Differential
Equations
& Applications
Volume 9, Number 4 (2017), 413–432 doi:10.7153/dea-2017-09-29

PRACTICAL STABILITY OF DIFFERENTIAL EQUATIONS WITH NON–INSTANTANEOUS IMPULSES

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(*Communicated by Michal Fečkan*)

Abstract. The concept of practical stability is generalized to nonlinear differential equations with non-instantaneous impulses. These type of impulses start their action abruptly at some points and then continue on given finite intervals. The practical stability and strict practical stability is studied using Lyapunov like functions and comparison results for scalar differential equations with non-instantaneous impulses. Several sufficient conditions for various types of practical stability, practical quasi stability and strict practical stability are established. Some examples are included to illustrate our theoretical results.

1. Introduction

Many evolutionary processes are characterized by the fact that at certain moments of time they experience changes to their state. For example, when a mass on a spring is given a blow by a hammer, it experiences a sharp change of velocity which could be modeled by an instantaneous impulse. When the introduction of the intravenous drug in the bloodstream and the consequent absorption for the body are gradual and continuous processes (for example, insulin), then we can interpret the situation as an impulsive action which starts abruptly and stays active on a finite time interval. In this case it is called a non-instantaneous impulse (see [3], [5], [6], [18], [19], [20]).

The theory of stability for differential equations with instantaneous impulses was discussed in the books [9], [12]. One type of stability is practical stability. The stability and even asymptotic stability themselves are neither necessary nor sufficient to ensure practical stability. The desired state of a system may be mathematically unstable; however, the system may oscillate sufficiently close to the desired state, and its performance is deemed acceptable. Practical stability is neither weaker nor stronger than the usual stability; an equilibrium can be stable in the usual sense, but not practically stable, and vice versa. Practical stability was studied for various types of differential equations (see, for example, [4], [7], [8], [10], [11], [14], [15], [16]). Also, the concept of strict stability (see, for example, [1], [2], [15]) gives information on the boundedness of solutions.

Mathematics subject classification (2010): 34A34, 34A08, 34D20.
Keywords and phrases: Non-instantaneous impulses, practical stability, Lyapunov like functions.
This research is supported by Fund MU17-FMI-007, University of Plovdiv Paisii Hilendarski.

Paper DEA-09-29

In this paper the impulses start abruptly at some points and their action continue on given finite intervals, the so called non-instantaneous impulses. The concept of practical stability is generalized to nonlinear differential equations with non-instantaneous impulses ([4]). Practical stability and strict practical stability is studied using Lyapunov like functions and comparison results for scalar differential equations with non-instantaneous impulses. Several sufficient conditions for various types of practical stability , uniform practical stability, practical quasi stability, strict practical stability, uniform strict practical stability as well as the corresponding uniform types of practical stability are established. Some examples are included to illustrate our theory.

2. Non-instantaneous impulses in differential equations

In this paper we will assume two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ are given such that $0 < s_0 < t_i \leqslant s_i < t_{i+1}$, $i = 1,2,\dots$, and $\lim_{k\to\infty} t_k = \infty$. The initial time $t_0 \in [0,s_0)\bigcup\cup_{i=1}^{\infty}[t_k,s_k)$ is a given point. Without loss of generality in this paper we assume $t_0 \in [0,s_0)$.

Consider the initial value problem (IVP) for the system of *non-instantaneous impulsive differential equations* (NIDE)

$$\begin{aligned} &x' = f(t,x) \quad \text{for } t \in \cup_{k=0}^{\infty}(t_k,s_k], \\ &x(t) = \phi_k(t,x(s_k-0)) \quad \text{for } t \in (s_k,t_{k+1}], \ i = 0,1,2,\dots, \\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

where $x,x_0 \in \mathbb{R}^n$, $f : \cup_{k=0}^{\infty}(t_k,s_k] \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_k : [s_k,t_{k+1}] \times \mathbb{R}^n \to \mathbb{R}^n$, $(k = 0,1,2,3,\dots)$.

REMARK 1. The intervals $(s_k,t_{k+1}]$, $k = 0,1,2,\dots$ are called intervals of non-instantaneous impulses and the functions $\phi_k(t,x)$, $k = 1,2,\dots$, are non-instantaneous impulsive functions.

Also consider the corresponding IVP for ordinary differential equations (ODE)

$$x' = f(t,x) \quad \text{for } t \in [\tau,s_k] \quad \text{with } x(\tau) = \tilde{x}_0. \tag{2}$$

We will use the following conditions:

(H1) The function $f \in C([0,s_0]\bigcup\cup_{k=1}^{\infty}(t_k,s_k] \times \mathbb{R}^n,\mathbb{R}^n)$ is such that $f(t,0) = 0$, $t \in [0,s_0]\bigcup\cup_{k=0}^{\infty}(t_k,s_k]$ and for any initial point $(\tau,\tilde{x}_0) \in [t_k,s_k) \times \mathbb{R}^n$, $k = 0,1,2,\dots$ the IVP for the system of ODE (2) has a solution $\tilde{x}(t;\tau,\tilde{x}_0) \in C^1([\tau,s_k],\mathbb{R}^n)$ (in the case $k = 0$ the interval $[t_k,s_k)$ is replaced by $[0,s_0]$).

(H2) The function $\phi_k \in C([s_k,t_{k+1}] \times \mathbb{R}^n,\mathbb{R}^n)$ and $\phi_k(t,0) = 0$, $t \in [s_k,t_{k+1}]$.

The solution $x(t;t_0,x_0)$ of IVP for NIDE (1) is given by

$$x(t;t_0,x_0) = \begin{cases} X_k(t), & \text{for } t \in (t_k,s_k], \ k = 0,1,2,\dots, \\ \phi_k(t,X_k(s_k-0)), & \text{for } t \in (s_k,t_{k+1}] \ \ k = 0,1,2,\dots \end{cases} \tag{3}$$

where

– on the interval $[t_0,s_0]$ the solution coincides with $X_0(t)$ which is the solution of IVP for ODE (2) for $\tau=t_0$, $k=0$ and $\tilde{x}_0=x_0$;

– on the interval $(s_k,t_{k+1}]$, $k=0,1,\dots$, the solution $x(t;t_0,x_0)=\phi_k(t,X_k(s_k-0))$;

– on the interval $(t_k,s_k]$, $k=1,2,\dots$, the solution $x(t;t_0,x_0)$ coincides with $X_k(t)$ which is the solution of IVP for ODE (2) for $\tau=t_k$ and $\tilde{x}_0=\phi_{k-1}(t_k,X_{k-1}(s_{k-1}-0))$.

Also, the solution $x(t;t_0,x_0)$ of (1) satisfies the following system of integral and algebraic equations

$$x(t;t_0,x_0)=\begin{cases} x_0+\int_{t_0}^t f(s,x(s;t_0,x_0))ds \\ \qquad\qquad \text{for } t\in[t_0,s_0], \\ \phi_k(t,x(s_k-0;t_0,x_0)) \\ \qquad\qquad \text{for } t\in(s_k,t_{k+1}],\ k=0,1,2,\dots, \\ \phi_{k-1}(t_k,x(s_{k-1}-0;t_0,x_0))+\int_{t_k}^t f(s,x(s;t_0,x_0))ds \\ \qquad\qquad \text{for } t\in[t_k,s_k],\ k=1,2,\dots,p. \end{cases} \tag{4}$$

REMARK 2. If $t_{k+1}=s_k$, $k=0,1,2,\dots$ then the IVP for NIDE (1) reduces to an IVP for impulsive differential equations (for example see the monographs [9], [12] and the cited references therein). In this case at any point of instantaneous impulse t_k the amount of jump of the solution $x(t)$ is given by $I_k=\phi_{k-1}(t_k,x(t_k-0))-x(t_k-0)$, $k=1,2,\dots$.

