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POSITIVE SOLUTIONS FOR A CLASS OF FRACTIONAL DIFFERENCE BOUNDARY VALUE PROBLEMS

JIAFA XU, DONAL O'REGAN, CHENGMIN HOU AND YUJUN CUI

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Abstract. In this paper using the fixed point index and the Leggett-Williams fixed point theorem we establish the existence and multiplicity of positive solutions for a class of fractional difference boundary value problems.

1. Introduction

In this paper we study the existence and multiplicity of positive solutions for the fractional difference boundary value problem

$$\begin{cases} -\Delta_{\nu-3}^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), \quad t \in [0, b+2]_{\mathbb{N}_0}, \\ y(\nu-3) = [\Delta_{\nu-3}^{\alpha} y(t)]|_{t=\nu-\alpha-2} = [\Delta_{\nu-3}^{\beta} y(t)]|_{t=\nu+b+2-\beta} = 0, \end{cases} \tag{1.1}$$

where $2 < \nu \leqslant 3$, $1 < \beta < 2$, $\nu - \beta > 1$, $0 < \alpha < 1$, $b > 3$ $(b \in \mathbb{N})$, $\Delta_{\nu-3}^{\nu}$ is a discrete fractional operator. For the nonlinear term f, we assume that

(H0) $f(t+\nu-1, \cdot) : [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function $(\mathbb{R}^+ := [0, +\infty))$.

Note that, in this paper we use $[a,b]_{\mathbb{N}_a}$ to represent $\{a, a+1, a+2, \cdots, b\}$ $(b - a \in \mathbb{N}_1)$, where $\mathbb{N}_a := \{a, a+1, a+2, \cdots\}$.

REMARK 1. If we delete α in (1.1), then $[\Delta_{\nu-3}^{\alpha} y(t)]|_{t=\nu-\alpha-2}$ is changed to $y(\nu-2)$. Indeed, the fractional condition at $\nu-\alpha-2$ and the Dirichlet condition at $\nu-2$ are equivalent. Note

$$\begin{aligned} \left[\Delta_{\nu-3}^{\alpha} y(t)\right]\Big|_{t=\nu-\alpha-2} &= \left[\frac{1}{\Gamma(-\alpha)} \sum_{s=\nu-3}^{t+\alpha} (t-\alpha-1)^{\underline{-\alpha-1}} y(s)\right]\Big|_{t=\nu-\alpha-2} \\ &= \frac{1}{\Gamma(-\alpha)} \sum_{s=\nu-3}^{\nu-2} (\nu-\alpha-s-3)^{\underline{-\alpha-1}} y(s) \\ &= \frac{1}{\Gamma(-\alpha)} (-\alpha)^{\underline{-\alpha-1}} y(\nu-3) + y(\nu-2). \end{aligned}$$

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As a result, $y(\nu-3)=0$ implies that $\left[\Delta^{\alpha}_{\nu-3}y(t)\right]\Big|_{t=\nu-\alpha-2}=y(\nu-2)$. Consequently, in (1.1), $[\Delta^{\alpha}_{\nu-3}y(t)]|_{t=\nu-\alpha-2}=0$ could be replaced by $y(\nu-2)=0$.

In [1, 2, 3, 4], the authors developed the fundamental theory of discrete delta and nabla fractional calculus and applications to various difference equations were presented in the literature (see for example [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and the references therein). In [5] the author studied the discrete fractional boundary value problems of order less than one

$$\begin{cases}({}_{\alpha-1}\Delta^{\alpha}u)(t)=f(t+\alpha-1,u(t+\alpha-1)), & t\in[0,T]_{\mathbb{N}}, \quad \alpha\in(0,1],\\ au(\alpha-1)+bu(\alpha+T)=c,\end{cases}$$

using the Banach fixed point theorem, and in [6, 7, 8] the authors used a similar method to study existence and uniqueness of solutions for some boundary value problems of fractional difference equations. In [9] the authors used the Guo-Krasnoselskii's fixed point theorem in a cone to study existence of positive solutions for the three-point boundary value problem of the nonlinear Caputo fractional difference equation

$$\begin{cases}\Delta^{\alpha}_{C}u(t)+a(t+\alpha-1)f(u(\theta(t+\alpha-1)))=0, & t\in\mathbb{N}_{0,T},\\ u(\alpha-3)=\Delta^{2}u(\alpha-3)=0,\\ u(T+\alpha)=\lambda\Delta^{-\beta}u(\eta+\beta),\end{cases}$$

under the superlinear and sublinear conditions:

$$f_0=\lim_{u\to0^+}\frac{f(u)}{u}=0 \quad \text{or} \quad \infty, \qquad f_\infty=\lim_{u\to\infty}\frac{f(u)}{u}=\infty \quad \text{or} \quad 0. \tag{1.2}$$

In [10, 11, 12, 13, 19] the authors used fixed point theorems and condition (1.2) to study many types of discrete fractional boundary value problems with nonnegative and semipositone nonlinearities.

In this paper, we use the fixed point index to obtain three existence and multiplicity theorems of positive solutions with a nonnegative nonlinearity. Our growth conditions on the nonlinearity improves that in (1.2) (see conditions (H1)–(H4) in section 3). We also use the Leggett-Williams fixed point theorem to obtain a result of twin positive solutions with a semipositone nonlinearity.

2. Preliminaries

We introduce some background materials from discrete fractional calculus; for more details we refer the reader to [1, 2, 3, 4, 16].

DEFINITION 1. We define $t^{\underline{\nu}}:=\frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for any $t,\nu\in\mathbb{R}$ for which the right-hand side is well-defined. We use the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}}=0$.

DEFINITION 2. For $\nu > 0$, the $\nu-$th fractional sum of a function f is

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s), \quad \text{for} \quad t \in \mathbb{N}_{a+\nu}.$$

We also define the $\nu-$th fractional difference for $\nu > 0$ by

$$\Delta_a^{\nu} f(t) = \Delta^N \Delta_a^{\nu-N} f(t), \quad \text{for} \quad t \in \mathbb{N}_{a+N-\nu},$$

where $N \in \mathbb{N}$ with $0 \leqslant N-1 < \nu \leqslant N$.

