^d}\frac{1}{|\square(x_1)|}\int_{\square(x_1)}\Big(\frac{1}{p}|\Phi(x,y)|^p-fu\Big)\,dyd\mu_*\\&+\int_{\mathbf{R}^d}\frac{1}{|\square(x_1)|}\int_{\square(x_1)}\frac{1}{p}|\Phi|^{p-2}\Phi\cdot(\nabla^{\mu_*}u+\nabla_y u_1-\Phi)\,dyd\mu_*.\end{aligned}$$

Taking a sequence of smooth functions $\Phi(x_1,y)$ converging to $\nabla^{\mu_*}u+\nabla_y u_1$ in $L^p(\mathbf{R}^d,\mu_\varepsilon)$ yields

$$\liminf_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) \geqslant \int_{\mathbf{R}^d} \frac{1}{|\square(x_1)|}\int_{\square(x_1)} \Big(\frac{1}{p}|\nabla^{\mu_*}u+\nabla_y u_1|^p - fu\Big)\,dy\,d\mu_*.$$

The last functional will be denoted by

$$J(v,v_1) = \int_{\mathbf{R}^d} \frac{1}{|\square(x_1)|}\int_{\square(x_1)} \Big(\frac{1}{p}|\nabla^{\mu_*}v+\nabla_y v_1|^p - fu\Big)\,dy\,d\mu_*. \tag{16}$$

Now we will derive an upper bound for (13). Since u_ε is a minimizer of (13),

$$J_\varepsilon(u_\varepsilon) \leqslant J_\varepsilon\Big(\phi(x_1)+\varepsilon\phi_1\Big(x_1,\frac{x}{\varepsilon}\Big)\Big)$$

for $\phi(x_1)\in C_0^\infty(\mathbf{R}^d)$, and $\phi_1\in C_0^\infty(\Omega_\varepsilon;C^\infty(\overline{\square(x_1)}))$ periodic in y_1.

Passing to the limit we get

$$\limsup_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) \leqslant \int_{\mathbf{R}^d} \frac{1}{|\square(x_1)|}\int_{\square(x_1)} \Big(\frac{1}{p}|\nabla^{\mu_*}\phi+\nabla_y \phi_1|^p - f\phi\Big)\,dy\,d\mu_*,$$

and thus

$$\limsup_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) \leqslant \inf_{(v,v_1)}\int_{\mathbf{R}^d} \frac{1}{|\square(x_1)|}\int_{\square(x_1)} \Big(\frac{1}{p}|\nabla^{\mu_*}v+\nabla_y v_1|^p - fv\Big)\,dy\,d\mu_*,$$

where infimum is taken over $v\in W^{1,p}(\mathbf{R}^d,\mu_*)$, $v_1\in L^p(\mathbf{R}^d,\mu_*;W^{1,p}(\square(x_1))\setminus\mathbf{R})$ (v_1 is periodic in y_1).

Finally

$$\begin{aligned}\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) &= \inf_{(v,v_1)}\int_{\mathbf{R}^d} \frac{1}{|\square(x_1)|}\int_{\square(x_1)} \Big(\frac{1}{p}|\nabla^{\mu_*}v+\nabla_y v_1|^p - fv\Big)\,dy\,d\mu_* \\ &= \inf_{(v,v_1)} J(v,v_1),\end{aligned} \tag{17}$$

where infimum is taken over $v\in W^{1,p}(\mathbf{R}^d,\mu_*)$, $v_1\in L^p(\mathbf{R}^d,\mu_*;W^{1,p}(\square(x_1))\setminus\mathbf{R})$, v_1 being periodic in y_1.

Solution (v,v_1) to the last minimization problem is unique due to the convexity properties.

The minimizer (v,v_1) of (17) satisfies the following integral identity:

$$\int_{\mathbf{R}^d} \frac{1}{|\square(x_1)|}\int_{\square(x_1)} |\nabla^{\mu_*}v+\nabla_y v_1|^{p-2}(\nabla^{\mu_*}v+\nabla_y v_1)\cdot(\nabla^{\mu_*}\Phi+\nabla_y\Phi_1)\,dy\,d\mu_* = \int_{\mathbf{R}^d} f\Phi\,d\mu_*, \tag{18}$$

for $\Phi\in W^{1,p}(\mathbf{R}^d,\mu_*)$ and $\Phi_1\in L^p(\mathbf{R}^d,\mu_*;W^{1,p}(\square(x_1))\setminus\mathbf{R})$ periodic in y_1.