Let $J\subset\mathbb{R}_+$ be a given interval. Introduce the following classes of functions

$$\begin{aligned} PC(J,\mathbb{R}^n)=\{&u:J\to\mathbb{R}^n:\ u\in C(J/\{s_k\}_{k=1}^{\infty},\mathbb{R}^n): \\ &u(s_k)=u(s_k-0)=\lim_{t\uparrow s_k}u(t)<\infty,\ \ u(s_k+0)=\lim_{t\downarrow s_k}u(t)<\infty,\ \ k:\ s_k\in J\}, \\ PC^1(J,\mathbb{R}^n)=\{&u:J\to\mathbb{R}^n:\ u\in C(J/\{s_k\}_{k=1}^{\infty},\mathbb{R}^n)\bigcap C^1(J\cap\left(\cup_{k=0}^{\infty}(t_k,s_k]\right),\mathbb{R}^n): \\ &u(s_k)=u(s_k-0)=\lim_{t\uparrow s_k}u(t)<\infty,\ \ u(s_k+0)=\lim_{t\downarrow s_k}u(t)<\infty \\ &\lim_{t\uparrow s_k}u'(t)<\infty,\ \ \lim_{t\downarrow s_k}u'(t)<\infty,\ \ k:\ s_k\in J\}. \end{aligned}$$

REMARK 3. According to the above description any solution of (1) is from the class $PC^1([t_0,b))$, $b\leqslant\infty$, i.e. any solution might have a discontinuity at any point s_k, $k=1,2,\dots$.

EXAMPLE 1. Consider the IVP for the scalar NIDE

$$\begin{aligned} &x'=A_kx \ \text{ for } \ t\in(t_k,s_k],\ \ k=0,1,2,\dots, \\ &x(t)=\phi_k(t,x(s_k-0)) \ \text{ for } \ t\in(s_k,t_{k+1}],\ \ k=0,1,2,\dots, \\ &x(t_0)=x_0, \end{aligned} \tag{5}$$

where $x,x_0\in\mathbb{R}$, A_k are constants, $0\leqslant t_0<s_0$.

The solution of (5) is given by

$$x(t;t_0,x_0)=\begin{cases}\phi_k(t,x(s_k-0)) & \text{for } t\in(s_k,t_{k+1}],\ k=0,1,2,\dots,\\ \phi_{k-1}\Big(t_k,x(s_{k-1}-0)\Big)e^{A_k(t-t_k)} & \text{for } t\in[t_k,s_k],\ k=0,1,2,3,\dots\end{cases}\tag{6}$$

where $\phi_0(t,x)\equiv x_0$.

Let $\phi_k(t,y)=a_k(t)y$, $a_k\in C([s_k,t_{k+1}],\mathbb{R})$, $k=0,1,2,3,\dots$. The solution of NIDE (5) is given by

$$x(t;t_0,x_0)=\begin{cases}x_0e^{A_k(t-t_k)+\sum_{i=0}^{k-1}A_i(s_i-t_i)}\prod_{i=1}^{k}a_i(t_i) & \text{for } t\in(t_k,s_k],\ k=0,1,2,\dots,\\ x_0a_k(t)e^{\sum_{i=0}^{k}A_i(s_i-t_i)}\prod_{i=1}^{k-1}a_i(t_i) & \text{for } t\in(s_k,t_{k+1}],\ k=0,1,2,\dots\end{cases}$$

3. Main definitions

First we give a definition for various types of practical stability of the zero solution of NIDE (1). In the definition below we denote by $x(t;t_0,x_0)\in PC^1([t_0,\infty),\mathbb{R}^n)$ any solution of the IVP for NIDE (1). Note the practical stability for non-instantaneous impulsive differential equation is defined and studied following the classical concept of the idea of practical stability ([14], [16]).

DEFINITION 1. Let positive constants $\lambda,A:\ \lambda<A$ be given. The zero solution of the system of NIDE (1) is said to be

(S1) *practically stable with respect to* (λ,A) if there exists $t_0\in[0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$ such that for any $x_0\in\mathbb{R}^n$ inequality $||x_0||<\lambda$ implies $||x(t;t_0,x_0)||<A$ for $t\geqslant t_0$;

(S2) *uniformly practically stable with respect to* (λ,A) if (S1) holds for all $t_0\in[0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$;

(S3) *practically quasi stable with respect to* (λ,A,T) if there exists an initial time $t_0\in[0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$ such that for any $x_0\in\mathbb{R}^n$ inequality $||x_0||<\lambda$ implies $||x(t;t_0,x_0)||<A$ for $t\geqslant t_0+T$, where the positive constant T is given;

(S4) *uniformly practically quasi stable with respect to* (λ,A,T) if (S3) holds for all $t_0\in[0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$.

Now following [15] we will generalize strict practical stability definitions to NIDE.

DEFINITION 2. Let positive constants $\lambda,A:\ \lambda<A$ be given. The zero solution of the system of NIDE (1) is said to be

(S5) *strictly practically stable* with respect to (λ,A) if there exists an initial time $t_0\in[0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$ such that for any $x_0\in\mathbb{R}^n$ the inequality $||x_0||<\lambda$ implies $||x(t;t_0,x_0)||<A$ for $t\geqslant t_0$ and for every $\lambda_1:\ 0<\lambda_1\leqslant\lambda$ there exists $A_1\leqslant\lambda_1$ such that the inequality $||x_0||>\lambda_1$ implies $||x(t;t_0,x_0)||>A_1$ for $t\geqslant t_0$;

(S6) *uniformly strictly practically stable* if (S5) holds for all initia times $t_0\in[0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$.

We will illustrate the concept of strict practical stability with an example.

EXAMPLE 2. (Strict practical stability) Consider the ODE $x' = -\frac{1}{(t+1)^2}x,\ x(t_0) = x_0$ with a solution $x(t) = x_0 e^{-\frac{1}{t_0+1}} e^{\frac{1}{t+1}}$. Since $1 < e^{\frac{1}{t+1}} \leqslant e$ and $e^{-1} < e^{-\frac{1}{t+1}} \leqslant 1$ for $t \in \mathbb{R}_+$ it follows that $|x_0|e^{-1} \leqslant |x_0|e^{-\frac{1}{t_0+1}} \leqslant x(t) \leqslant |x_0|e^{-\frac{1}{t_0+1}} e \leqslant |x_0|e$

Then the zero solution is uniformly strictly practically stable w.r.t. $(1,e)$ because if $\lambda > 1$ and $|x_0| > \lambda$ then $|x(t)| \geqslant |x_0|e^{-1} \leqslant \lambda e^{-1} = A$.

The presence of non-instantaneous impulses can cause a change on the behavior of the solution and the stability properties and it will be illustrated in the next example:

EXAMPLE 3. Let $t_k = 2k,\ s_k = 2k+1,\ k = 0,1,2,\dots$.

Case 1. Consider the IVP for the scalar linear NIDE

$$\begin{aligned} &x' = x \text{ for } t \in \cup_{k=0}^{\infty}(2k, 2k+1], \\ &x(t) = \phi_k(t, x(2k+1-0)) \text{ for } t \in (2k+1, 2k+2],\ k = 1,2,\dots, \\ &x(0) = x_0, \end{aligned} \tag{7}$$

The zero solution of the corresponding ODE $x' = x$ is neither periodic nor stable.

Case 1.1. Let $\phi_k(t,x) = e^{-1}x$. The solution of (7) is given by

$$x(t;0,x_0) = \begin{cases} x_0 e^{t-2k} & \text{for } t \in (2k, 2k+1],\ k = 0,1,2,\dots, \\ x_0 & \text{for } t \in (2k+1, 2k+2],\ k = 0,1,2,\dots. \end{cases} \tag{8}$$

It is a periodic function (see Figure 1). The estimate $|x_0| \leqslant |x(t;0,x_0)| \leqslant |x_0|e$ is valid and the zero solution of NIDE (8) is uniformly strictly practically stable w.r.t. to (λ, A) where $\lambda > 0$ is a given number, $A = e\lambda > \lambda$. Also, the zero solution of NIDE (8) is uniformly stable.

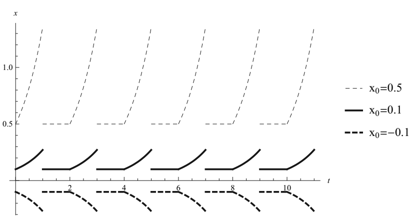

Figure 1. Example 3. Case 1.1. Graphs of solutions for $\phi_k(t,x) = e^{-1}x$ and initial values $x_0 = 1,\ x_0 = 0.5\ x_0 = -0.3$.

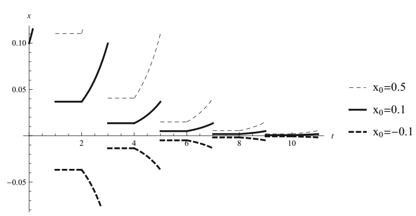

Figure 2. Example 3. Case 1.2. Graphs of solutions for $\phi_k(t,x) = e^{-2}x$ and $x_0 = 1,\ x_0 = 0.5\ x_0 = -0.1$.