LEMMA 1. *For $t, \nu \in \mathbb{R}$, we have $\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}$ if $t^{\underline{\nu}}$, $t^{\underline{\nu-1}}$ are well-defined.*

LEMMA 2. *Let $N \in \mathbb{N}$ with $0 \leqslant N-1 < \nu \leqslant N$. Then*

$$\Delta_0^{-\nu} \Delta_{\nu-N}^{\nu} y(t) = y(t) + c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + \cdots + c_N t^{\underline{\nu-N}} \quad (c_i \in \mathbb{R},\ 1 \leqslant i \leqslant N).$$

LEMMA 3. *For $\alpha > 0$ and $\nu \in \mathbb{R}$, we have $\Delta_\nu^\alpha t^{\underline{\nu}} = \frac{\Gamma(\nu+1) t^{\underline{\nu-\alpha}}}{\Gamma(\nu+1-\alpha)}$.*

We now construct the Green's function associated with (1.1). For this, we let $h : [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \to \mathbb{R}$ be a continuous function. Then we consider the fractional difference boundary value problem

$$\begin{cases} -\Delta_{\nu-3}^{\nu} y(t) = h(t+\nu-1), \quad t \in [0, b+2]_{\mathbb{N}_0}, \\ y(\nu-3) = [\Delta_{\nu-3}^{\alpha} y(t)]|_{t=\nu-\alpha-2} = [\Delta_{\nu-3}^{\beta} y(t)]|_{t=\nu+b+2-\beta} = 0, \end{cases} \tag{2.1}$$

where ν, α, β, b are as in (1.1). The following two lemmas are in [19] (for completeness we present their proofs).

LEMMA 4. (see [19, Theorem 2.1]) *The problem (2.1) has a unique solution*

$$y(t) = \sum_{s=0}^{b+2} G(t,s) h(s+\nu-1), \quad t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{t^{\underline{\nu-1}} (\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}} - (t-s-1)^{\underline{\nu-1}}, & 0 \leqslant s < t-\nu+1 \leqslant b+2, \\ \frac{t^{\underline{\nu-1}} (\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}, & 0 \leqslant t-\nu+1 \leqslant s \leqslant b+2. \end{cases} \tag{2.3}$$

Proof. From Lemma 2 we have

$$y(t) = -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} h(s+\nu-1) + c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + c_3 t^{\underline{\nu-3}}, \ (c_i \in \mathbb{R},\ 1 \leqslant i \leqslant 3).$$

The boundary condition $y(\nu-3)=0$ guarantees that $c_3=0$. Therefore, we have

$$\begin{aligned}\Delta_{\nu-3}^{\alpha}y(t)&=c_1\Delta_{\nu-3}^{\alpha}t^{\underline{\nu-1}}+c_2\Delta_{\nu-3}^{\alpha}t^{\underline{\nu-2}}-\Delta_0^{-(\nu-\alpha)}h(t+\nu-1)\\&=c_1\frac{\Gamma(\nu)t^{\underline{\nu-\alpha-1}}}{\Gamma(\nu-\alpha)}+c_2\frac{\Gamma(\nu-1)t^{\underline{\nu-\alpha-2}}}{\Gamma(\nu-\alpha-1)}\\&\quad-\frac{1}{\Gamma(\nu-\alpha)}\sum_{s=0}^{t-\nu+\alpha}(t-s-1)^{\underline{\nu-\alpha-1}}h(s+\nu-1).\end{aligned}$$

Using this and $[\Delta_{\nu-3}^{\alpha}y(t)]|_{t=\nu-\alpha-2}=0$ gives $c_2=0$. Moreover, we obtain

$$\begin{aligned}&[\Delta_{\nu-3}^{\beta}y(t)]|_{\nu+b+2-\beta}\\&=\left[c_1\frac{\Gamma(\nu)t^{\underline{\nu-\beta-1}}}{\Gamma(\nu-\beta)}-\frac{1}{\Gamma(\nu-\beta)}\sum_{s=0}^{t-\nu+\beta}(t-s-1)^{\underline{\nu-\beta-1}}h(s+\nu-1)\right]\Big|_{t=\nu+b+2-\beta}\\&=0,\end{aligned}$$

and

$$c_1=\frac{1}{\Gamma(\nu)(\nu+b+2-\beta)^{\underline{\nu-\beta-1}}}\sum_{s=0}^{b+2}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}h(s+\nu-1).$$

As a result, we have

$$\begin{aligned}y(t)&=-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h(s+\nu-1)\\&\quad+\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)(\nu+b+2-\beta)^{\underline{\nu-\beta-1}}}\sum_{s=0}^{b+2}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}h(s+\nu-1).\end{aligned}$$

Thus (2.2) holds. This completes the proof. □

LEMMA 5. (see [19, Theorem 2.2]) *The Green function* (2.3) *has the properties*
(i) $G(t,s)>0,\ (t,s)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\times[0,b+2]_{\mathbb{N}_0}$,
(ii) $q(t)G(b+\nu+1,s)\leqslant G(t,s)\leqslant G(b+\nu+1,s),\ (t,s)\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$ $\times[0,b+2]_{\mathbb{N}_0}$, *where* $q(t)=\frac{t^{\underline{\nu-1}}}{(b+\nu+1)^{\underline{\nu-1}}}$.

