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EXISTENCE OF SOLUTION FOR FUNCTIONAL COUPLED SYSTEMS WITH FULL NONLINEAR TERMS AND APPLICATIONS TO A COUPLED MASS–SPRING MODEL

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Abstract. In this paper we consider some boundary value problems composed by coupled systems of second order differential equations with full nonlinearities and general functional boundary conditions verifying some monotone assumptions.

The arguments apply lower and upper solutions method and fixed point theory. Due to an adequate auxiliary problem, including a convenient truncature, there is no need of sign, bound, monotonicity or other growth assumptions on the nonlinearities, besides the Nagumo condition.

An application to a coupled mass-spring system with functional behavior at the final instant is shown.

1. Introduction

In this paper we consider the boundary value problem composed by the coupled system of the second order differential equations with full nonlinearities

$$\begin{cases} u''(t) = f(t,u(t),v(t),u'(t),v'(t)), \ t \in [a,b], \\ v''(t) = h(t,u(t),v(t),u'(t),v'(t)) \end{cases} \tag{1.1}$$

with $f,h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ continuous functions, and the functional boundary conditions

$$\begin{cases} u(a) = v(a) = 0 \\ L_1(u,u(b),u'(b)) = 0 \\ L_2(v,v(b),v'(b)) = 0, \end{cases} \tag{1.2}$$

where $L_1, L_2 : C[a,b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions verifying some monotone assumptions.

Ordinary differential systems have been studied by many authors, like, for instance, [1, 5, 11, 12, 13, 14, 15, 17, 18, 20, 21, 24, 28]. In particular, coupled second order ordinary differential systems can be applied to several real phenomena, such as, Lokta-Volterra models, reaction diffusion processes, prey-predator or other interaction

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systems, Sturm-Liouville problems, mathematical biology, chemical systems (see, for example, [2, 3, 4, 16, 19, 26] and the references therein).

In [23] the authors study the existence of solutions for the nonlinear second order coupled system

$$\begin{cases} -u''(t) = f_1(t, v(t)) \\ -v''(t) = f_2(t, v(t)), \end{cases} t \in [0,1],$$

with $f_1, f_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous functions, together with the nonlinear boundary conditions

$$\begin{aligned} \varphi\left(u(0), v(0), u'(0), v'(0), u'(1), v'(1)\right) &= (0,0), \\ \psi\left(u(0), v(0)\right) + \left(u(1), v(1)\right) &= (0,0), \end{aligned}$$

where $\varphi : \mathbb{R}^6 \to \mathbb{R}^2$ and $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous functions.

In [7] it is provided some growth conditions on the nonnegative nonlinearities of the system

$$\begin{cases} -x''(t) = f_1(t, x(t), y(t)) \\ -y''(t) = f_2(t, x(t), y(t)), t \in (0,1), \\ x(0) = y(0) = 0, \\ x(1) = \alpha\,[y], \\ y(1) = \beta\,[x], \end{cases}$$

where $f_1, f_2 : (0,1) \times [0,+\infty)^2 \to [0,+\infty)$ are continuous and may be singular at $t = 0, 1$, and $\alpha[x]$, $\beta[x]$ are bounded linear functionals on $C[0,1]$ given by

$$\alpha\,[y] = \int_0^1 y(t)dA(t), \quad \beta\,[x] = \int_0^1 x(t)dB(t),$$

involving Stieltjes integrals, and A, B are functions of bounded variation with positive measures.

Motivated by these works we consider the second order coupled fully differential equations (1.1) together with the functional boundary conditions (1.2). To the best of our knowledge, it is the first time where these coupled differential systems embrace functional boundary conditions. Remark that, the functional dependence includes and generalizes the classical boundary conditions such as separated, multi-point, nonlocal, integro-differential, with maximum or minimum arguments,...More details on such conditions and their application potentialities can be seen, for instance, in [6, 9, 10, 22, 25] and the references therein. Our main result is applied to a coupled mass-spring systems subject to a new type of global boundary data.

The arguments in this paper follow lower and upper solutions method and fixed point theory. Therefore, the main result is an existence and localization theorem, as it provides not only the existence of solution, but a strip where the solution varies, as well. Due to an adequate auxiliary problem, including a convenient truncature, there is no need of sign, bound, monotonicity or other growth assumptions on the nonlinearities, besides the Nagumo condition.

The paper is organized as it follows: Section 2 contains the space framework, lower an upper solutions definition and some *a priori* estimates on the first derivative of the unknown functions. In sections 3 and 4 we present an existence and localization result and an example to show the applicability of the main theorem. An application to a real phenomenon is shown in the last section: a coupled mass-spring system together with functional behavior at the final instant.

2. Definitions and preliminaries

Let $E = C^1[a,b]$ be the Banach space equipped with the norm $\|\cdot\|_{C^1}$, defined by

$$\|w\|_{C^1} := \max\left\{\|w\|, \|w'\|\right\},$$

where

$$\|y\| := \max_{t\in[a,b]} |y(t)|$$

and $E^2 = \left(C^1[a,b]\right)^2$ with the norm

$$\|(u,v)\|_{E^2} = \|u\|_{C^1} + \|v\|_{C^1}.$$

Forward in this work, we consider the following assumption

(A) The functions $L_1, L_2 : C[a,b]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous, nonincreasing in the first variable and nondecreasing in the second one.

To apply lower and upper solutions method we consider next definition:

DEFINITION 1. A pair of functions $(\alpha_1,\alpha_2)\in\left(C^2[a,b]\right)^2$ is a coupled lower solution of problem (1.1), (1.2) if

$$\begin{aligned}
\alpha_1''(t) &\leqslant f(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t))\\
\alpha_2''(t) &\leqslant h(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t))\\
\alpha_1(a) &\leqslant 0\\
\alpha_2(a) &\leqslant 0\\
L_1(\alpha_1,\alpha_1(b),\alpha_1'(b)) &\geqslant 0\\
L_2(\alpha_2,\alpha_2(b),\alpha_2'(b)) &\geqslant 0.
\end{aligned}$$

A pair of functions $(\beta_1,\beta_2)\in\left(C^2[a,b]\right)^2$ is a coupled upper solution of problem (1.1), (1.2) if it verifies the reverse inequalities.