Case 1.2. Let $\phi_k(t,x) = e^{-2}x$. The solution of (7) is given by

$$x(t;0,x_0) = \begin{cases} x_0 e^{t-3k} & \text{for } t \in (2k, 2k+1],\ k = 0,1,2,\dots, \\ x_0 e^{-k} & \text{for } t \in (2k-1, 2k],\ k = 1,2,\dots \end{cases}$$

The estimate $|x_0|e^{-k} \leqslant |x(t;0,x_0)| \leqslant |x_0|e^{1-k},\ t \in [2k-1, 2k+1],\ k = 1,2,\dots$ is valid. The zero solution of NIDE (8) is uniformly practically stable w.r.t. (λ, A), $A = \lambda e$ (see Figure 2).

Case 2. Consider the IVP for the scalar linear NIDE

$$\begin{aligned} &x' = -x \text{ for } t \in \cup_{k=0}^{\infty}(2k, 2k+1], \\ &x(t) = \phi_k(t, x(2k+1-0)) \ \text{ for } t \in (2k+1, 2k+2],\ k = 1,2,\dots, \\ &x(0) = x_0, \end{aligned} \tag{9}$$

The zero solution of the corresponding ODE $x' = -x$ is asymptotically stable.

Case 2.1. Let $\phi_k(t,x) = xt$. The solution of the IVP for NIDE (9) is given by

$$x(t;0,x_0) = \begin{cases} x_0 e^{k-t} \prod_{i=1}^{k}(2i) & \text{for } t \in (2k, 2k+1],\ k = 0,1,2,\dots, \\ x_0 e^{-(k+1)} t \prod_{i=1}^{k}(2i) & \text{for } t \in (2k+1, 2k+2],\ k = 0,1,2,\dots \end{cases}$$

The zero solution of of the IVP for NIDE (9) (with $x_0 = 0$) is neither stable nor practically stable (see Figure 3). The estimate $x(t;0,x_0) \leqslant |x_0| \prod_{i=1}^{k+1}(2ie^{-1}),\ t \in (2k, 2k+2],\ k = 0,1,2,\dots$ is valid.

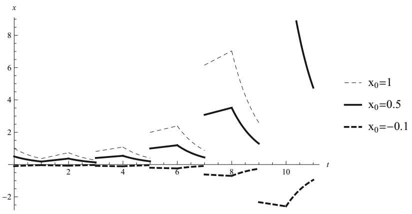

Figure 3. Example 3. Case 2.1. Graphs of solutions for $\phi_k(t,x) = xt$ and initial values $x_0 = 1,\ x_0 = 0.5\ x_0 = -0.1$.

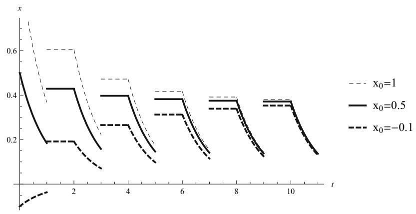

Figure 4. Example 3. Case 2.2. Graphs of solutions for $\phi_k(t,x) = \sqrt{|x|}$ and initial values $x_0 = 1,\ x_0 = 0.5\ x_0 = -0.1$.

Case 2.2. Let $\phi_k(t,x) = \sqrt{|x|}$. The zero solution of the IVP for NIDE (9) (with $x_0 = 0$) is stable. (see Figure 4).

Case 2.3. Let $\phi_k(t,x) = e^{-2}x$. The solution of the IVP for NIDE (9) is given by

$$x(t;0,x_0) = \begin{cases} x_0 e^{-k-t} & \text{for } t \in (2k, 2k+1],\ k = 0,1,2,\dots, \\ x_0 e^{-3k} & \text{for } t \in (2k-1, 2k],\ k = 1,2,\dots \end{cases}$$

For any $k = 1,2,\dots$ the estimate $|x(t;0,x_0)| \leqslant x_0 e^{-3k},\ \ t \in (2k-1, 2k+1]$ is valid.

The zero solution of scalar NIDE (9) is practically quasi stable w.r.t. (λ, A, T) : $A = \frac{\lambda}{e^3} < \lambda,\ T = 1$.

Case 3. Consider the IVP for the scalar linear NIDE

$$\begin{aligned} &x' = 0 \text{ for } t \in \cup_{k=0}^{\infty}(2k, 2k+1], \\ &x(t) = e^{-2}x(2k+1-0) \ \ \text{ for } t \in (2k+1, 2k+2],\ k = 1,2,\dots, \\ &x(0) = x_0, \end{aligned} \tag{10}$$

The solution of (10) is given by

$$x(t;0,x_0) = \begin{cases} x_0 & \text{for } t \in [0,1], \\ x_0 e^{-2k} & \text{for } t \in (2k-1, 2k+1],\ k = 1,2,\dots \end{cases}$$

The zero solution of scalar NIDE (10) is practically quasi stable w.r.t. $(\lambda,A,T):\ A=\frac{\lambda}{e^2}<\lambda,\ T=1$.

The zero solution of the corresponding ODE $x'=0$ is not asymptotically stable but it is stable.

In this paper we will use the followings sets:

$$\mathscr{K} = \{a\in C[\mathbb{R}_+,\mathbb{R}_+]: a \text{ is strictly increasing and } a(0)=0\},$$
$$S(\lambda) = \{x\in\mathbb{R}^n: \|x\|\leqslant\lambda\},\quad \lambda=const>0.$$

In our investigations we will use the IVP for scalar non-instantaneous impulsive differential equations of the type

$$\begin{aligned} &u'=g(t,u), \quad \text{for } t\in\cup_{k=0}^{\infty}(t_k,s_k],\\ &u(t)=\Xi_k(t,u(s_k-0)) \text{ for } t\in(s_k,t_{k+1}],\quad k\in\mathbb{Z}_+\\ &u(t_0)=u_0\end{aligned} \tag{11}$$

and the IVP for its corresponding scalar ODE

$$u'=g(t,u),\quad t\in[\tau,s_k],\quad u(\tau)=\tilde{u}_0 \tag{12}$$

where $u,\tilde{u}_0\in\mathbb{R}$.

We will use minimal/maximal solutions of the IVP for ODE (12). For details and some existence conditions see Definition 1.3.1 and Theorem 1.3.1 in [13].

We will use the following conditions:

(H3) The function $g\in C(J\times\mathbb{R},\mathbb{R})$, $J\subset[0,s_0]\bigcup\cup_{k=1}^{\infty}(t_k,s_k]$ is such that $g(t,0)=0$, $t\in J$ and for any initial point $(\tau,\tilde{u}_0):\ \tau\in[t_k,s_k)\cap J$, $k=0,1,2,\dots$, and $\tilde{u}_0\in\mathbb{R}$ the IVP for ODE (12) has a maximal solution $\tilde{u}(t;\tau,\tilde{u}_0)\in C^1([\tau,s_k]\cap J,\mathbb{R}^n)$ (in the case $k=0$ the interval $[t_k,s_k)$ is replaced by $[0,s_0]$).

(H4) The function $g\in C(J\times\mathbb{R},\mathbb{R})$, $J\subset[0,s_0]\bigcup\cup_{k=1}^{\infty}(t_k,s_k]$ is such that $g(t,0)=0$, $t\in J$ and for any initial point $(\tau,\tilde{u}_0):\ \tau\in[t_k,s_k)\cap J$, $k=0,1,2,\dots$, and $\tilde{u}_0\in\mathbb{R}$ the IVP for ODE (12) has a minimal solution $\tilde{u}(t;\tau,\tilde{u}_0)\in C^1([\tau,s_k]\cap J,\mathbb{R}^n)$ (in the case $k=0$ the interval $[t_k,s_k)$ is replaced by $[0,s_0]$).

(H5) The function $\Xi_k\in C([s_k,t_{k+1}]\times\mathbb{R},\mathbb{R})$, $\Xi_k(t,0)=0$ for $t\in[s_k,t_{k+1}]$ and $\Xi_k(t,u)\leqslant\Xi_k(t,v)$ for $u\leqslant v$, $t\in[s_k,t_{k+1}]$.