Proof. Clearly $G(t,s)>0$ when $0\leqslant t-\nu+1\leqslant s\leqslant b+2$. If $0\leqslant s<t-\nu+1$ $\leqslant b+2$, we have

$$\begin{aligned}\Delta_tG(t,s)&=\frac{(\nu-1)t^{\underline{\nu-2}}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{\Gamma(\nu)(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}-\frac{(\nu-1)(t-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)}\\&=\frac{(\nu-1)(t-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)}\left[\frac{t^{\underline{\nu-2}}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(t-s-1)^{\underline{\nu-2}}(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}-1\right].\end{aligned}$$

Let

$$F(t,s,\beta)=\frac{t^{\underline{\nu-2}}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(t-s-1)^{\underline{\nu-2}}(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}$$
$$=\frac{t^{\underline{\nu-2}}\Gamma(b+4)}{(t-s-1)^{\underline{\nu-2}}\Gamma(b-s+3)}(b+\nu-\beta-s+1)^{\underline{-s-1}}.$$

Then $F(t,s,\beta)$ is nondecreasing in β $(1<\beta<2)$ since $\Delta_\beta F(t,s,\beta)>0$. This implies

$$F(t,s,\beta)>F(t,s,1)=\frac{t^{\underline{\nu-2}}(\nu+b-s)^{\underline{\nu-2}}}{(t-s-1)^{\underline{\nu-2}}(\nu+b+1)^{\underline{\nu-2}}}$$
$$=\frac{t(t-1)\cdots(t-s)(b+3)(b+2)\cdots(b-s+3)}{(t-\nu+2)(t-\nu+1)\cdots(t-\nu+2-s)(\nu+b+1)(\nu+b)\cdots(\nu+b+1-s)}.$$

Since $\frac{(t-i)(b+3-i)}{(t-\nu+2-i)(\nu+b+1-i)}\geqslant 1(0\leqslant i\leqslant s)$, we see that $F(t,s,\beta)>F(t,s,1)\geqslant 1$ and thus $\Delta_t G(t,s)\geqslant 0$. As a result $G(t,s)$ is nondecreasing in t, i.e.,

$$G(t,s)\geqslant G(s+\nu-1,s)=\frac{\Gamma(s+\nu)\Gamma(b+\nu-\beta-s+2)\Gamma(b+4)}{\Gamma(\nu)\Gamma(s+1)\Gamma(b+3-s)\Gamma(\nu+b-\beta+3)}>0.$$

Thus (i) of Lemma 5 is true.

When $0\leqslant t-\nu+1\leqslant s\leqslant b+2$, we have

$$\Delta G_t(t,s)=\frac{(\nu-1)t^{\underline{\nu-2}}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{\Gamma(\nu)(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}>0,$$

and thus $G(t,s)\leqslant G(s+\nu-1,s)$. Moreover,

$$\frac{G(t,s)}{G(b+\nu+1,s)}=\frac{\frac{t^{\underline{\nu-1}}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}}{\frac{(b+\nu+1)^{\underline{\nu-1}}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}-(b+\nu-1)^{\underline{\nu-1}}}$$
$$\geqslant\frac{\frac{t^{\underline{\nu-1}}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}}{\frac{(b+\nu+1)^{\underline{\nu-1}}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}}$$
$$=\frac{t^{\underline{\nu-1}}}{(b+\nu+1)^{\underline{\nu-1}}}.$$

When $0\leqslant s<t-\nu+1\leqslant b+2$, from (i) we have $\Delta G_t(t,s)\geqslant 0$ and then $G(s+\nu-1,s)<G(t,s)\leqslant G(b+\nu+1,s)$. Moreover,

$$\frac{G(t,s)}{G(b+\nu+1,s)}=\frac{\frac{t^{\underline{\nu-1}}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}-(t-s-1)^{\underline{\nu-1}}}{\frac{(b+\nu+1)^{\underline{\nu-1}}(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}-(b+\nu-1)^{\underline{\nu-1}}}$$
$$=\frac{t^{\underline{\nu-1}}\left[\frac{(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}-\frac{(t-s-1)^{\underline{\nu-1}}}{t^{\underline{\nu-1}}}\right]}{(b+\nu+1)^{\underline{\nu-1}}\left[\frac{(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}}-\frac{(b+\nu-s)^{\underline{\nu-1}}}{(b+\nu+1)^{\underline{\nu-1}}}\right]},$$

where

$$\frac{\frac{(t-s-1)^{\underline{\nu-1}}}{t^{\underline{\nu-1}}}}{\frac{(b+\nu-s)^{\underline{\nu-1}}}{(b+\nu+1)^{\underline{\nu-1}}}} = \frac{(t-s-1)^{\underline{\nu-1}}(b+\nu+1)^{\underline{\nu-1}}}{t^{\underline{\nu-1}}(b+\nu-s)^{\underline{\nu-1}}}$$
$$= \frac{(t-\nu+1)(t-\nu)\cdots(t-\nu+1-s)(\nu+b+1)(\nu+b)\cdots(\nu+b+1-s)}{t(t-1)\cdots(t-s)(b+2)(b+1)\cdots(b-s+2)}$$
$$\leqslant 1,$$

i.e.,

$$\frac{(t-s-1)^{\underline{\nu-1}}}{t^{\underline{\nu-1}}} \leqslant \frac{(b+\nu-s)^{\underline{\nu-1}}}{(b+\nu+1)^{\underline{\nu-1}}}.$$

Therefore, we have

$$\frac{(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}} - \frac{(t-s-1)^{\underline{\nu-1}}}{t^{\underline{\nu-1}}} \geqslant \frac{(\nu+b-\beta-s+1)^{\underline{\nu-\beta-1}}}{(\nu+b-\beta+2)^{\underline{\nu-\beta-1}}} - \frac{(b+\nu-s)^{\underline{\nu-1}}}{(b+\nu+1)^{\underline{\nu-1}}},$$

and

$$\frac{G(t,s)}{G(b+\nu+1,s)} \geqslant \frac{t^{\underline{\nu-1}}}{(b+\nu+1)^{\underline{\nu-1}}}.$$

Consequently, (ii) of Lemma 5 holds. This completes the proof. □

Let $\varphi(s+\nu-1) = G(b+\nu+1,s)$ for $s \in [0,b+2]_{\mathbb{N}_0}$. Then $\varphi(t) = G(b+\nu+1,t-\nu+1)$ for $t \in [\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$. From Lemma 5 we have the inequalities

$$\sum_{t=\nu-1}^{b+\nu+1} q(t)\varphi(t)\cdot\varphi(s+\nu-1) \leqslant \sum_{t=\nu-1}^{b+\nu+1} G(t,s)\varphi(t) \leqslant \sum_{t=\nu-1}^{b+\nu+1} \varphi(t)\cdot\varphi(s+\nu-1), \quad (2.4)$$

for $s \in [0,b+2]_{\mathbb{N}_0}$. For convenience, we let $\kappa_1 = \sum_{t=\nu-1}^{b+\nu+1} q(t)\varphi(t)$ and $\kappa_2 = \sum_{t=\nu-1}^{b+\nu+1} \varphi(t)$.