A Nagumo-type condition is useful to obtain *a priori* bounds on the first derivatives of the unknown functions:

DEFINITION 2. Let $\alpha_1(t), \beta_1(t), \alpha_2(t)$ and $\beta_2(t)$ be continuous functions such that

$$\alpha_1(t) \leqslant \beta_1(t),\ \alpha_2(t) \leqslant \beta_2(t),\ \forall t \in [a,b].$$

The continuous functions $f,h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy a Nagumo type condition relative to intervals $[\alpha_1(t), \beta_1(t)]$ and $[\alpha_2(t), \beta_2(t)]$, if, there are $N_1 > r_1$, $N_2 > r_2$, with

$$r_1 := \max\left\{ \frac{\beta_1(b) - \alpha_1(a)}{b-a}, \frac{\beta_1(a) - \alpha_1(b)}{b-a} \right\}, \tag{2.1}$$

$$r_2 := \max\left\{ \frac{\beta_2(b) - \alpha_2(a)}{b-a}, \frac{\beta_2(a) - \alpha_2(b)}{b-a} \right\}, \tag{2.2}$$

and continuous positive functions $\varphi, \psi : [0,+\infty) \to (0,+\infty)$, such that

$$|f(t,x,y,z,w)| \leqslant \varphi(|z|), \tag{2.3}$$

and

$$|h(t,x,y,z,w)| \leqslant \psi(|w|), \tag{2.4}$$

for

$$\alpha_1(t) \leqslant x \leqslant \beta_1(t), \alpha_2(t) \leqslant y \leqslant \beta_2(t),\ \forall t \in [a,b], \tag{2.5}$$

verifying

$$\int_{r_1}^{N_1} \frac{ds}{\varphi(s)} > b-a, \int_{r_2}^{N_2} \frac{ds}{\psi(s)} > b-a. \tag{2.6}$$

LEMMA 1. *Let $f,h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous functions satisfying a Nagumo type condition relative to intervals $[\alpha_1(t), \beta_1(t)]$ and $[\alpha_2(t), \beta_2(t)]$.*

Then for every solution $(u,v) \in \left(C^2[a,b]\right)^2$ verifying (2.5), there are $N_1, N_2 > 0$, given by (2.6), such that

$$\|u'\| \leqslant N_1 \text{ and } \|v'\| \leqslant N_2. \tag{2.7}$$

Proof. Let $(u(t), v(t))$ be a solution of (1.1) satisfying (2.5).

By Lagrange Theorem, there are $t_0, t_1 \in [a,b]$ such that

$$u'(t_0) = \frac{u(b) - u(a)}{b-a} \text{ and } v'(t_1) = \frac{v(b) - v(a)}{b-a}.$$

Suppose, by contradiction, that $|u'(t)| > r_1$, $\forall t \in [a,b]$, with r_1 given by (2.1). If $u'(t) > r_1$, $\forall t \in [a,b]$, by (2.5) we obtain the following contradiction with (2.1):

$$u'(t_0) = \frac{u(b) - u(a)}{b-a} \leqslant \frac{\beta_1(b) - \alpha_1(a)}{b-a} \leqslant r_1.$$

If $u'(t) < -r_1$, $\forall t \in [a,b]$, the contradiction is similar.

In the case where $|u'(t)| \leqslant r_1$, $\forall t \in [a,b]$, the proof will be finished.

So, assume that there are $t_2, t_3 \in [a,b]$ such that $t_2 < t_3$, and

$$u'(t_2) < r_1 \text{ and } u'(t_3) > r_1.$$

By continuity, there is $t_4 \in [t_2, t_3]$ such that

$$u'(t_4) = r_1 \text{ and } u'(t_3) > r_1,\ \forall t \in]t_4, t_3].$$

So, by a convenient change of variable, by (2.3), (2.4), and (2.6), we obtain

$$\begin{aligned}\int_{u'(t_4)}^{u'(t_3)} \frac{ds}{\varphi(|s|)} &= \int_{t_4}^{t_3} \frac{u''(t)}{\varphi(|u'(t)|)}dt \leqslant \int_a^b \frac{|u''(t)|}{\varphi(|u'(t)|)}dt \\ &= \int_a^b \frac{|f(t,u(t),v(t),u'(t),v'(t)|}{\varphi(|u'(t)|)}dt \leqslant b-a < \int_{r_1}^{N_1} \frac{ds}{\varphi(|s|)}.\end{aligned}$$

Therefore $u'(t_3) < N_1$, and, as t_3 is taken arbitrarily, $u'(t_3) < N_1$, for values of t where $u'(t) > r_1$.

If $t_2 > t_3$, the technique is analogous for $t_4 \in [t_3, t_2]$.

The same conclusion can be achieved if there are $t_2, t_3 \in [a,b]$ such that

$$u'(t_2) > -r_1 \text{ and } u'(t_3) < -r_1.$$

Therefore $\|u'\| \leqslant N_1$ and, by similar arguments, it can be proved that $\|v'\| \leqslant N_2$. $\square$

For the reader's convenience we present Schauder's fixed point theorem:

THEOREM 1. ([27]) *Let Y be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P as at least one fixed point in Y.*

3. Main result

Along this work we denote $(a,\, b) \leqslant (c,\, d)$ meaning that $a \leqslant c$ and $b \leqslant d$, for $a,b,c,d \in \mathbb{R}$.

THEOREM 2. *Let $f, h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous functions, and assume that hypothesis (A) holds. If there are coupled lower and upper solutions of (1.1), (1.2), (α_1, α_2) and (β_1, β_2), respectively, according Definition 1, such that*

$$(\alpha_1(t), \alpha_2(t)) \leqslant (\beta_1(t), \beta_2(t)), \tag{3.1}$$

and f and h verify the Nagumo conditions relative to intervals $[\alpha_1(t), \beta_1(t)]$ and $[\alpha_2(t), \beta_2(t)]$, then there is at least a pair $(u(t), v(t)) \in \left(C^2[a,b], \mathbb{R}\right)^2$ solution of (1.1), (1.2) and, moreover,

$$\alpha_1(t) \leqslant u(t) \leqslant \beta_1(t),\ \ \alpha_2(t) \leqslant v(t) \leqslant \beta_2(t),\ \forall t \in [a,b].$$

and

$$\left\|u'\right\| \leqslant N_1 \text{ and } \left\|v'\right\| \leqslant N_2,$$

with N_1 *and* N_2 *given by Lemma* 1.