DEFINITION 3. Let p be a natural number and $T\in(t_p,s_p]$ be a given number. The function $u^*(t)$ will be called a maximal solution (minimal solution) of the IVP for NIDE (11) on the interval $[t_0,T]$ if

- it is a solution of the IVP for NIDE(11) on $[t_0,T]$;
- for any $k=0,1,2,\dots,p$ and any solution $u(t)\in C^1([t_k,s_k],\mathbb{R})$ of IVP for ODE (12) with $\tau=t_k$, $\tilde{u}_0=u^*(t_k)$ the inequalities

$$u^*(t)\geqslant(\leqslant)u(t)\ \text{ for } t\in[t_k,s_k]\cap[t_0,T]$$

and for any $k = 0, 1, 2, \dots, p-1$

$$\Xi_k(t, u^*(s_k - 0)) \geqslant (\leqslant) \Xi_k(t, u(s_k)) \text{ for } t \in (s_k, t_{k+1}]$$

hold.

LEMMA 1. *Let:*
1. Condition (H3) be satisfied for $J = \cup_{k=0}^{p}(t_k, s_k]$ *where* $p \leqslant \infty$ *is a positive integer.*
2. Condition (H5) be satisfied for all $k = 0, 1, 2, \dots, p-1$.
Then there exist a maximal solution of IVP for NIDE (11) on the interval $[t_0, s_p]$.

Proof. We will use induction to prove the claim.

Let $t \in [t_0, s_0]$. According to condition (H3) there exists a maximal solution $u_0^*(t)$ of IVP for ODE (12) with $\tau = t_0$ and $\tilde{u}_0 = u_0$.

Let $t \in (s_0, t_1]$. According to condition (H5) for the function $\Xi_0(t, u)$ the inequality $\Xi_0(t, u_0^*(s_0)) \geqslant \Xi_0(t, u(s_0))$ for $t \in (s_0, t_1]$ holds where $u(t)$ is any solution of IVP for ODE (12) with $\tau = t_0$, $\tilde{u}_0 = u_0$ which exists on $[t_0, s_0]$.

Let $t \in (t_1, s_1]$. According to condition (H3) there exists a maximal solution $u_1^*(t)$ of IVP for ODE (12) with $\tau = t_1$ and $\tilde{u}_0 = \Xi_0(t_1, u_0^*(s_0))$.

Let $t \in (s_1, t_2]$. According to condition (H5) for Ξ_1 the inequality $\Xi_1(t, u_1^*(s_1)) \geqslant \Xi_1(t, u(s_1))$ for $t \in (s_1, t_2]$ holds where $u(t)$ is any solution of IVP for ODE (12) with $\tau = t_1$, $\tilde{u}_0 = \Xi_0(t_1, u_0^*(s_0)) = u_1^*(t_1)$ which exists on $[t_1, s_1]$.

Following the same idea we construct the function

$$u^*(t; t_0, u_0) = \begin{cases} u_k^*(t) & \text{for } t \in (t_k, s_k],\ k = 0, 1, 2, \dots, p \\ \Xi_k(t, u_k^*(s_k - 0)) & \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, p-1, \end{cases}$$

where $u_k^*(t)$ is the maximal solution of the IVP for ODE (12) on $[t_k, s_k]$ with $\tau = t_k$ and $\tilde{u}_0 = \Xi_{k-1}(t_k, u_{k-1}^*(s_{k-1}))$ (in the case $k = 0$ it is denoted $\Xi_{-1}(t_0, u_{-1}^*(s_{-1})) = u_0$).

According to Definition 3 the function $u^*(t; t_0, u_0)$ is a maximal solution of IVP for NIDE (11). $\square$

LEMMA 2. *Let:*
1. Condition (H4) be satisfied for $J = \cup_{k=0}^{p}(t_k, s_k]$ *where* $p \leqslant \infty$ *is a positive integer.*
2. Condition (H5) be satisfied for all $k = 0, 1, 2, \dots, p-1$.
Then there exist a minimal solution of IVP for NIDE (11) on the interval $[t_0, s_p]$.

Also, in our study we will use a comparison couple of scalar NIDE

$$\begin{aligned} & u' = g_1(t, u), && v' = g_2(t, v), && t \in \cup_{k=0}^{\infty}(t_k, s_k], \\ & u(t) = \Phi_k(t, u(s_k - 0)), && v(t) = \Psi_k(t, v(s_k - 0)), && t \in (s_k, t_{k+1}], \quad k = 0, 1, 2, \dots, \\ & u(t_0) = u_0, && v(t_0) = v_0, && \end{aligned} \tag{13}$$

where $u, v \in \mathbb{R}$.

We will introduce the strict practical stability of the couple of NIDE as follows (for ODEs see Definition 4.1 in [15]):

DEFINITION 4. The zero solution of the couple of NIDE (13) is said to be

(S7) *strictly practically stable in couple* if for given $(\lambda_1, A_1): 0 < \lambda_1 < A_1$ there exists $t_0 \in [0, s_0) \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$ and for any $\lambda_2 < \lambda_1$ there exists $A_2 < \lambda_2$ such that the inequalities $|u_0| < \lambda_1$ and $\lambda_2 < |v_0|$ imply $|u(t; t_0, u_0)| < A_1$ and $A_2 < |v(t; t_0, v_0)|$ for $t \geqslant t_0$ where the couple of functions $\big(u(t; t_0, u_0), v(t; t_0, u_0)\big)$ is a solution of the IVP for the couple of NIDE (13).

(S8) *uniformly strictly practically stable in couple* if (S7) is satisfied for every initial time $t_0 \in [0, s_0) \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$.

In this paper we study the connection between the practical stability of the system NIDE (1) and the practical stability of the scalar NIDE (11).

We use the class Λ of Lyapunov-like functions:

DEFINITION 5. Let $J \subset \mathbb{R}_+$ be a given interval, and $\Delta \subset \mathbb{R}^n$ be a given set. We say that the function $V(t,x): J \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda(J, \Delta)$ if

1. The function $V(t,x)$ is continuous on $J/\{s_k \in J\} \times \Delta$ and it is locally Lipschitzian with respect to its second argument;
2. For each $s_k \in Int(J)$ and $x \in \Delta$ there exist finite limits

$$V(s_k, x) = V(s_k - 0, x) = \lim_{t \uparrow s_k} V(t,x) < \infty, \quad \text{and } V(s_k + 0, x) = \lim_{t \downarrow s_k} V(t,x) < \infty.$$

We use the following derivative of Lyapunov functions $V(t,x) \in \Lambda(J, \Delta)$

$$D^+_{(1)} V(t,x) = \limsup_{h \to 0+} \frac{1}{h} \Big[V(t,x) - V(t-h, x - hf(t,x) \Big] \text{ for } t \in J \cap \Big(\cup_{k=0}^{\infty} [t_k, s_k) \Big),\ x \in \Delta, \tag{14}$$

where for any $t \in (t_k, s_k)$ there exists $h_t > 0$ such that $t - h \in (t_k, s_k)$, $x - hf(t,x) \in \Delta$ for $0 < h \leqslant h_t$.

4. Main results

Now we give some comparison results.

LEMMA 3. (Theorem 1.4.1 [13]) *Let $E \subset \mathbb{R}^2$ be open set and $g \in C(E, \mathbb{R})$. Suppose $[\tau, \tau + a)$ be the largest interval in which the maximal solution $r(t)$ of (12) exists. Let $m \in C([\tau, \tau + a), \mathbb{R})$ be such that $(t, m(t)) \in E$ for $t \in [\tau, \tau + a)$, $m(\tau) \leqslant u_0$ and Dini derivative $D_+ m(t) \leqslant g(t, m(t))$ for $t \in [\tau, \tau + a)$.*

Then $m(t) \leqslant r(t)$ for $t \in [\tau, \tau + a)$.

We give the comparison result for the scalar NIDE (11) which is a generalization of Lemma 1 in [3] .

LEMMA 4. *Suppose:*

1. *The function* $x^*(t) = x(t;t_0,x_0) \in PC^1([t_0,\Theta)],\Delta)$ *is a solution of the NIDE (1) where* $\Delta \subset \mathbb{R}^n$, $\Theta \in (t_p,s_p]$ *is a given number,* p *is a natural number.*

2. *For all* $k = 0,1,\ldots,p-1$ *condition (H5) is satisfied.*

3. *Condition (H3) is satisfied on the interval* $J = [t_p,\Theta]\bigcup\cup_{k=0}^{p-1}[t_k,s_k]$.