Let E be the collection of all maps from $[\nu-3,b+\nu+1]_{\mathbb{N}_{\nu-3}}$ to $\mathbb{R}$, which is equipped with the max norm, $\|\cdot\|$. Then E is a Banach space. Define a set $P \subset E$ by

$$P = \left\{y \in E : y(t) \geqslant 0,\ t \in [\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}\right\}.$$

Then P is a cone on E.

From Lemma 4, we have that (1.1) is equivalent to the sum equation

$$y(t) = \sum_{s=0}^{b+2} G(t,s)f(s+\nu-1,y(s+\nu-1)) := (Ay)(t), \quad t \in [\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}, \quad (2.5)$$

where G is defined in (2.3). From (H0) it is immediate that $A : P \to P$ is completely continuous. It is clear that $y \in P\setminus\{0\}$ is a positive solution for (1.1) if and only if $y \in P\setminus\{0\}$ is a fixed point of A.

LEMMA 6. *Let* $P_0 = \{y \in P : y(t) \geqslant q(t)\|y\|, \forall y \in t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}\}$. *Then* $A(P) \subset P_0$.

This is a direct result from (ii) of Lemma 5.

LEMMA 7. (see [20]) *Let* E *be a real Banach space and* P *a cone on* E. *Suppose that* $\Omega \subset E$ *is a bounded open set and that* $A : \overline{\Omega} \cap P \to P$ *is a continuous compact operator. If there exists* $\omega_0 \in P \setminus \{0\}$ *such that*

$$\omega - A\omega \neq \lambda\omega_0, \ \ \forall\lambda \geqslant 0, \ \ \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$, *where* i *denotes the fixed point index on* P.

LEMMA 8. (see [20]) *Let* E *be a real Banach space and* P *a cone on* E. *Suppose that* $\Omega \subset E$ *is a bounded open set with* $0 \in \Omega$ *and that* $A : \overline{\Omega} \cap P \to P$ *is a continuous compact operator. If*

$$\omega - \lambda A\omega \neq 0, \ \ \forall\lambda \in [0,1], \ \ \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 1$.

DEFINITION 3. Given a cone P in a real Banach space E, a functional $\alpha : P \to \mathbb{R}^+$ is said to be nonnegative continuous concave on P, provided $\alpha(tx + (1-t)y) \geqslant t\alpha(x) + (1-t)\alpha(y)$, for all $x, y \in P$ with $t \in [0,1]$.

Let $a, b, r > 0$ be constants and let the functional α be as defined above. Let $P_r = \{y \in P : \|y\| < r\}$ and $P\{\alpha, a, b\} = \{y \in P : \alpha(y) \geqslant a, \|y\| \leqslant b\}$.

LEMMA 9. (Leggett-Williams fixed point theorem, see [20]) *Let* E *be a real Banach space,* $P \subset E$ *a cone in* E. *Suppose that* $A : \overline{P}_c \to \overline{P}_c$ *(* $c > 0$ *is a constant) is a continuous compact operator, and* α *is a nonnegative continuous concave functional on* P *such that* $\alpha(y) \leqslant \|y\|$ *for* $y \in \overline{P}_c$. *Assume there exist* $0 < a < b < d \leqslant c$ *such that*

(i) $\{y \in P(\alpha, b, d) | \alpha(y) > b\} \neq \emptyset$ *and* $\alpha(Ay) > b$ *for all* $y \in P(\alpha, b, d)$,

(ii) $\|Ay\| < a$ *for all* $\|y\| \leqslant a$,

(iii) $\alpha(Ay) > b$ *for all* $y \in P(\alpha, b, c)$ *with* $\|Ay\| > d$.

Then A *has at least three fixed points* y_i $(i = 1, 2, 3)$ *with*

$$\|y_1\| < a, \ b < \alpha(y_2), \ \|y_3\| > a, \ \alpha(y_3) < b.$$

3. Main results

Now, we list our assumptions on f in this section.

(H1) $\liminf_{y \to +\infty} \frac{f(t,y)}{y} > \kappa_1^{-1}$ uniformly on $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$.

(H2) $\limsup_{y \to 0^+} \frac{f(t,y)}{y} < \kappa_2^{-1}$ uniformly on $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$.

(H3) $\liminf_{y \to 0^+} \frac{f(t,y)}{y} > \kappa_1^{-1}$ uniformly on $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$.

(H4) $\limsup_{y \to +\infty} \frac{f(t,y)}{y} < \kappa_2^{-1}$ uniformly on $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$.

(H5) There exists $M > 0$ such that $f(t,y) < \kappa_2^{-1}M$ for $(t,y) \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \times [0,M]$.

Let $B_\rho := \{y \in E : \|y\| < \rho\}$ for $\rho > 0$.

THEOREM 1. *Suppose that (H0)–(H2) hold. Then* (1.1) *has at least one positive solution.*

Proof. From (H1), there exist $\varepsilon_1 > 0$ and $d_1 > 0$ such that

$$f(t,y) \geqslant (\kappa_1^{-1} + \varepsilon_1)y - d_1, \quad \text{for} \quad y \in \mathbb{R}^+, \quad t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}. \tag{3.1}$$

This implies

$$\begin{aligned}(Ay)(t) &\geqslant \sum_{s=0}^{b+2} G(t,s)[(\kappa_1^{-1}+\varepsilon_1)y(s+\nu-1)-d_1]\\ &\geqslant (\kappa_1^{-1}+\varepsilon_1)\sum_{s=0}^{b+2} G(t,s)y(s+\nu-1) - d_1\kappa_2, \quad \text{for} \quad y \in \mathbb{R}^+,\\ &\quad t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}.\end{aligned} \tag{3.2}$$

In what follows, we prove that there is a $R > 0$ such that

$$y \neq Ay + \lambda\varphi^*, \ \forall y \in \partial B_R \cap P, \ \lambda \geqslant 0, \tag{3.3}$$

where φ^* is a fixed element on P_0. If the claim is false, there exists $y_0 \in \partial B_R \cap P$ and $\lambda_0 \geqslant 0$ such that

$$y_0 = Ay_0 + \lambda_0\varphi^*.$$

This implies:

(i) $y_0 \in P_0$ since $A(P) \subset P_0$ and $\varphi^* \in P_0$,

(ii) $y_0(t) \geqslant (Ay_0)(t)$, for $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$.