Proof. Consider the auxiliary functions $F(t,x,y,z,w) := F$ defined as

- $f(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)) - \frac{x-\beta_1(t)}{1+|x-\beta_1(t)|} - \frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $x>\beta_1(t),\ y>\beta_2(t)$,
- $f(t,u(t),v(t),u'(t),v'(t)) - \frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $\alpha_1(t)\leqslant x\leqslant\beta_1(t),\ y>\beta_2(t)$,
- $f(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)) - \frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|} + \frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $x<\alpha_1(t),\ y>\beta_2(t)$,
- $f(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)) - \frac{x-\beta_1(t)}{1+|x-\beta_1(t)|}$, if $x>\beta_1(t),\ \alpha_2(t)\leqslant y\leqslant\beta_2(t)$,
- $f(t,u(t),v(t),u'(t),v'(t))$ if $\alpha_1(t)\leqslant x\leqslant\beta_1(t),\ \alpha_2(t)\leqslant y\leqslant\beta_2(t)$,
- $f(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)) - \frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|}$ if $x<\alpha_1(t),\ \alpha_2(t)\leqslant y\leqslant\beta_2(t)$,
- $f(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)) - \frac{x-\beta_1(t)}{1+|x-\beta_1(t)|} + \frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $x>\beta_1(t),\ y<\alpha_2(t)$,
- $f(t,u(t),v(t),u'(t),v'(t)) - \frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $\alpha_1(t)\leqslant x\leqslant\beta_1(t),\ y<\alpha_2(t)$,
- $f(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)) - \frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|} - \frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $x<\alpha_1(t),\ y<\alpha_2(t)$,

and $H(t,x,y,z,w) := H$ given by

- $h(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)) - \frac{x-\beta_1(t)}{1+|x-\beta_1(t)|} - \frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $x>\beta_1(t),\ y>\beta_2(t)$,
- $h(t,u(t),v(t),u'(t),v'(t)) - \frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $\alpha_1(t)\leqslant x\leqslant\beta_1(t),\ y>\beta_2(t)$,
- $h(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)) - \frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|} + \frac{y-\beta_2(t)}{1+|y-\beta_2(t)|}$ if $x<\alpha_1(t),\ y>\beta_2(t)$,
- $h(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)) - \frac{x-\beta_1(t)}{1+|x-\beta_1(t)|}$, if $x>\beta_1(t),\ \alpha_2(t)\leqslant y\leqslant\beta_2(t)$,
- $h(t,u(t),v(t),u'(t),v'(t))$ if $\alpha_1(t)\leqslant x\leqslant\beta_1(t),\ \alpha_2(t)\leqslant y\leqslant\beta_2(t)$.
- $h(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)) - \frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|}$ if $x<\alpha_1(t),\ \alpha_2(t)\leqslant y\leqslant\beta_2(t)$,
- $h(t,\beta_1(t),\beta_2(t),\beta_1'(t),\beta_2'(t)) - \frac{x-\beta_1(t)}{1+|x-\beta_1(t)|} + \frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $x>\beta_1(t),\ y<\alpha_2(t)$,
- $h(t,u(t),v(t),u'(t),v'(t)) - \frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $\alpha_1(t)\leqslant x\leqslant\beta_1(t),\ y<\alpha_2(t)$,
- $h(t,\alpha_1(t),\alpha_2(t),\alpha_1'(t),\alpha_2'(t)) - \frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|} - \frac{y-\alpha_2(t)}{1+|y-\alpha_2(t)|}$ if $x<\alpha_1(t),\ y<\alpha_2(t)$,

and the auxiliary problem

$$\begin{cases} u''(t) = F(t,u(t),v(t),u'(t),v'(t)) \\ v''(t) = H(t,u(t),v(t),u'(t),v'(t)) \\ u(a) = v(a) = 0 \\ u(b) = \delta_1\left(b,u(b) + L_1(u,u(b),u'(b))\right) \\ v(b) = \delta_2\left(b,v(b) + L_2(v,v(b),v'(b))\right), \end{cases} \tag{3.2}$$

where, for each $i = 1,2,$

$$\delta_i(t,w) = \begin{cases} \beta_i(t)\ , & w > \beta_i(t) \\ w\ , & \alpha_i(t) \leqslant w \leqslant \beta_i(t) \\ \alpha_i(t)\ , & w < \alpha_i(t). \end{cases} \tag{3.3}$$

Claim 1: *Solutions of problem* (3.2) *can be written as*

$$\begin{aligned} u(t) &= \frac{t-a}{b-a}\delta_1\left(b,u(b) + L_1(u,u(b),u'(b))\right) \\ &\quad + \int_a^b G(t,s)F(s,u(s),v(s),u'(s),v'(s))ds \\ v(t) &= \frac{t-a}{b-a}\delta_2\left(b,v(b) + L_2(v,v(b),v'(b))\right) \\ &\quad + \int_a^b G(t,s)H(s,u(s),v(s),u'(s),v'(s))ds, \end{aligned}$$

where

$$G(t,s) = \frac{1}{b-a}\begin{cases} (a-s)(b-t), & a \leqslant t \leqslant s \leqslant b \\ (a-t)(b-s), & a \leqslant s \leqslant t \leqslant b. \end{cases} \tag{3.4}$$

In fact, for the equation $u''(t) = F(t)$, the solution is,

$$u(t) = At + B + \int_a^t (t-s)F(s)ds, \tag{3.5}$$

for some $A, B \in \mathbb{R}$.

By the boundary conditions, it follows that,

$$\begin{aligned} A &= \frac{1}{b-a}\delta_1\left(b,u(b) + L_1(u,u(b),u'(b))\right) - \frac{1}{b-a}\int_a^b (b-s)F(s)ds \\ B &= -\frac{a}{b-a}\delta_1\left(b,u(b) + L_1(u,u(b),u'(b))\right) + \frac{a}{b-a}\int_a^b (b-s)F(s)ds. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), then

$$\begin{aligned}
u(t) &= \frac{t-a}{b-a}\delta_1\left(b,u(b)+L_1(u,u(b),u'(b))\right) \\
&\quad -\frac{t-a}{b-a}\int_a^b (b-s)F(s)ds+\int_a^t (t-s)F(s)ds \\
&= \frac{t-a}{b-a}\delta_1\left(b,u(b)+L_1(u,u(b),u'(b))\right) \\
&\quad +\int_a^t \left(\frac{(a-t)(b-s)}{b-a}+t-s\right)F(s)ds+\int_t^b \frac{(a-t)(b-s)}{b-a}F(s)ds \\
&= \frac{t-a}{b-a}\delta_1\left(b,u(b)+L_1(u,u(b),u'(b))\right) \\
&\quad +\int_a^t \left(\frac{(a-s)(b-t)}{b-a}+t-s\right)F(s)ds+\int_t^b \frac{(a-t)(b-s)}{b-a}F(s)ds \\
&= \frac{t-a}{b-a}\delta_1\left(b,u(b)+L_1(u,u(b),u'(b))\right)+\int_a^t G(t,s)F(s)ds,
\end{aligned}$$

with $G(t,s)$ given by (3.4).