4. *The function* $V \in \Lambda([t_0,\Theta],\Delta)$ *and*

   (i) the inequality

$$D^+_{(1)}V(t,x^*(t)) \leqslant g(t,V(t,x^*(t))) \quad \textit{for } t \in (t_p,\Theta]\bigcup\cup_{k=0}^{p-1}(t_k,s_k]$$

   holds;

   (ii) for any number $k = 0,1,2,\ldots,p-1$ *the inequality*

$$V(t,\phi_k(t,x^*(s_k-0))) \leqslant \Xi_k(t,V(s_k-0,x^*(s_k-0))) \ \textit{for}\, t \in (s_k,t_{k+1}]$$

   holds.

Then the inequality $V(t_0,x_0) \leqslant u_0$ *implies* $V(t,x^*(t)) \leqslant u^*(t)$ *on* $[t_0,\Theta]$ *where* $u^*(t)$ *is the maximal solution of IVP for NIDE (11) on* $[t_0,\Theta]$.

Proof. Note that according to Lemma 1 from conditions 2 and 3 of Lemma 4 there exists a maximal solution $u^*(t)$ of IVP for NIDE (11) on $[t_0,\Theta]$.

Let $V(t_0,x_0) \leqslant u_0$. We use induction to prove Lemma 4.

Denote $m(t) = V(t,x^*(t))$ for $t \in [t_0,\Theta]$. Let $t \in (t_p,\Theta]\bigcup\cup_{k=0}^{p-1}(t_k,s_k]$. Using condition 4(i) we obtain

$$\begin{aligned} D_+m(t) &\leqslant D^+_{(1)}V(t,x^*(t)) + L\limsup_{h\to 0+}\frac{1}{h}||x^*(t)-x^*(t-h)-hf(t,x^*(t))|| \\ &= D^+_{(1)}V(t,x^*(t)) \leqslant g(t,m(t)), \end{aligned} \tag{15}$$

where $L > 0$ is the Lipshitz constant of the Lyapunov function $V(t,x)$.

Let $t \in [t_0,s_0]$. The function $m(t) \in C([t_0,s_0],\mathbb{R}^n)$ and $m(t_0) = V(t_0,x_0) \leqslant u_0$. From Definition 3 the function $u^*(t)$ is a maximal solution of IVP for ODE (12) with $\tau = t_0$, $\tilde{u}_0 = u_0$ on $[t_0,s_0]$. According to Lemma 3 the inequality $m(t) = V(t,x^*(t)) \leqslant u^*(t)$ holds on $[t_0,s_0]$.

Let $t \in (s_0,t_1]$. Then $x^*(t) = \phi_0(t,x^*(s_0-0))$. From conditions 4(i) and (H5) for $k=0$, the monotonicity of Ξ_0 and the above we get $V(t,x^*(t)) = V(t,\phi_0(t,x^*(s_0-0))) \leqslant \Xi_0(t,V(s_0-0,x^*(s_0-0)) \leqslant \Xi_0(t,u^*(s_0-0)) = u^*(t)$, $t \in (s_0,t_1]$.

Let $t \in (t_1, s_1]$ (if $\Theta < s_1$ then we consider the interval $(t_1, \Theta]$). Define the function $m(t) = V(t, x^*(t))$ for $t \in (t_1, s_1]$ and $m(t_1) = V(t_1, \phi_0(t_1, x^*(s_0 - 0)))$. The function $m(t) \in C([t_1, s_1], \mathbb{R}^n)$, satisfies the inequality (15) and $m(t_1) = V(t_1, \phi_0(t_1, x^*(s_0 - 0))) = V(t_1, x^*(t_1)) \leqslant u^*(t_1)$. Since the function $u^*(t)$ is also the maximal solution of (12) with $\tau = t_1$, $\tilde{u}_0 = u^*(t_1)$ on the interval $[t_1, s_1]$, according to Lemma 3 we obtain $m(t) = V(t, x^*(t)) \leqslant u^*(t)$ for $t \in (t_1, s_1]$.

Continue this process and an induction argument proves the claim in Lemma 4 is true for $t \in [t_0, \Theta]$. $\square$

LEMMA 5. *Suppose:*

1. *The conditions 1 and 2 of Lemma 4 are satisfied.*
2. *The condition (H4) is satisfied on the interval* $J = [t_p, \Theta] \bigcup \cup_{k=0}^{p-1} [t_k, s_k]$.
3. *There exists a function* $V \in \Lambda([t_0, \Theta], \Delta)$ *such that condition 4 of Lemma 4 is satisfied where inequalities in 4(i) and 4(ii) are replaced by* $\geqslant$.

Then the inequality $V(t_0, x_0) \geqslant u_0$ *implies* $V(t, x^*(t)) \geqslant u^*(t)$ *on* $[t_0, \Theta]$ *where* $u^*(t)$ *is the maximal solution of IVP for NIDE (11) on* $[t_0, \Theta]$. .

Proof. The proof is similar to the one in Lemma 4 so we omit it. $\square$

REMARK 4. The results of Lemma 4 and Lemma 5 are also true on the half line, i.e. $\Theta = \infty$.

We now obtain sufficient conditions for practical stability of the system NIDE (1).

THEOREM 1. *Let the following conditions be satisfied:*
1. The conditions (H1) and (H3) are satisfied on $J = [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k]$.
2. The conditions (H2) and (H5) are satisfied for all $k = 0, 1, 2, \ldots$.
3. There exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that*

(i) the inequality

$$D^+_{(1)} V(t, x) \leqslant g(t, V(t, x)) \quad \text{for } t \in [0, s_0] \bigcup \cup_{k=1}^{\infty} (t_k, s_k],\ x \in S(A) \tag{16}$$

holds where A *is a given constant;*

(ii) for any number $k = 0, 1, 2, \ldots$ *the inequalities*

$$V(t, \phi_k(t, x)) \leqslant \Xi_k(t, V(s_k - 0, x)) \quad \text{for } t \in (s_k, t_{k+1}],\ x \in S(A)$$

hold;

(iii) $a(||x||) \leqslant V(t, x) \leqslant b(||x||)$ *for* $t \in \mathbb{R}_+$, $x \in S(A)$, *where* $a, b \in \mathscr{K}$.

4. The zero solution of the scalar NIDE (11) is practically stable (uniformly practically stable) w.r.t. $(b(\lambda),a(A))$ *where the constant* $\lambda:\ 0<\lambda<A,\ b(\lambda)<a(A)$ *is given.*

Then the zero solution of the system of NIDE (1) is practical stable (uniformly practically stable) w.r.t. (λ,A)*.*

Proof. Let the zero solution of the scalar NIDE (11) be practically stable w.r.t. $(b(\lambda),a(A))$. From condition 4 there exists a point $t_0\in[0,s_0]\bigcup\cup_{k=1}^{\infty}[t_k,s_k]$ such that $|u_0|<b(\lambda)$ implies

$$|u(t;t_0,u_0)|<a(A)\quad \text{for } t\geqslant t_0, \tag{17}$$

where $u(t;t_0,u_0)$ is a solution of (11). Without loss of generality we assume $t_0\in[0,s_0]$.

Choose a point $x_0\in\mathbb{R}^n$ with $||x_0||<\lambda$ and let $x(t;t_0,x_0)$ be a solution of the IVP for NIDE (1) for the chosen x_0 and the above t_0. Let $u_0=V(t_0,x_0)$ and $u^*(t;t_0,u_0)$ be the maximal solution of the scalar NIDE (11) defined for $t\geqslant t_0$ (Note it exists because of conditions (H3), (H5) and Lemma 1). Since $|u_0|=V(t_0,x_0)<b(\lambda)$ the solution $u^*(t;t_0,u_0)$ satisfies the inequality (17).

Assume inequality

$$||x(t;t_0,x_0)||<A\ \text{ for } t\geqslant t_0 \tag{18}$$

is not true. There are three cases to consider.