From (3.2), we have

$$y_0(t) \geqslant (\kappa_1^{-1}+\varepsilon_1)\sum_{s=0}^{b+2} G(t,s)y_0(s+\nu-1) - d_1\kappa_2, \quad \text{for} \quad t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}. \tag{3.4}$$

Multiplying both sides of (3.4) by $\varphi(t)$, and from (2.4) we have

$$\begin{aligned}\sum_{t=\nu-1}^{b+\nu+1} y_0(t)\varphi(t) &\geqslant \sum_{t=\nu-1}^{b+\nu+1} \varphi(t)\left[(\kappa_1^{-1}+\varepsilon_1)\sum_{s=0}^{b+2} G(t,s)y_0(s+\nu-1) - d_1\kappa_2\right]\\ &\geqslant (\kappa_1^{-1}+\varepsilon_1)\kappa_1\sum_{s=0}^{b+2} y_0(s+\nu-1)\varphi(s+\nu-1) - d_1\kappa_2^2\\ &= (1+\varepsilon_1\kappa_1)\sum_{t=\nu-1}^{b+\nu+1} y_0(t)\varphi(t) - d_1\kappa_2^2.\end{aligned} \tag{3.5}$$

As a result, we have

$$\sum_{t=\nu-1}^{b+\nu+1} y_0(t)\varphi(t) \leqslant \varepsilon_1^{-1} d_1 \kappa_1^{-1} \kappa_2^2.$$

Note that $y_0 \in P_0$, and then

$$\|y_0\| \sum_{t=\nu-1}^{b+\nu+1} q(t)\varphi(t) \leqslant \varepsilon_1^{-1} d_1 \kappa_1^{-1} \kappa_2^2, \quad \text{and} \quad \|y_0\| \leqslant \varepsilon_1^{-1} d_1 \kappa_1^{-2} \kappa_2^2.$$

Therefore, we can choose $R > \varepsilon_1^{-1} d_1 \kappa_1^{-2} \kappa_2^2$ so (3.3) is true. From Lemma 7 we have

$$i(A, B_R \cap P, P) = 0. \tag{3.6}$$

From (H2), there exist $r \in (0,R)$ and $\varepsilon_2 \in (0, \kappa_2^{-1})$ such that

$$f(t,y) \leqslant (\kappa_2^{-1} - \varepsilon_2) y, \quad \text{for} \quad y \in [0,r], \quad t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}. \tag{3.7}$$

This implies

$$(Ay)(t) \leqslant (\kappa_2^{-1} - \varepsilon_2) \sum_{s=0}^{b+2} G(t,s) y(s+\nu-1), \quad \text{for} \quad y \in \overline{B}_r, \quad t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}. \tag{3.8}$$

Now we prove

$$y \neq \lambda A y, \ \ \forall y \in \partial B_r \cap P, \ \ \lambda \in [0,1]. \tag{3.9}$$

Suppose there exists $y_1 \in \partial B_r \cap P$, $\lambda_1 \in [0,1]$ such that

$$y_1(t) = \lambda_1 (Ay_1)(t) \leqslant (\kappa_2^{-1} - \varepsilon_2) \sum_{s=0}^{b+2} G(t,s) y_1(s+\nu-1), \ \text{for } t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}. \tag{3.10}$$

Multiplying both sides of the above inequality by $\varphi(t)$, and from (2.4) we have

$$\begin{aligned}
\sum_{t=\nu-1}^{b+\nu+1} y_1(t)\varphi(t) &\leqslant \sum_{t=\nu-1}^{b+\nu+1} \varphi(t) \left[(\kappa_2^{-1} - \varepsilon_2) \sum_{s=0}^{b+2} G(t,s) y_1(s+\nu-1) \right] \\
&\leqslant \kappa_2 (\kappa_2^{-1} - \varepsilon_2) \sum_{s=0}^{b+2} y_1(s+\nu-1) \varphi(s+\nu-1) \qquad (3.11) \\
&= (1 - \varepsilon_2 \kappa_2) \sum_{t=\nu-1}^{b+\nu+1} y_1(t)\varphi(t).
\end{aligned}$$

This implies $\sum\limits_{t=\nu-1}^{b+\nu+1} y_1(t)\varphi(t) \equiv 0$ and so $y_1(t) \equiv 0$ for $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$. This contradicts $y_1 \in \partial B_r \cap P$. Hence, (3.9) is true. From Lemma 8 we have

$$i(A, B_r \cap P, P) = 1. \tag{3.12}$$

From (3.6) and (3.12) we have

$$i(A,(B_R\setminus\overline{B}_r)\cap P,P)=i(A,B_R\cap P,P)-i(A,B_r\cap P,P)=0-1=-1\neq 0. \tag{3.13}$$

Consequently, the operator A has a fixed point in $(B_R\setminus\overline{B}_r)\cap P$, i.e., (1.1) has at least one positive solution in $(B_R\setminus\overline{B}_r)\cap P$. This completes the proof. $\square$

THEOREM 2. *Suppose that (H0), (H3)–(H4) hold. Then* (1.1) *has at least one positive solution.*

Proof. From (H3), there exist $\varepsilon_3>0$ and $r>0$ such that

$$f(t,y)\geqslant(\kappa_1^{-1}+\varepsilon_3)y,\quad \text{for}\quad y\in[0,r],\quad t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}. \tag{3.14}$$