The integral form of $v(t)$ can be achieved by the same arguments.

Claim 2: *Every solution* (u,v) *of* (3.2) *satisfies*

$$\left\|u'\right\| \leqslant N_1 \text{ and } \left\|v'\right\| \leqslant N_2,$$

with N_1 *and* N_2 *given by Lemma* 1.

This claim is a direct consequence of Lemma 1, as $F(t,x,y,z,w)$ and $H(t,x,y,z,w)$ are defined on bounded arguments of x and y.

Define the operators $T_1 : \left(C^1[a,b]\right)^2 \to C^1[a,b]$ and $T_2 : \left(C^1[a,b]\right)^2 \to C^1[a,b]$ such that

$$\begin{cases}
T_1(u,v)(t) = \frac{t-a}{b-a}\delta_1\left(b,u(b)+L_1(u,u(b),u'(b))\right) \\
\qquad\qquad\quad +\int_a^b G(t,s)F(s,u(s),v(s),u'(s),v'(s))ds \\
T_2(u,v)(t) = \frac{t-a}{b-a}\delta_2\left(b,v(b)+L_2(v,v(b),v'(b))\right) \\
\qquad\qquad\quad +\int_a^b G(t,s)H(s,u(s),v(s),u'(s),v'(s))ds,
\end{cases} \tag{3.7}$$

where $G(t,s)$ is given by (3.4), and $T : \left(C^1[a,b]\right)^2 \to \left(C^1[a,b]\right)^2$ by

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)). \tag{3.8}$$

By Claim 1, fixed points of the operator $T := (T_1,T_2)$ are solutions of problem (3.2) .

Claim 3: *The operator* T, *given by* (3.8) *has a fixed point* (u_0,v_0).

In order to apply Theorem 1, we will prove the following steps for operator $T_1(u,v)$. The proof for the operator $T_2(u,v)$ is analogous.

(i) $T_1 : \left(C^1[a,b]\right)^2 \to C^1[a,b]$ *is well defined.*

The function F is bounded and the Green function $G(t,s)$ is continuous in $[a,b]^2$, then the operator $T_1(u,v)$ is continuous. Moreover, as $\frac{\partial G}{\partial t}(t,s)$ is bounded in $[a,b]^2$ and

$$\begin{aligned}(T_1(u,v))'(t) = {} & \frac{1}{b-a}\delta_1\left(b,u(b)+L_1(u,u(b),u'(b))\right) \\ & + \int_a^b \frac{\partial G}{\partial t}(t,s)\, F(s,u(s),v(s),u'(s),v'(s))ds,\end{aligned}$$

with

$$\frac{\partial G}{\partial t}(t,s) = \frac{1}{b-a}\begin{cases} s-a, & a \leqslant t \leqslant s \leqslant b \\ s-b, & a \leqslant s \leqslant t \leqslant b,\end{cases}$$

verifying

$$\left|\frac{\partial G}{\partial t}(t,s)\right| \leqslant 1,\ \forall (t,s) \in [a,b]^2,$$

therefore, $(T_1(u,v))'$ is continuous on $[a,b]$. So, $T_1 \in C^1[a,b]$.

(ii) *TB is uniformly bounded, for B a bounded set in $\left(C^1[a,b]\right)^2$.*

Let B be a bounded set of $\left(C^1[a,b]\right)^2$. Then there exists $K > 0$ such that

$$\|(u,v)\|_{E^2} = \|u\|_{C^1} + \|v\|_{C^1} \leqslant K,\ \forall (u,v) \in B.$$

By (3.3), and taking into account that F and H are bounded, then there are M_1, M_2, $M_3 > 0$ such that

$$\begin{gathered}\delta_1 \leqslant \max\{\|\alpha_1\|, \|\beta_1\|\} := M_1, \\ \int_a^b \max_{t\in[a,b]} |G(t,s)| \left|F(s,u(s),v(s),u'(s),v'(s))\right| ds \leqslant M_2, \\ \int_a^b \max_{t\in[a,b]} \left|\frac{\partial G}{\partial t}(t,s)\right| \left|F(s,u(s),v(s),u'(s),v'(s))\right| ds \leqslant M_3.\end{gathered}$$

Moreover,

$$\begin{aligned}\|T_1(u,v)(t)\| = {} & \max_{t\in[a,b]}\left|\frac{t-a}{b-a}\delta_1\left(b,u(b)+L_1(u,u(b),u'(b))\right)\right. \\ & \left. + \int_a^b G(t,s)F(s,u(s),v(s),u'(s),v'(s))ds\right| \\ \leqslant {} & \max_{t\in[a,b]}\left|\frac{t-a}{b-a}\right| \left|\delta_1\left(b,u(b)+L_1(u,u(b),u'(b))\right)\right| \\ & + \int_a^b \max_{t\in[a,b]} |G(t,s)| \left|F(s,u(s),v(s),u'(s),v'(s))\right| ds \\ \leqslant {} & M_1 + M_2 < +\infty,\ \forall (u,v) \in B,\end{aligned}$$

and

$$\begin{aligned}\|(T_1(u,v))'(t)\| &= \max_{t\in[a,b]}\left|\frac{1}{b-a}\delta_1\left(b,u(b)+L_1(u,u(b),u'(b))\right)\right.\\&\quad\left.+\int_a^b \frac{\partial G}{\partial t}(t,s)F(s,u(s),v(s),u'(s),v'(s))ds\right|\\&\leqslant \frac{M_1}{b-a}+\int_a^b \max_{t\in[a,b]}\left|\frac{\partial G}{\partial t}(t,s)\right|\left|F(s,u(s),v(s),u'(s),v'(s))\right|ds\\&\leqslant \frac{M_1}{b-a}+M_3<+\infty,\ \forall (u,v)\in B.\end{aligned}$$

So, TB is uniformly bounded, for B a bounded set in $\left(C^1[a,b]\right)^2$.

(iii) *TB is equicontinuous on* $\left(C^1[a,b]\right)^2$.