Case 1. There exists a point $t^*>t_0,\ t^*\neq s_k,\ k=0,1,2,\dots$ such that

$$||x(t;t_0,x_0)||<A\ \text{ for } t\in[t_0,t^*)\quad\text{and}\quad ||x(t^*;t_0,x_0)||=A. \tag{19}$$

From Lemma 4 with $\Theta=t^*$ and $\Delta=S(A)$ we obtain

$$V(t,x(t;t_0,x_0))\leqslant u^*(t;t_0,u_0)\quad \text{for } t\in[t_0,t^*]. \tag{20}$$

From inequality (20) and condition 3(*iii*) we get

$$a(A)=a(||x(t^*;t_0,x_0)||)\leqslant V(t^*,x(t^*;t_0,x_0))\leqslant u^*(t^*;t_0,u_0)<a(A). \tag{21}$$

We obtain a contradiction.

Case 2. There exists an integer $k\geqslant 0$ such that

$$||x(t;t_0,x_0)||<A\ \text{ for } t\in[t_0,s_k)\quad\text{and}\quad ||x(s_k;t_0,x_0)||=A. \tag{22}$$

As in Case 1 with $t^*=s_k$ we obtain a contradiction.

Case 3. There exists an integer $k\geqslant 0$ such that

$$||x(t;t_0,x_0)||<A\ \text{ for } t\in[t_0,s_k]\quad\text{and}\quad ||x(s_k+0;t_0,x_0)||\geqslant A. \tag{23}$$

From Lemma 4 with $\Theta=s_k$ and $\Delta=S(A)$ we obtain inequality (20) for $t\in[t_0,s_k]$.

Then $x(s_k+0;t_0,x_0)=\phi_k(s_k,x(s_k-0;t_0,x_0))$ and according to conditions 3(*ii*), 3(*iii*) and inequality (22) we get

$$\begin{aligned}a(A)&\leqslant a(||x(s_k+0;t_0,x_0)||)=a(||\phi_k(s_k,x(s_k-0;t_0,x_0)||)\\&\leqslant V(s_k,\phi_k(s_k,x(s_k-0;t_0,x_0))\ \leqslant\Psi_k(s_k,V(s_k-0,x(s_k-0;t_0,x_0))\\&\leqslant\Psi_k(s_k,u^*(s_k-0;t_0,u_0))=u^*(s_k+0;t_0,u_0))<a(A).\end{aligned} \tag{24}$$

The contradictions obtained above prove inequality (18) is true and therefore the zero solution of the system of NIDE (1) is practical stable w.r.t. (λ, A).

The proof of the uniform practical stability of the zero solution of NIDE (1) is similar. $\square$

THEOREM 2. *Let the following conditions be satisfied:*
1. The conditions (H1) and (H3) are satisfied on $J = [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k]$.
2. The conditions (H2) and (H5) are satisfied for all $k = 0, 1, 2, \ldots$.
3. There exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that*

(i) the inequality

$$D^+_{(1)} V(t,x) \leqslant g(t, V(t,x)) \quad \text{for } t \in [0, s_0] \bigcup \cup_{k=1}^{\infty} (t_k, s_k),\ x \in \mathbb{R}^n \tag{25}$$

holds;

(ii) for any number $k = 0, 1, 2, \ldots$ *the inequalities*

$$V(t, \phi_k(t,x)) \leqslant \Xi_k(t, V(s_k - 0, x)) \ \text{for } t \in (s_k, t_{k+1}],\ x \in \mathbb{R}^n$$

hold;

(iii) $a(||x||) \leqslant V(t,x) \leqslant b(||x||)$ *for* $t \in \mathbb{R}_+,\ x \in \mathbb{R}^n$, *where* $a, b \in \mathscr{K}$.

4. The zero solution of the scalar NIDE (11) is practically quasi stable (uniformly practically quasi stable) w.r.t. $(b(\lambda), a(A), T)$ *where the positive constants* T, λ, A : $\lambda < A,\ b(\lambda) < a(A)$ *are given.*

Then the zero solution of the system NIDE (1) is practical quasi stable (uniformly quasi practically stable) w.r.t. (λ, A, T).

Proof. Let the zero solution of scalar NIDE (11) be practically quasi stable w.r.t. $(b(\lambda), a(A), T)$. From condition 4 there exists a point $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k]$ such that $|u_0| < b(\lambda)$ implies

$$|u(t; t_0, u_0)| < a(A) \quad \text{ for } t \geqslant t_0 + T, \tag{26}$$

where $u(t; t_0, u_0)$ is a solution of (11).

Choose a point $x_0 \in \mathbb{R}^n$ with $||x_0|| < \lambda$ and let $x(t; t_0, x_0)$ be a solution of the IVP for NIDE (1) for the chosen x_0 and the above t_0. Assume inequality

$$||x(t; t_0, x_0)|| < A \ \text{ for } t \geqslant t_0 + T \tag{27}$$

is not true.

A proof similar to that in Theorem 1 by applying Lemma 4 with $\Delta = \mathbb{R}^n$ yields a contradiction. $\square$

We obtain sufficient conditions for strict practical stability of the system NIDE (1).

THEOREM 3. (Strict practical stability of NIDE) *Let the following conditions be satisfied:*

1. *The condition (H1) is satisfied on* $[0,s_0]\bigcup\cup_{k=1}^{\infty}(t_k,s_k)$.

2. *The condition (H2) is satisfied for all* $k=0,1,2,\dots$

2. *The conditions (H3) and (H4) are satisfied on* $J=[0,s_0]\bigcup\cup_{k=1}^{\infty}(t_k,s_k)$ *for the functions* g_1 *and* g_2 *respectively* .

4. *The functions* Φ_k *and* Ψ_k, $k=0,1,2,\dots$ *satisfy the condition (H5).*

5. *There exists a function* $V_1\in\Lambda(\mathbb{R}_+,\mathbb{R}^n)$ *such that*

   (i) the inequality

   $$D_+^q V_1(t,x)\leqslant g_1(t,V_1(t,x)) \quad \textit{for } t\in[0,s_0]\bigcup\cup_{k=1}^{\infty}(t_k,s_k),\ x\in S(A_1)$$

   holds where $A_1>0$ *is a given constant;*

   (ii) for any $k=0,1,2,\dots$ *the inequality*

   $$V_1(t,\phi_k(t,x))\leqslant\Phi_k(t,V_1(s_k-0,x)) \ \textit{for } t\in(s_k,t_{k+1}],\ x\in S(A_1)$$

   holds;

   (iii) $a_1(||x||)\leqslant V_1(t,x)\leqslant b_1(||x||)$ *for* $t\in\mathbb{R}_+,\ \ x\in S(A_1)$, *where* $a_1,b_1\in\mathscr{K}$.

6. *For each* $\eta\in(0,A_1)$ *there exists a function* $V_\eta\in\Lambda(\mathbb{R}_+,\mathbb{R}^n)$ *such that*

   (iv) the inequality

   $$D_+^q V_\eta(t,x)\geqslant g_2(t,V_\eta(t,x)) \quad \textit{for } t\in[0,s_0]\bigcup\cup_{k=1}^{\infty}(t_k,s_k),\ ||x||\geqslant\eta$$

   holds;

   (v) for any $k=0,1,2,\dots$ *the inequality*

   $$V_2(t,\phi_k(t,x))\geqslant\Psi_k(t,V_2(s_k-0,x)) \ \textit{for } t\in(s_k,t_{k+1}],\ ||x||\geqslant\eta$$

   holds;

   (vi) $a_2(||x||)\leqslant V_\eta(t,x)\leqslant b_2(||x||)$ *for* $t\in\mathbb{R}_+,\ \ ||x||\geqslant\eta$, *where* $a_2,b_2\in\mathscr{K}$.

7. *The zero solution of the couple of NIDE (13) is strictly practically stable (uniformly strictly practically stable) in couple w.r.t.* $(b_1(\lambda_1),a_1(A_1))$ *where the constant* $\lambda_1:\ 0<\lambda_1<A_1,\ b_1(\lambda_1)<a_1(A_1)$ *is given.*

Then the zero solution of the system NIDE (1) is strictly practically stable (uniformly strictly practically stable) with respect to (λ_1,A_1).