Consequently, we have

$$(Ay)(t)\geqslant(\kappa_1^{-1}+\varepsilon_3)\sum_{s=0}^{b+2}G(t,s)y(s+\nu-1),\quad \text{for}\quad y\in[0,r],\quad t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}. \tag{3.15}$$

We prove that

$$y\neq Ay+\lambda\varphi^*,\ \ \forall y\in\partial B_r\cap P,\ \ \lambda\geqslant 0, \tag{3.16}$$

where φ^{**} is a fixed element on P. Suppose there exists $y_2\in\partial B_r\cap P$ and $\lambda_2\geqslant 0$ such that

$$y_2=Ay_2+\lambda_2\varphi^{**}.$$

This implies $y_2(t)\geqslant(Ay_2)(t)$, for $t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$. Now with (3.15), we have

$$y_2(t)\geqslant(\kappa_1^{-1}+\varepsilon_3)\sum_{s=0}^{b+2}G(t,s)y_2(s+\nu-1),\quad \text{for}\quad t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}. \tag{3.17}$$

Multiplying both sides of (3.17) by $\varphi(t)$, and from (2.4) we have

$$\begin{aligned}\sum_{t=\nu-1}^{b+\nu+1}y_2(t)\varphi(t)&\geqslant\sum_{t=\nu-1}^{b+\nu+1}\varphi(t)\left[(\kappa_1^{-1}+\varepsilon_3)\sum_{s=0}^{b+2}G(t,s)y_2(s+\nu-1)\right]\\&\geqslant(\kappa_1^{-1}+\varepsilon_3)\kappa_1\sum_{s=0}^{b+2}y_2(s+\nu-1)\varphi(s+\nu-1)\\&=(1+\varepsilon_3\kappa_1)\sum_{t=\nu-1}^{b+\nu+1}y_2(t)\varphi(t).\end{aligned} \tag{3.18}$$

Hence, we have $\sum\limits_{t=\nu-1}^{b+\nu+1}y_2(t)\varphi(t)=0$, and thus $y_2(t)\equiv 0$ for $t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$. This contradicts $y_2\in\partial B_r\cap P$. Thus (3.16) holds. From Lemma 7 we have

$$i(A,B_r\cap P,P)=0. \tag{3.19}$$

From (H4), there exists $d_2 > 0$ and $\varepsilon_4 \in (0, \kappa_2^{-1})$ such that

$$f(t,y) \leqslant (\kappa_2^{-1} - \varepsilon_4)y + d_2, \quad \text{for} \quad y \in \mathbb{R}^+, \quad t \in [\nu - 1, b + \nu + 1]_{\mathbb{N}_{\nu-1}}. \tag{3.20}$$

Thus

$$(Ay)(t) \leqslant (\kappa_2^{-1} - \varepsilon_4) \sum_{s=0}^{b+2} G(t,s)y(s+\nu-1) + d_2\kappa_2, \text{ for } y \in \mathbb{R}^+, \quad t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}. \tag{3.21}$$

Now we prove there is a $R > r$ (r is defined in (3.14)) such that

$$y \neq \lambda Ay, \ \forall y \in \partial B_R \cap P, \ \lambda \in [0,1]. \tag{3.22}$$

Suppose there exists $y_3 \in \partial B_R \cap P$, $\lambda_3 \in [0,1]$ such that

$$y_3(t) = \lambda_3(Ay_3)(t) \leqslant (\kappa_2^{-1} - \varepsilon_4) \sum_{s=0}^{b+2} G(t,s)y_3(s+\nu-1) + d_2\kappa_2, \text{ for } t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}. \tag{3.23}$$

Multiplying both sides of the above inequality by $\varphi(t)$, and from (2.4) we have

$$\begin{aligned} \sum_{t=\nu-1}^{b+\nu+1} y_3(t)\varphi(t) &\leqslant \sum_{t=\nu-1}^{b+\nu+1} \varphi(t) \left[(\kappa_2^{-1} - \varepsilon_4) \sum_{s=0}^{b+2} G(t,s)y_3(s+\nu-1) + d_2\kappa_2 \right] \\ &\leqslant \kappa_2(\kappa_2^{-1} - \varepsilon_4) \sum_{s=0}^{b+2} y_3(s+\nu-1)\varphi(s+\nu-1) + d_2\kappa_2^2 \\ &= (1 - \varepsilon_4\kappa_2) \sum_{t=\nu-1}^{b+\nu+1} y_3(t)\varphi(t) + d_2\kappa_2^2. \end{aligned} \tag{3.24}$$

This implies

$$\sum_{t=\nu-1}^{b+\nu+1} y_3(t)\varphi(t) \leqslant \varepsilon_4^{-1} d_2 \kappa_2.$$

Note that $y_3 = \lambda_3 Ay_3 \in P_0$ from Lemma 6. Hence,

$$\|y_3\| \leqslant \varepsilon_4^{-1} d_2 \kappa_1^{-1} \kappa_2.$$

Taking $R > \max\{r, \varepsilon_4^{-1} d_2 \kappa_1^{-1} \kappa_2\}$ we have (3.22). From Lemma 8 we have

$$i(A, B_R \cap P, P) = 1. \tag{3.25}$$

As a result, from (3.19) and (3.25) we have

$$i(A, (B_R \setminus \overline{B}_r) \cap P, P) = i(A, B_R \cap P, P) - i(A, B_r \cap P, P) = 1 - 0 = 1 \neq 0. \tag{3.26}$$

Consequently, the operator A has a fixed point in $(B_R \setminus \overline{B}_r) \cap P$, i.e., (1.1) has at least one positive solution in $(B_R \setminus \overline{B}_r) \cap P$. This completes the proof. $\square$

THEOREM 3. *Suppose that (H0), (H1), (H3), (H5) hold. Then* (1.1) *has at least two positive solutions.*