Let t_1 and $t_2\in[a,b]$. Without loss of generality suppose $t_1\leqslant t_2$. As $G(t,s)$ is uniformly continuous and F is bounded, then

$$\begin{aligned}&|T_1(u,v)(t_1)-T_1(u,v)(t_2)|\\&=\left|\frac{(t_1-a)-(t_2-a)}{b-a}\delta_1\left(b,u(b)+L_1(u,u(b),u'(b))\right)\right.\\&\quad\left.+\int_a^b[G(t_1,s)-G(t_2,s)]F(s,u(s),v(s),u'(s),v'(s))ds\right|\\&\leqslant\left|\frac{t_1-t_2}{b-a}\right|M_1+\left|\int_a^b[G(t_1,s)-G(t_2,s)]\right|F(s,u(s),v(s),u'(s),v'(s))ds\\&\to 0,\text{ as } t_1\to t_2,\end{aligned}$$

and

$$\begin{aligned}&\left|(T_1(u,v)(t_1))'-(T_1(u,v)(t_2))'\right|\\&=\left|\int_a^b\left[\frac{\partial G}{\partial t}(t_1,s)-\frac{\partial G}{\partial t}(t_2,s)\right]F(s,u(s),v(s),u'(s),v'(s))ds\right|\\&\leqslant\int_a^{t_1}\left|\frac{\partial G}{\partial t}(t_1,s)-\frac{\partial G}{\partial t}(t_2,s)\right|\left|F(s,u(s),v(s),u'(s),v'(s))\right|ds\\&\quad+\int_{t_1}^{t_2}\left|\frac{\partial G}{\partial t}(t_1,s)-\frac{\partial G}{\partial t}(t_2,s)\right|\left|F(s,u(s),v(s),u'(s),v'(s))\right|ds\\&\quad+\int_{t_2}^{b}\left|\frac{\partial G}{\partial t}(t_1,s)-\frac{\partial G}{\partial t}(t_2,s)\right|\left|F(s,u(s),v(s),u'(s),v'(s))\right|ds.\end{aligned}$$

As the function $\frac{\partial G}{\partial t}(t,s)$ has only a jump discontinuity at $t=s$, therefore, as previously, the first and third integrals tend to 0, as $t_1\to t_2$. For the second integral, as the functions $\frac{\partial G}{\partial t}(t_1,s)$ and $\frac{\partial G}{\partial t}(t_2,s)$ are uniformly continuous, for $s\in[a,t_1[\cup]t_1,b]$ and $s\in[a,t_2[\cup]t_2,b]$, respectively, and

$$\left|\frac{\partial G}{\partial t}(t_1,s)-\frac{\partial G}{\partial t}(t_2,s)\right|\left|F(s,u(s),v(s),u'(s),v'(s))\right|$$

is bounded, then

$$\int_{t_1}^{t_2} \left| \frac{\partial G}{\partial t}(t_1,s) - \frac{\partial G}{\partial t}(t_2,s) \right| \left| F(s,u(s),v(s),u'(s),v'(s)) \right| ds \to 0, \text{ as } t_1 \to t_2.$$

By the Arzèla-Ascoli Theorem $T_1(u,v)$ is compact in $\left(C^1[a,b]\right)^2$.

Following similar arguments with K_1, K_2, $K_3 > 0$ such that

$$\begin{gathered} \delta_2 \leqslant \max\{\|\alpha_2\|, \|\beta_2\|\} := K_1, \\ \int_a^b \max_{t\in[a,b]} |G(t,s)| \left| H(s,u(s),v(s),u'(s),v'(s)) \right| ds \leqslant K_2, \\ \int_a^b \max_{t\in[a,b]} \left| \frac{\partial G}{\partial t}(t,s) \right| \left| H(s,u(s),v(s),u'(s),v'(s)) \right| ds \leqslant K_3, \end{gathered}$$

it can be shown that $T_2(u,v)$ is compact in $\left(C^1[a,b]\right)^2$, too.

(iv) $TD \subset D$ for some $D \subset \left(C^1[a,b]\right)^2$ *a closed and bounded set.*

Suppose $D \subset \left(C^1[a,b]\right)^2$ defined by

$$D = \left\{ (u,v) \in \left(C^1[a,b]\right)^2 : \|(u,v)\|_{E^2} \leqslant 2\rho \right\},$$

where ρ is such that

$$\rho := \max\left\{ M_1 + M_2, \frac{M_1}{b-a} + M_3, K_1 + K_2, \frac{K_1}{b-a} + K_3, N_1, N_2 \right\},$$

with N_1, N_2 given by ((2.7)).

Arguing as in Claim 3 (ii), it can be shown that

$$\begin{gathered} \|T_1(u,v)\| \leqslant M_1 + M_2 \leqslant \rho, \\ \left\|(T_1(u,v))'\right\| \leqslant \frac{M_1}{b-a} + M_3 \leqslant \rho \end{gathered}$$

and, therefore, $\|T_1(u,v)\|_{C^1} \leqslant \rho$.

Analogously $\|T_2(u,v)\|_{C^1} \leqslant \rho$ and, so,

$$\begin{aligned} \|T(u,v)\|_{E^2} &= \|(T_1(u,v), T_2(u,v))\|_{E^2} \\ &= \|T_1(u,v)\|_{C^1} + \|T_2(u,v)\|_{C^1} \leqslant 2\rho. \end{aligned}$$

By Theorem 1, the operator T, given by (3.8) has a fixed point (u_0, v_0).

Claim 4: *This fixed point* (u_0, v_0) *is also solution of the initial problem* (1.1), (1.2), *if every solution of* (3.2) *verifies*

$$\alpha_1(t) \leqslant u_0(t) \leqslant \beta_1(t),\ \alpha_2(t) \leqslant v_0(t) \leqslant \beta_2(t),\ \forall t \in [a,b],$$

$$\alpha_1(b) \leqslant u_0(b) + L_1(u_0, u_0(b), u_0'(b)) \leqslant \beta_1(b), \tag{3.9}$$

$$\alpha_2(b) \leqslant v_0(b) + L_2(v_0, v_0(b), v_0'(b)) \leqslant \beta_2(b). \tag{3.10}$$

Let (u_0, v_0) be a fixed point of T, that is (u_0, v_0) is a fixed point of T_1 and T_2.

By Claim 1, (u_0, v_0) is solution of problem (3.2).

In the following we will prove the estimations for u_0, as for v_0 the procedure is analogous.