Proof. Let the zero solution of the couple of NIDE (13) be strictly practically stable. Then there exists $t_0\in[0,s_0]\bigcup\cup_{k=1}^{\infty}[t_k,s_k]$ and for any $\tilde{\lambda}_2<b_1(\lambda_1)$ there exists $\tilde{A}_2<\tilde{\lambda}_2$ such that the inequalities $|u_0|<b_1(\lambda_1)$ and $\tilde{\lambda}_2<|v_0|$ imply

$$|u(t;t_0,u_0)|<a_1(A_1) \quad \text{for } t\geqslant t_0, \tag{28}$$

$$|v(t;t_0,v_0)| > \tilde{A_2} \quad \text{for } t \geqslant t_0, \tag{29}$$

where the couple $\big(u(t;t_0,u_0), v(t;t_0,v_0)\big)$ is a solution of (13).

Let $\lambda_2 < \lambda_1$ be an arbitrary positive number and let $\tilde{\lambda_2} = a_2(\lambda_2)$. Choose $x_0 \in \mathbb{R}^n$ with $\lambda_2 < ||x_0|| < \lambda_1$ and let $x^*(t) = x(t;t_0,x_0)$ be a solution of the IVP for NIDE (1) for the initial data (t_0,x_0).

Let $\big(u^*(t), v^*(t)\big)$ be the solution of the IVP for the couple of NIDE (13) with initial values (u_0,v_0) : $u_0 = V_1(t_0,x_0)$, $v_0 = V_2(t_0,x_0)$ where $u^*(t) = u(t;t_0,u_0)$, $v^*(t) = v(t;t_0,v_0)$ are the maximal solution and the minimal solution of the first and second equation of (13), respectively defined for $t \geqslant t_0$ (Note they exist because of Lemma 1 and Lemma 2). From the choice of x_0 and conditions 5(iii) and 6(vi) it follows that $\tilde{\lambda_2} = b_1(\lambda_2) < b_1(\lambda_1)$ and $|u_0| \leqslant b_1(||x_0||) < b_1(\lambda_1)$ and $|v_0| \geqslant a_2(||x_0||) > a_2(\lambda_2) = \tilde{\lambda_2}$. Therefore there exists $\tilde{A_2} < a_2(\lambda_2)$ such that inequalities (28), (29) hold.

As in Theorem 1 we can prove inequality (18) holds for $t \geqslant t_0$ by replacing A by A_1.

Let the constant $A_2 > 0$ be such that $b_2(A_2) = \tilde{A_2}$. Then $b_2(A_2) = \tilde{A_2} < a_2(\lambda_2) \leqslant b_2(\lambda_2)$, and therefore $A_2 < \lambda_2 < \lambda_1 < A_1$. From condition 6 for $\eta = A_2$ there exists a function $V_{A_2}(t,x)$ such that conditions 6(iv), 6(v) and 6(vi) are satisfied for $||x|| \geqslant A_2$.

From the choice of x_0 it follows that $||x_0|| > \lambda_2 > A_2$.

Assume inequality

$$||x(t;t_0,x_0)|| > A_2 \ \text{ for } t \geqslant t_0 \tag{30}$$

is not true. There are three cases to consider.

Case 1. There exists a point $t^* > t_0$, $t \neq s_k$, $k = 0,1,2,\dots$ such that

$$||x(t;t_0,x_0)|| > A_2 \ \text{ for } t \in [t_0,t^*) \quad \text{and} \quad ||x(t^*;t_0,x_0)|| = A_2. \tag{31}$$

From Lemma 5 applied to the second component $v^*(t) = v(t;t_0,v_0)$ of the solution of the couple of scalar NIDE (13) with $V(t,x) = V_{A_2}(t,x)$, $\tau = t_0$, $\Theta = t^*$ and $\Delta = \{x \in \mathbb{R}^n : ||x|| \geqslant A_2\}$ we obtain

$$V_{A_2}(t, x(t;t_0,x_0)) \geqslant |v(t;t_0,u_0)| \quad \text{for } t \in [t_0,t^*]. \tag{32}$$

From inequalities (29), (32) and condition 6(*vi*) we get

$$\tilde{A_2} = b_2(A_2) = b_2(||x(t^*;t_0,x_0)||) \geqslant V_{A_2}(t^*, x(t^*;t_0,x_0)) \geqslant |v^*(t^*;t_0,u_0)| > \tilde{A_2}. \tag{33}$$

We obtain a contradiction.

Case 2. There exists an integer $k \geqslant 0$ such that

$$||x(t;t_0,x_0)|| > A_2 \ \text{ for } t \in [t_0,s_k) \quad \text{and} \quad ||x(s_k;t_0,x_0)|| = A_2. \tag{34}$$

As in Case 1 with $t^* = s_k$ we obtain a contradiction.

Case 3. There exists an integer $k \geqslant 0$ such that

$$||x(t;t_0,x_0)|| > A_2 \ \text{ for } t \in [t_0,s_k] \quad \text{and} \quad ||x(s_k+0;t_0,x_0)|| \leqslant A_2. \tag{35}$$

From Lemma 5 applied to the second component $v^*(t) = v(t;t_0,v_0)$ of the solution of the couple of scalar NIDE (13) with $V(t,x) = V_{A_2}(t,x)$, $\tau = t_0$, $\Theta = s_k$ and $\Delta = \{x \in \mathbb{R}^n : \ ||x|| \geqslant A_2\}$ we obtain inequality (32) for $t \in [t_0, s_k]$.

Then $x(s_k+0;t_0,x_0) = \phi_k(s_k, x(s_k-0;t_0,x_0)$ and according to inequality (29), conditions 4, 6(v), 6(vi) we get

$$\begin{aligned}\tilde{A_2} = b_2(A_2) &\geqslant b_2(||x(s_k+0;t_0,x_0)||) = b_2(||\phi_k(s_k,x(s_k-0;t_0,x_0)||) \\ &\geqslant V(s_k,\phi_k(s_k,x(s_k-0;t_0,x_0)) \geqslant \Psi_k(s_k,V(s_k-0,x(s_k-0;t_0,x_0)) \\ &\geqslant \Psi_k(s_k,v(s_k-0;t_0,v_0)) = v(s_k+0;t_0,v_0)) > \tilde{A_2}.\end{aligned} \tag{36}$$

The contradictions obtained above proves inequality (30) is true and therefore the zero solution of the system of NIDE (1) is strictly practical stable w.r.t. (λ_1, A_1).

The proof of the uniform strict practical stability of the zero solution of NIDE (1) is similar. $\square$

5. Applications

We will give some applications of the obtained sufficient conditions to nonlinear systems of NIDE.

Let $t_k = 2k,\ s_k = 2k+1,\ k = 0,1,2,\dots$.

EXAMPLE 4. . Consider the initial value problem for the system of differential equations with non-instantaneous impulses

$$\begin{aligned}&x'(t) = 0.5x - 0.5y^3 \\ &y'(t) = xy^2 - 2x^2y + 0.5y \quad \text{for} \quad t \in \cup_{k=0}^{\infty}(t_k,s_k], \\ &x(t) = Ax(s_k-0), \qquad y(t) = By(s_k-0) \quad \text{for } t \in (s_k,t_{k+1}],\ k = 0,1,2,\dots, \\ &x(0) = x_0, \quad y(0) = y_0\end{aligned} \tag{37}$$

where $A, B \in \mathbb{R} : \ |A| < e^{-1}, |B| < e^{-1}$ are constants.

Consider the Lyapunov function $V(x,y) = x^2 + 0.5y^2$. Then $a(s) = 0.5s^2$ and $b(s) = s^2$ in condition 3(iii) of Theorem 1.

Then

$$D^+_{(37)}V(t,x) = x^2 - xy^3 + xy^3 - 2x^2y^2 + 0.5y^2 \leqslant V(t,x)$$

and

$$V(Ax,By) = A^2x^2 + 0.5B^2y^2 \leqslant e^{-2}V(x,y)$$

The comparison scalar equation is

$$\begin{aligned}&u'(t) = u \quad \text{for} \quad t \in \cup_{k=0}^{\infty}(t_k,s_k], \\ &u(t) = e^{-2}u(s_k-0) \quad \text{for } t \in (s_k,t_{k+1}],\ k = 0,1,2,\dots, \\ &u(0) = V(x_0,y_0).\end{aligned} \tag{38}$$

From Example 3, Case 1.2 the zero solution of NIDE (38) is uniformly practically stable w.r.t. (λ, A), $A \geqslant \lambda e^{-1}$.

According to Theorem 1 the solution of the system of NIDE (37) is uniformly practically stable w.r.t. $(\sqrt{\lambda}, \sqrt{2A})$.