Taking $\Phi=0$ and $\Phi_1=\varphi(x)\psi(y)$ we obtain an auxiliary cell problem

$$\begin{aligned}-\mathrm{div}_y\big(|\nabla^{\mu_*}v+\nabla_y v_1|^{p-2}(\nabla^{\mu_*}v+\nabla_y v_1)\big) &= 0, \quad y\in\square(x_1),\\ |\nabla^{\mu_*}v+\nabla_y v_1|^{p-2}(\nabla^{\mu_*}v+\nabla_y v_1)\cdot n &= 0, \quad y\in\partial\square(x_1).\end{aligned}$$

Denote $\xi=\nabla^{\mu_*}v$, and let us analyse the following cell problem:

$$\begin{aligned} -\mathrm{div}_y\big(|\xi+\nabla_y v_1|^{p-2}(\xi+\nabla_y v_1)\big)&=0, \quad y\in\square(x_1),\\ |\xi+\nabla_y v_1|^{p-2}(\xi+\nabla_y v_1)\cdot n&=0, \quad y\in\partial\square(x_1). \end{aligned} \tag{19}$$

For each ξ there exists a unique, defined up to a function depending on x_1, solution $v_1^{\xi}(x_1,y)\in C^{1,\alpha}(\bar{I};W^{1,p}(\square(x_1)))$. Take $\xi=e_1=\{1,0,\cdots,0\}$. Then a constant function $v_1(x_1,y)=\mathrm{Const}$ solves (19).

For any $\xi\in\mathbf{R}^d$, we set

$$A^{\mathrm{eff}}(x_1,\xi)=\int_{\square(x_1)}|\xi+\nabla_y v_1^{\xi}|^{p-2}(\xi+\nabla_y v_1^{\xi})\,dy.$$

Multiplying (19) by v_1^{ξ} and integrating by parts yields

$$\int_{\square(x_1)}|\xi+\nabla_y v_1^{\xi}|^{p-2}(\xi+\nabla_y v_1^{\xi})\cdot\nabla_y v_1^{\xi}\,dy=0,$$

and thus

$$A^{\mathrm{eff}}(x_1,\xi)\cdot\xi=\int_{\square(x_1)}|\xi+\nabla_y v_1^{\xi}|^{p}\,dy.$$

For $\xi=e_1$, $A^{\mathrm{eff}}(x_1,e_1)=\{|\square(x_1)|,0,\cdots,0\}$.

Taking in (18) a test function with zero trace $\Phi(x_1,0)=0$ and non-zero gradient $\nabla^{\mu_*}\phi$ (for example $\phi(x)=\sum_{j\neq1}x_j\psi_j(x_1)$ with arbitrary $\psi_j\in L^p(\mathbf{R})\setminus\{0\}$), we see that the choice of the gradient with respect to μ_* is uniquely determined by the condition that the flux is orthogonal in L^p to the gradients of zero and that all the components of $A(x_1,\nabla^{\mu_*}v)$ except for the first one are zeroes:

$$A_m^{\mathrm{eff}}(x_1,\nabla^{\mu_*}v)=\int_{\square(x_1)}|\nabla^{\mu_*}v+\nabla_y v_1|^{p-2}(\partial_m^{\mu_*}v+\nabla_{y_m}v_1)\,dy=0, \quad m\neq1.$$

The last condition is obviously satisfied for $\nabla^{\mu_*}v=\{v'(x_1),0,\cdots,0\}$ and the corresponding $v_1=\mathrm{Const}$, and we conclude immediately that

$$A^{\mathrm{eff}}(x_1,\nabla^{\mu_*}v)=|\square(x_1)|\,|v'(x_1)|^{p-2}v'(x_1).$$

Thus (18) is the weak formulation of

$$\begin{gathered} (|\square(x_1)|\,|v'(x_1)|^{p-2}v'(x_1))'=|\square(x_1)|f(x_1), \quad x_1\in I,\\ v(\pm L)=0. \end{gathered}$$

The strong convergence of u_ε and ∇u_ε follows from the convergence of energy functionals. $\square$

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