Suppose, by contradiction, that the first inequality is not true. So, there exists $t \in [a,b]$ such that $\alpha_1(t) > u_0(t)$ and it can be defined

$$\max_{t\in[a,b]} (\alpha_1(t) - u_0(t)) := \alpha_1(t_0) - u_0(t_0) > 0. \tag{3.11}$$

Remark that, by (3.2), Definition 1 and (3.3), $t_0 \neq a$, as $\alpha_1(a) - u_0(a) \leqslant 0$, and $t_0 \neq b$, because

$$\alpha_1(b) - u_0(b) = \alpha_1(b) - \delta_1\left(b, u_0(b) + L_1(u_0, u_0(b), u_0'(b))\right) \leqslant 0.$$

Then $t_0 \in]a,b[$,

$$\alpha_1'(t_0) - u_0'(t_0) = 0 \text{ and } \alpha_1''(t_0) - u_0''(t_0) \leqslant 0. \tag{3.12}$$

There are three possibilities for the value of $v_0(t_0)$:

- If $v_0(t_0) > \beta_2(t_0)$, then, by (3.2) and Definition 1, the following contradiction with (3.12) is obtained

$$\begin{aligned}
u_0''(t_0) &= F(t_0, u_0(t_0), v_0(t_0), u_0'(t_0), v_0'(t_0)) \\
&= f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) - \frac{u_0(t_0) - \alpha_1(t_0)}{1 + |u_0(t_0) - \alpha_1(t_0)|} \\
&\quad + \frac{v_0(t_0) - \beta_2(t_0)}{1 + |v_0(t_0) - \beta_2(t_0)|} \\
&> f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) \geqslant \alpha_1''(t_0).
\end{aligned}$$

- If $\alpha_2(t_0) \leqslant v_0(t_0) \leqslant \beta_2(t_0)$, then

$$\begin{aligned}
u_0''(t_0) &= F(t_0, u_0(t_0), v_0(t_0), u_0'(t_0), v_0'(t_0)) \\
&= f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) - \frac{u_0(t_0) - \alpha_1(t_0)}{1 + |u_0(t_0) - \alpha_1(t_0)|} \\
&> f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) \geqslant \alpha_1''(t_0).
\end{aligned}$$

- If $\alpha_2(t_0) < v_0(t_0)$, the contradiction is

$$\begin{aligned}
u_0''(t_0) &= F(t_0, u_0(t_0), v_0(t_0), u_0'(t_0), v_0'(t_0)) \\
&= f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) - \frac{u_0(t_0) - \alpha_1(t_0)}{1 + |u_0(t_0) - \alpha_1(t_0)|} \\
&\quad + \frac{v_0(t_0) - \alpha_2(t_0)}{1 + |v_0(t_0) - \alpha_2(t_0)|} \\
&> f\left(t_0, \alpha_1(t_0), \alpha_2(t_0), \alpha_1'(t_0), \alpha_2'(t_0)\right) \geqslant \alpha_1''(t_0).
\end{aligned}$$

Therefore $\alpha_1(t) \leqslant u_0(t), \ \forall t \in [a,b]$.

By a similar technique it can be shown that $u_0(t) \leqslant \beta_1(t), \ \forall t \in [a,b]$, and so,

$$\alpha_1(t) \leqslant u_0(t) \leqslant \beta_1(t), \forall t \in [a,b]. \tag{3.13}$$

Assume now that, to prove the first inequality of (3.9),

$$\alpha_1(b) > u_0(b) + L_1(u_0, u_0(b), u_0'(b)). \tag{3.14}$$

Then, by (3.2) and (3.3),

$$u_0(b) = \delta_1\left(b, u_0(b) + L_1(u_0, u_0(b), u_0'(b))\right) = \alpha_1(b),$$

and

$$u_0'(b) \leqslant \alpha_1'(b).$$

By (3.14), (A) and Definition 1, we have the contradiction

$$\begin{aligned} 0 &> L_1(u_0, u_0(b), u_0'(b)) + u_0(b) - \alpha_1(b) \\ &\geqslant L_1(u_0, u_0(b), u_0'(b)) \geqslant L_1(\alpha_1, \alpha_1(b), \alpha_1'(b)) \geqslant 0. \end{aligned}$$

Then $\alpha_1(b) \leqslant u_0(b) + L_1(u_0, u_0(b), u_0'(b))$.

To prove the second inequality of (3.9), assume that

$$u_0(b) + L_1(u_0, u_0(b), u_0'(b)) > \beta_1(b). \tag{3.15}$$

Then, by (3.2) and (3.3),

$$u_0(b) = \delta_1\left(b, u_0(b) + L_1(u_0, u_0(b), u_0'(b))\right) = \beta_1(b), \tag{3.16}$$

and

$$u_0'(b) \geqslant \beta_1'(b).$$

By (3.14), (3.16), (A) and Definition 1, we have the contradiction

So, $u_0(b) + L_1(u_0, u_0(b), u_0'(b)) \leqslant \beta_1(b)$, and, therefore, (3.9) holds.

To prove (3.10) the technique is analogous.

So, the fixed point (u_0, v_0) of T, solution of problem (3.2), is a solution of problem (1.1), (1.2), too. □

4. Example

Consider the boundary value problem composed by the coupled system of the second order differential equations with full nonlinearities

$$\begin{cases} u''(t) = -u'(t)\,v(t) + \arctan(u(t)\,v'(t)) + t, \\ v''(t) = t^2\left[-e^{-|u'(t)|}\,v(t) + u(t)\,(v'(t) - 2)\right] \end{cases} \tag{4.1}$$

with $t \in [0,1]$, and the functional boundary conditions

$$\begin{cases} u(0) = v(0) = 0 \\ u(1) = \max\limits_{t\in[0,1]} u(t) - (u'(1))^2 \\ v(1) = 2\int\limits_0^1 v(s)ds - \frac{(v'(1))^3}{8}. \end{cases} \tag{4.2}$$

This problem is a particular case of system (1.1), (1.2) with

$$\begin{aligned} f(t,x,y,z,w) &= -z\,y + \arctan(x\,w) + t, \\ h(t,x,y,z,w) &= t^2\left[-e^{-|z|}\,y + x\,(w-2)\right], \end{aligned} \tag{4.3}$$

continuous functions, $t \in [0,1]$, $a = 0,\ b = 1,$ and

$$\begin{aligned} L_1(w,x,y) &= x - \max_{t\in[0,1]} w(t) + y^2, \\ L_2(w,x,y) &= x - 2\int\limits_0^1 w(s)ds + \frac{y^3}{8}. \end{aligned} \tag{4.4}$$

Remark that, L_1 and L_2 are continuous functions, verifying (A).