Proof. From (H5), for $(t,y) \in [\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}} \times [0,M]$, we have

$$\begin{aligned}(Ay)(t) &= \sum_{s=0}^{b+2} G(t,s) f(s+\nu-1, y(s+\nu-1)) \\ &< \sum_{s=0}^{b+2} G(t,s)\kappa_2^{-1} M \leqslant \kappa_2^{-1} M \sum_{t=\nu-1}^{b+\nu+1} \varphi(t) = M.\end{aligned} \tag{3.27}$$

This implies

$$\|Ay\| < \|y\| \quad \text{for} \quad y \in \partial B_M \cap P. \tag{3.28}$$

We now show

$$y \neq \lambda Ay, \ \ \forall y \in \partial B_M \cap P, \ \ \lambda \in [0,1]. \tag{3.29}$$

Suppose there exists $y_4 \in \partial B_M \cap P$ and $\lambda_4 \in [0,1]$ such that $y_4 = \lambda_4 A y_4$. Hence,

$$(Ay_4)(t) \geqslant y_4(t), \quad \text{for} \quad t \in [\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}.$$

Consequently, $\|Ay_4\| \geqslant \|y_4\|$, and this contradicts (3.28). Thus (3.29) holds. From Lemma 8 we have

$$i(A, B_M \cap P, P) = 1. \tag{3.30}$$

Note that we can choose $R > M > r$ such that (3.6) and (3.19) are satisfied. Now with (3.30), we obtain

$$\begin{aligned} &i(A,(B_R \setminus \overline{B}_M) \cap P, P) = i(A, B_R \cap P, P) - i(A, B_M \cap P, P) = 0 - 1 = -1 \neq 0, \\ &i(A,(B_M \setminus \overline{B}_r) \cap P, P) = i(A, B_M \cap P, P) - i(A, B_r \cap P, P) = 1 - 0 = 1 \neq 0. \end{aligned} \tag{3.31}$$

As a result, the operator A has two fixed points in $(B_R \setminus \overline{B}_M) \cap P$ and $(B_M \setminus \overline{B}_r) \cap P$, respectively. Therefore, (1.1) has at least two positive solutions. This completes the proof. □

Consider the semipositone condition:

(H0)$'$ $f(t+\nu-1,\cdot) : [\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, and moreover, there exists a positive constant $M_1 > 0$ such that

$$f(t,y) \geqslant -M_1, \text{ for all } (t,y) \in [\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R}^+.$$

In the following we replace (H0) with (H0)$'$.

Let $q_0 := \min_{t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}} q(t) = \min_{t\in[\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}} \frac{t^{\underline{\nu-1}}}{(b+\nu+1)^{\underline{\nu-1}}} > 0$.

THEOREM 4. *Suppose that (H0)$'$ holds. Moreover, there exist positive constants e,a,c,N with $M_1\kappa_2 q_0^{-1} < e < e + M_1\kappa_2 < a < q_0^2 c$, $q_0^{-1} < N < q_0 \frac{c}{a}$ such that*

(H6) $f(t,y) < \frac{e}{\kappa_2} - M_1$ for $t \in [\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$, $0 \leqslant y \leqslant e$,

(H7) $f(t,y) \geqslant \frac{a}{q_0\kappa_2} N - M_1$ for $t \in [\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$, $a - M_1\kappa_2 \leqslant y \leqslant \frac{a}{q_0^2}$,

(H8) $f(t,y) \leqslant \frac{c}{\kappa_2} - M_1$ for $t \in [\nu-1,b+\nu+1]_{\mathbb{N}_{\nu-1}}$, $0 \leqslant y \leqslant c$.

Then (1.1) *has at least two positive solutions.*

Proof. Let ω be a solution of

$$\begin{cases} -\Delta_{\nu-3}^{\nu} y(t) = 1, \quad t \in [0, b+2]_{\mathbb{N}_0}, \\ y(\nu-3) = [\Delta_{\nu-3}^{\alpha} y(t)]|_{t=\nu-\alpha-2} = [\Delta_{\nu-3}^{\beta} y(t)]|_{t=\nu+b+2-\beta} = 0, \end{cases} \tag{3.32}$$

where ν, α, β, b are as in (1.1). Define $z = M_1 \omega$, and then from Lemma 4, we have

$$z(t) = M_1 \omega(t) = M_1 \sum_{s=0}^{b+2} G(t,s) \leqslant M_1 \sum_{s=0}^{b+2} \varphi(s+\nu-1) = M_1 \sum_{s=\nu-1}^{b+\nu+1} \varphi(s) = M_1 \kappa_2.$$

We note that (1.1) (under condition (H0)$'$) has a positive solution y if and only if $y + z = \widetilde{y}$ is a solution of the problem

$$\begin{cases} -\Delta_{\nu-3}^{\nu} y(t) = \widetilde{f}(t+\nu-1, y(t+\nu-1) - z(t+\nu-1)), \quad t \in [0, b+2]_{\mathbb{N}_0}, \\ y(\nu-3) = [\Delta_{\nu-3}^{\alpha} y(t)]|_{t=\nu-\alpha-2} = [\Delta_{\nu-3}^{\beta} y(t)]|_{t=\nu+b+2-\beta} = 0, \end{cases} \tag{3.33}$$

and $\widetilde{y}(t) \geqslant z(t)$ for $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$, where ν, α, β, b are as in (1.1) and

$$\widetilde{f}(t,y) = \begin{cases} f(t,y) + M_1, & (t,y) \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R}^+, \\ f(t,0) + M_1, & (t,y) \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \times (-\infty, 0). \end{cases}$$

For $y \in P$, we define the operator

$$(By)(t) = \sum_{s=0}^{b+2} G(t,s) \widetilde{f}(s+\nu-1, y(s+\nu-1) - z(s+\nu-1)), \text{ for } t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}.$$

Note that $q_0 > 0$. Then from Lemma 6 we have $B(P) \subset P_1$, where $P_1 = \{y \in P : y(t) \geqslant q_0 \|y\|, \forall t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}\}$. We now show that all the conditions of Lemma 9 are satisfied. We first define the nonnegative, continuous concave functional $\alpha : P \to \mathbb{R}^+$ by $\alpha(y) = \min_{t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}} |y(t)|$. For each $y \in P$, we see that $\alpha(y) \leqslant \|y\|$. Next we prove that $B(\overline{P}_c) \subset \overline{P}_c$. Let $y \in \overline{P}_c$. Then for $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$, we have

(i) if $y(t) \geqslant z(t)$, then $0 \leqslant y(t) - z(t) \leqslant y(t) \leqslant c$ and $\widetilde{f}(t, y(t) - z(t)) = f(t, y(t) - z(t)) + M_1 \geqslant 0$. From (H8) we have $\widetilde{f}(t, y(t) - z(t)) \leqslant \frac{c}{\kappa_2}$.