Now, consider the corresponding system without any impulses:

$$x'(t) = 0.5x - 0.5y^3, \quad y'(t) = xy^2 - 2x^2y + 0.5y \quad \text{for} \quad t \geqslant 0. \tag{39}$$

Its solution is neither bounded nor practically stable (see Figure 5 and Figure 6).

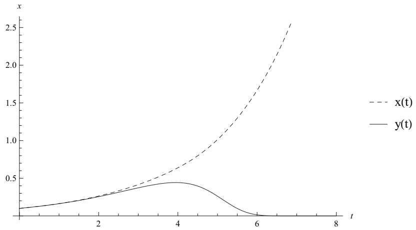

Figure 5. Example 4. Graphs of solutions of (37) with $x_0 = y_0 = 0.1$.

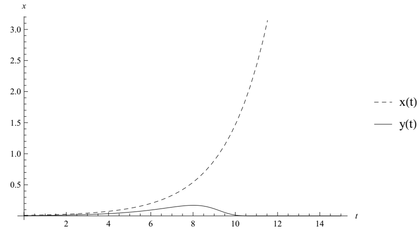

Figure 6. Example 4. Graphs of solutions for (37) with $x_0 = 0.01,\ y_0 = 0.005$.

The presence of impulses can change the behavior of the solution of the differential equation.

EXAMPLE 5. Consider the initial value problem for the system of differential equations with non-instantaneous impulses

$$\begin{aligned}
&x'(t) = y \\
&y'(t) = -x - 100x^3 \quad \text{for} \quad t \in \cup_{k=0}^{\infty}(t_k, s_k], \\
&x(t) = A_k t x(s_k - 0), \qquad y(t) = B_k t y(s_k - 0) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0,1,2,\ldots, \\
&x(0) = x_0, \quad y(0) = y_0
\end{aligned} \tag{40}$$

where $A_k, B_k \in \mathbb{R}:\ |A_k|(2k+2) < e^{-1}, |B_k|(2k+2) < e^{-1}$ are constants.

Consider the Lyapunov function $V(x,y) = 0.5x^2 + 25x^4 + 0.5y^2$. Then $a(s) = 0.5s^2$ and $b(s) = 0.5s^2 + 25s^4$ in condition 3(iii) of Theorem 2.

Then

$$D^+_{(40)}V(t,x) = xx' + 100x^3x' + yy' = xy + 100yx^3 - xy - 100yx^3 = 0$$

and for any $k = 0,1,2,\ldots$

$$\begin{aligned}
V(A_k t x, B_k t y) &= 0.5A_k^2 t^2 x^2 + 25A_k^4 t^4 x^4 + 0.5B_k^2 y^2 \\
&\leqslant 0.5A_k^2 t_{k+1}^2 x^2 + 25A_k^4 t_{k+1}^4 x^4 + 0.5B_k^2 t_{k+1}^2 y^2 \\
&\leqslant e^{-2}V(x,y).
\end{aligned}$$

The comparison scalar equation is

$$\begin{aligned}
&u'(t) = 0 \quad \text{for} \quad t \in \cup_{k=0}^{\infty}(t_k, s_k], \\
&u(t) = e^{-2}u(t_k - 0) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0,1,2,\ldots, \\
&u(0) = V(x_0, y_0).
\end{aligned} \tag{41}$$

From Example 3, Case 3 the zero solution of (41) is practically quasi stable w.r.t. $(\lambda, A, T):\ A = \frac{\lambda}{e^2} < \lambda,\ T = 1$.

According to Theorem 2 the solution of NIDE (40) is practically quasi stable.

Now, consider the corresponding system without any type of impulses:

$$x'(t) = -y, \quad y'(t) = -x - 100x^3 \quad \text{for} \quad t \geqslant 0. \tag{42}$$

Its solution is a periodic function (see Figure 7 and Figure 8).

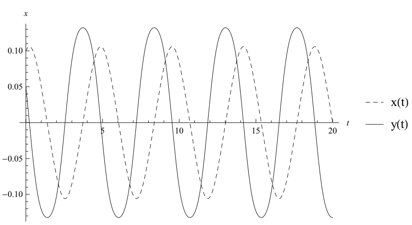

Figure 7. Example 5. Graphs of solutions of (40) with $x_0 = 0.1, y_0 = 0.05$.

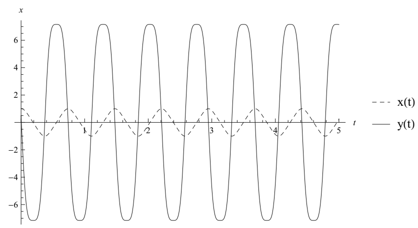

Figure 8. Example 5. Graphs of solutions for (40) with $x_0 = 1, y_0 = 0.5$.

EXAMPLE 6. Consider the initial value problem for the system of differential equations with non-instantaneous impulses

$$\begin{aligned}
&x'(t) = -2y + yz - x \\
&y'(t) = x - xz - y \\
&z'(t) = xy - z \quad \text{for} \quad t \in \cup_{k=0}^{\infty}(t_k, s_k], \\
&x(t) = \frac{A_k}{t}x(s_k - 0), \qquad y(t) = B_k y(s_k - 0), \quad z(t) = C_k t z(s_k - 0) \\
&\qquad\qquad\qquad \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, \\
&x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0
\end{aligned} \tag{43}$$

where $A_k, B_k, C_k \in \mathbb{R},\ k = 0, 1, 2, \ldots,$ are constants such that $e|A_k| < s_k,\ |B_k| < e^{-1}$, and $|C_k|t_{k+1} < e^{-1}$.

Consider the Lyapunov function $V(x,y,z) = 0.5x^2 + y^2 + 0.5z^2$. Then $a(s) = 0.5s^2$ and $b(s) = s^2$ in condition 3(iii) of Theorem 2.

Then

$$D^+_{(43)}V(t,x) = xx' + 4yy' + zz' = -V(x,y,z)$$

and for any $k = 0, 1, 2, \ldots$

$$\begin{aligned}
V(A_k t x, B_k t y) &= 0.5\frac{A_k^2}{t^2}x^2 + B_k^2 y^2 + 0.5C_k^2 t^2 z^2 \\
&\leqslant 0.5\frac{A_k^2}{s_k^2}x^2 + B_k^2 y^2 + 0.5C_k^2 t_{k+1}^2 z^2 \leqslant e^{-2}V(x,y)
\end{aligned}$$

The comparison scalar equation is

$$\begin{aligned} &u'(t) = -u \quad \text{for} \quad t \in \cup_{k=0}^{\infty}(t_k, s_k], \\ &u(t) = e^{-2}u(t_k - 0) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0,1,2,\dots, \\ &u(0) = V(x_0, y_0). \end{aligned} \tag{44}$$

From Example 3, Case 2.3 the zero solution of scalar NIDE (44) is practically quasi stable w.r.t. $(\lambda, A, T):\ A = \frac{\lambda}{e^3} < \lambda,\ T = 1$.

According to Theorem 1 the solution of NIDE (43) is practically quasi stable w.r.t. $(\sqrt{\lambda}, \sqrt{2\frac{\lambda}{e^3}}, 1)$.

Now, consider the corresponding system without any type of impulses:

$$x'(t) = -2y + yz - x, \quad y'(t) = x - xz - y, \quad z'(t) = xy - z \quad \text{for} \quad t \geqslant 0. \tag{45}$$

Its zero solution is stable (see Figure 9 and Figure 10).

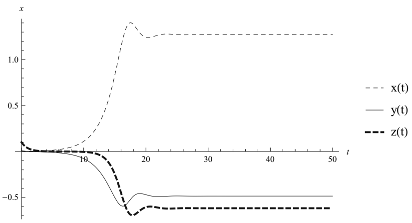

Figure 8. Example 6. Graphs of solutions of (43) with $x_0 = 0.01$, $y_0 = 0.005$, $z_0 = 0.1$.

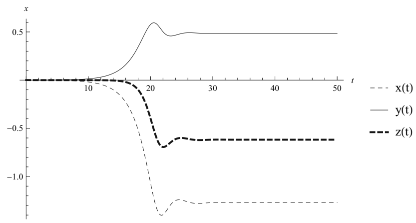

Figure 10. Example 6. Graphs of solutions of (43) with $x_0 = 0.0001$, $y_0 = 0.0005$, $z_0 = 0.001$.

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(Received April 23, 2017)

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