The functions given by

$$(\alpha_1(t),\, \alpha_2(t)) = (-1,\, -1) \text{ and } (\beta_1(t),\, \beta_2(t)) = (3+t,\, 2t+2)$$

are, respectively, lower and upper solutions of problem (4.1)–(4.2), satisfying (3.1), because, by Definition (1), we have

$$\begin{aligned} \alpha_1''(t) &= 0 \leqslant f(t,-1,-1,0,0) = t,\ \ t\in[0,1] \\ \alpha_2''(t) &= 0 \leqslant h(t,-1,-1,0,0) = 3t^2,\ \ t\in[0,1] \\ L_1(\alpha_1,\alpha_1(1),\alpha_1'(1)) &= 0 \geqslant 0 \\ L_2(\alpha_2,\alpha_2(b),\alpha_2'(b)) &= \frac{7}{8} \geqslant 0, \end{aligned}$$

and

$$\begin{aligned} \beta_1(t)''(t) = 0 &\geqslant f(t,3+t,2t+2,1,2) \\ &= -(2t+2) + \arctan(6t+2) + t,\ \ t\in[0,1] \\ \beta_2''(t) = 0 &\geqslant h(t,3+t,2t+2,1,2) = -\frac{2t^3+2t^2}{e},\ \ t\in[0,1] \\ L_1(\beta_1,\beta_1(1),\beta_1'(1)) &= 0 \leqslant 0 \\ L_2(\beta_2,\beta_2(b),\beta_2'(b)) &= -1 \leqslant 0. \end{aligned}$$

Furthermore, the functions f and h, given by (4.3), satisfy a Nagumo condition relative to the intervals $[-1, 2+t]$ and $[-1, 2t+1]$, for $t \in [0,1]$, with $r_1 = r_2 = 4$,

$$|f(t,x,y,z,w)| \leqslant 3\,|z| + \frac{\pi}{2} + 1 := \varphi(|z|),$$

$$|h(t,x,y,z,w)| \leqslant 3\,(1+|w-2|) := \psi\,(|w|)\,,$$

and

$$\int_4^{N_1} \frac{ds}{\varphi(|s|)} = \int_4^{N_1} \frac{1}{3s+\frac{\pi}{2}+1} ds > 1,$$

for $N_1 \geqslant 100$ and

$$\int_4^{N_2} \frac{ds}{\psi(|s|)} = \int_4^{N_2} \frac{1}{3\,(1+|s-2|)\,ds} > 1,$$

for $N_2 \geqslant 80$.

Then, by Theorem (2), there is at least a pair $(u(t),v(t)) \in \left(C^2[0,1],\mathbb{R}\right)^2$ solution of (4.1), (4.2) and, moreover,

$$-1 \leqslant u(t) \leqslant 2+t, \quad -1 \leqslant v(t) \leqslant 2t+1, \ \forall t \in [0,1],$$
$$\left\|u'\right\| \leqslant N_1 \text{ and } \left\|v'\right\| \leqslant N_2.$$

5. Coupled mass-spring system

Consider the mass-spring system composed by two springs with constants of proportionality k_1 and k_2, and two weights of mass m_1 and m_2. The mass m_1 is suspended vertically from a fixed support by a spring with constant k_1 and the mass m_2 is attached to the first weight by a spring with constant k_2. The system described is illustrated in the Figure 1.

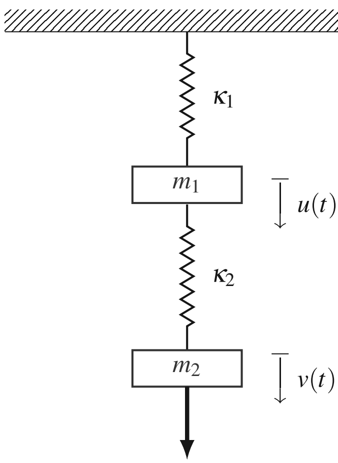

Figure 1: *The coupled springs.*

Let us call $u(t)$ and $v(t)$ the displacements of the weights of mass m_1 and m_2, respectively, in relation to their respective equilibrium positions. Thus, at time t, the position of the displacement of the mass m_1 is $u(t)$ and the displacement of mass m_2 is $v(t)$.

For simplicity we consider $t \in [0,1]$, and, therefore, $u(0)$ and $u(1)$ are the initial and final displacement of mass m_1, and $v(0)$ and $v(1)$ are the similar displacements of mass m_2.

As it can be seen in [8], the above system is modelled by the second order nonlinear system of differential equations forced and with friction

$$\begin{cases} m_1 u''(t) = -\delta_1 u'(t) - \kappa_1 u(t) + \mu_1 (u(t))^3 - \kappa_2 (u(t) - v(t)) \\ \qquad\qquad + \mu_2 (u(t) - v(t))^3 + F_1 \cos(\omega_1 t), \\ \\ m_2 v''(t) = -\delta_2 v'(t) - \kappa_2 (v(t) - u(t)) + \mu_2 (v(t) - u(t))^3 \\ \qquad\qquad + F_2 \cos(\omega_2 t) \end{cases} \tag{5.1}$$

where $t \in [0,\, 1]$,

- δ_1, δ_2 are the damping coefficients;
- μ_1, μ_2 are the coefficients of the nonlinear terms of each system equation;
- $\kappa_2(u(t) - v(t)) + \mu_2(u(t) - v(t))^3$ and $\kappa_2(v(t) - u(t)) + \mu_2(v(t) - u(t))^3$ are the nonlinear restoring forces;
- F_1, F_2 are the forcing amplitudes of the sinusoidal forces $F_1 \cos(\omega_1 t)$ and $F_2 \cos(\omega_2 t)$, where ω_1, ω_2 are the forcing frequencies.

In this work we add to the system the functional boundary conditions

$$\begin{cases} u(0) = v(0) = 0 \\ u(1) = \max\limits_{t \in [0,1]} u(t) + 2u'(1) \\ v(1) = \max\limits_{t \in [0,1]} v(t) + 2\,(v'(1))^3 . \end{cases} \tag{5.2}$$

The functional conditions (5.2) can have a physical meaning such as, for example, the first one can be seen as the displacement of mass 1 at the final moment given by the sum of the maximum displacement in this period of time, with the double of the velocity of the displacement at the end point.

Clearly, the above model (5.1), (5.2) is a particular case of system (1.1), (1.2) with

$$\begin{aligned} f(t,x,y,z,w) &= \tfrac{1}{m_1} \begin{bmatrix} -\delta_1 z - \kappa_1 x + \mu_1 x^3 - \kappa_2 (x-y) + \mu_2 (x-y)^3 \\ +F_1 \cos(\omega_1 t) \end{bmatrix}, \\ h(t,x,y,z,w) &= \tfrac{1}{m_2} \left[-\delta_2 w - \kappa_2 (y-x) + \mu_2 (y-x)^3 + F_2 \cos(\omega_2 t) \right]. \end{aligned} \tag{5.3}$$

These functions are continuous in $[0,1] \times \mathbb{R}^4$, and

$$\begin{aligned} L_1(w,x,y) &= x - \max_{t \in [0,1]} w(t) - 2y, \\ L_2(w,x,y) &= x - \max_{t \in [0,1]} w(t) - 2y^3 \end{aligned} \tag{5.4}$$

verify (A).