(ii) if $y(t) < z(t)$, then $y(t) - z(t) < 0$ and $\widetilde{f}(t, y(t) - z(t)) = f(t, 0) + M_1 \geqslant 0$. From (H8) we have $\widetilde{f}(t, y(t) - z(t)) \leqslant \frac{c}{\kappa_2}$.

Therefore, we have proved that, if $y \in \overline{P}_c$, then $\widetilde{f}(t, y(t) - z(t)) \leqslant \frac{c}{\kappa_2}$ for $t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$. Then,

$$\begin{aligned} \|By\| &= \max_{t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b+2} G(t,s) \widetilde{f}(s+\nu-1, y(s+\nu-1) - z(s+\nu-1)) \\ &\leqslant \frac{c}{\kappa_2} \sum_{s=0}^{b+2} \varphi(s+\nu-1) \\ &= c. \end{aligned}$$

This implies $B(\overline{P}_c) \subseteq \overline{P}_c$. If $y \in \overline{P}_e$, then (H6) yields $\widetilde{f}(t, y(t) - z(t)) \leqslant \frac{e}{\kappa_2}$ for $t \in [\nu - 1, b + \nu + 1]_{\mathbb{N}_{\nu-1}}$. Thus $B : \overline{P}_e \to P_e$, i.e., assumption (ii) of Lemma 9 holds.

Let $y(t) = \frac{a}{q_0^2}$, for $t \in [\nu - 1, b + \nu + 1]_{\mathbb{N}_{\nu-1}}$. Then $y \in P$, $\alpha(y) = a/q_0^2 > a$, i.e., $\{y \in P(\alpha, a, \frac{a}{q_0^2}) : \alpha(y) > a\} \neq \emptyset$. Moreover, if $y \in P(\alpha, a, \frac{a}{q_0^2})$, then $\alpha(y) \geqslant a$, and $a \leqslant \|y\| \leqslant \frac{a}{q_0^2}$. Thus, $0 < a - M_1\kappa_2 \leqslant y(t) - z(t) \leqslant y(t) \leqslant \frac{a}{q_0^2}, t \in [0,1]$. From (H7) we obtain $\widetilde{f}(t, y(t) - z(t)) \geqslant \frac{a}{q_0\kappa_2}N$ for $t \in [\nu - 1, b + \nu + 1]_{\mathbb{N}_{\nu-1}}$. From the definition of α, we have

$$\begin{aligned}
\alpha(By) &= \min_{t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}} (By)(t) \geqslant q_0\|By\| \\
&\geqslant q_0 \max_{t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}} \sum_{s=0}^{b+2} G(t,s)\widetilde{f}(s + \nu - 1, y(s + \nu - 1) - z(s + \nu - 1)) \\
&\geqslant q_0 \frac{a}{q_0\kappa_2} N \sum_{s=0}^{b+2} G(t,s) \geqslant q_0 \frac{a}{q_0\kappa_2} N \sum_{s=0}^{b+2} q(t)\varphi(s + \nu - 1) > a.
\end{aligned}$$

Therefore, condition (i) of Lemma 9 is satisfied with $d = a/q_0^2$.

Finally let $y \in P(\alpha, a, c)$ with $\|By\| > a/q_0^2$. Then we have $\alpha(By) \geqslant q_0\|By\| \geqslant \frac{a}{q_0} > a$. Hence, condition (iii) of Lemma 9 holds with $\|By\| > a/q_0^2$.

As a result all the conditions in Lemma 9 are satisfied. Hence B has at least three positive fixed points $\widetilde{y}_1$, $\widetilde{y}_2$ and $\widetilde{y}_3$ such that

$$\|\widetilde{y}_1\| < e, \quad a < \alpha(\widetilde{y}_2), \quad \|\widetilde{y}_3\| > e, \quad \alpha(\widetilde{y}_3) < a.$$

Furthermore, $\widetilde{y}_i = y_i + z$ $(i = 1,2,3)$ are solutions of (3.33) and moreover,

$$\begin{gathered}
\widetilde{y}_2(t) \geqslant q_0\|\widetilde{y}_2\| \geqslant q_0\alpha(\widetilde{y}_2) > q_0 a > q_0 M_1 \kappa_2 q_0^{-1} \geqslant z(t),\ t \in [\nu - 1, b + \nu + 1]_{\mathbb{N}_{\nu-1}}, \\
\widetilde{y}_3(t) \geqslant q_0\|\widetilde{y}_3\| > q_0 e > q_0 M_1 \kappa_2 q_0^{-1} \geqslant z(t),\ t \in [\nu - 1, b + \nu + 1]_{\mathbb{N}_{\nu-1}}.
\end{gathered}$$

Thus $y_2 = \widetilde{y}_2 - z$, $y_3 = \widetilde{y}_3 - z$ are two positive solutions of (1.1). This completes the proof. □

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Jiafa Xu
School of Mathematical Sciences
Chongqing Normal University
Chongqing 401331, China
e-mail: xujiafa292@sina.com

Donal O'Regan
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland
Galway, Ireland
e-mail: donal.oregan@nuigalway.ie

Chengmin Hou
Department of Mathematics
Yanbian University
Yanji 133002, China
e-mail: cmhou@foxmail.com

Yujun Cui
State Key Laboratory of Mining Disaster Prevention and Control Co-founded by Shandong Province and the Ministry of Science and Technology
Shandong University of Science and Technology
Qingdao 266590, China
e-mail: cyj720201@163.com