The functions given by

$$(\alpha_1(t), \alpha_2(t)) = (-t, -t) \text{ and } (\beta_1(t), \beta_2(t)) = (t, t)$$

are, respectively, lower and upper solutions of problem (5.1), (5.2), satisfying (3.1), for every positive $m_1,\ m_2$, non negative $\delta_1,\ \delta_2,\ F_1,\ F_2,\ \omega_1,\ \omega_2$, and any real $\kappa_1,\ \kappa_2,\ \mu_1,\ \mu_2$, such that

$$\begin{aligned} F_1 &\leqslant \delta_1, \\ F_2 &\leqslant \delta_2, \\ \mu_1 &\leqslant 0. \end{aligned} \tag{5.5}$$

Indeed, Definition 1 holds because,

$$\begin{aligned}
\alpha_1''(t) = 0 &\leqslant \frac{1}{m_1}(\delta_1 - F_1) \\
&\leqslant \frac{1}{m_1}\left[\delta_1 - \mu_1 t^3 + F_1 \cos(\omega_1 t)\right],\ \forall t \in [0,\, 1] \\
&\leqslant \frac{1}{m_1}\left[\delta_1 + \kappa_1 t - \mu_1 t^3 + F_1 \cos(\omega_1 t)\right],\ \forall t \in [0,\, 1] \\
\alpha_2''(t) = 0 &\leqslant \frac{1}{m_2}(\delta_2 - F_2) \leqslant \frac{1}{m_2}\left[\delta_2 + F_2 \cos(\omega_2 t)\right],\ \forall t \in [0,\, 1]
\end{aligned}$$

$$\begin{aligned}
L_1(\alpha_1, \alpha_1(1), \alpha_1'(1)) &= \alpha_1(1) - \max_{t\in[0,1]} \alpha_1(t) - 2\alpha_1'(1) = 1 \geqslant 0 \\
L_2(\alpha_2, \alpha_2(b), \alpha_2'(b)) &= \alpha_2(1) - \max_{t\in[0,1]} \alpha_2(t) - 2(\alpha_2'(1))^3 = 1 \geqslant 0,
\end{aligned}$$

and

$$\begin{aligned}
\beta_1''(t) = 0 &\geqslant \frac{1}{m_1}(-\delta_1 + F_1) \\
&\geqslant \frac{1}{m_1}\left[-\delta_1 - \kappa_1 t + \mu_1 t^3 + F_1 \cos(\omega_1 t)\right],\ \forall t \in [0,\, 1] \\
\beta_2''(t) = 0 &\geqslant \frac{1}{m_2}(-\delta_2 + F_2) \geqslant \frac{1}{m_2}\left[-\delta_2 + F_2 \cos(\omega_2 t)\right],\ \forall t \in [0,\, 1]
\end{aligned}$$

$$\begin{aligned}
L_1(\beta_1, \beta_1(1), \beta_1'(1)) &= \beta_1(1) - \max_{t\in[0,1]} \beta_1(t) - 2\beta_1'(1) = -2 \leqslant 0 \\
L_2(\beta_2, \beta_2(b), \beta_2'(b)) &= \beta_2(1) - \max_{t\in[0,1]} \beta_2(t) - 2(\beta_2'(1))^3 = -2 \leqslant 0.
\end{aligned}$$

Furthermore, f and h, given by (5.3), verify a Nagumo condition relative to the interval $[-t,t]$, for $t \in [0,\, 1]$, with $r_1 = r_2 = 1$, for

$$\begin{aligned}
&-t \leqslant x \leqslant t,\ \ -t \leqslant y \leqslant t, \forall t \in [0,1], \\
|f(t,x,y,z,w)| &\leqslant \frac{1}{m_1}\left(\delta_1 |z| + \kappa_1 + |\mu_1| + 2\kappa_2 + 8|\mu_2| + F_1\right) := \varphi(|z|), \\
|h(t,x,y,z,w)| &\leqslant \frac{1}{m_2}\left(\delta_2 |w| + 2\kappa_2 + 8|\mu_2| + F_2\right) := \psi(|w|),
\end{aligned}$$

and for N_1 and N_2 positive, and large enough such that

$$\int_1^{N_1} \frac{ds}{\varphi(s)} = \int_1^{N_1} \left(\frac{m_1}{\delta_1 s + \kappa_1 + |\mu_1| + 2\kappa_2 + 8|\mu_2| + F_1} \right) ds > 1,$$

and

$$\int_1^{N_2} \frac{ds}{\psi(s)} = \int_1^{N_2} \left(\frac{m_2}{\delta_2 s + 2\kappa_2 + 8|\mu_2| + F_2} \right) ds > 1.$$

So, by Theorem 2, there is a solution $(u(t), v(t))$ of the mass-spring system (5.1), (5.2), for the values of coefficients verifying (5.5), such that

$$-t \leqslant u(t) \leqslant t,$$
$$-t \leqslant v(t) \leqslant t, \forall t \in [0,1],$$

that is, both with values in the strip and

$$\|u'\| \leqslant N_1 \text{ and } \|v'\| \leqslant N_2. \quad \square$$

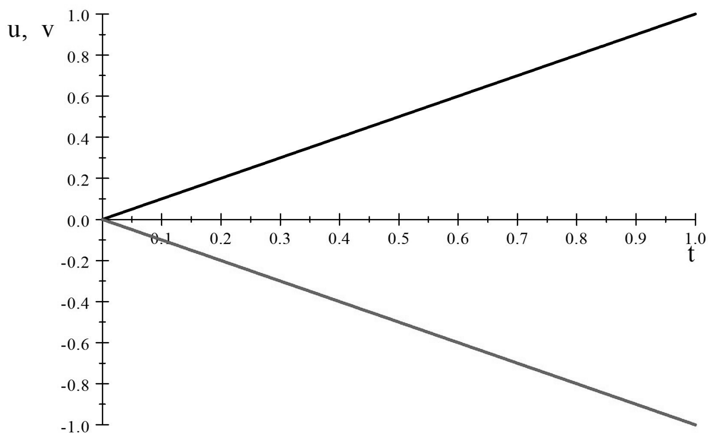

Figure 2: $u(t)$, $v(t)$ *in the strip*

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