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THE (p,q)–ELLIPTIC SYSTEMS WITH CONCAVE–CONVEX NONLINEARITIES

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Abstract. Multiple positive solutions for the (p,q)-elliptic systems with the concave-convex nonlinearities are obtained by using the Nehari manifold and the fibering method.

1. Introduction and Main Result

In this paper, we consider the existence of weak solutions for the following (p,q)-elliptic systems

$$\begin{cases} -\Delta_p u = \lambda \alpha a(x)|u|^{\alpha-2}u|v|^{\beta} + \gamma b(x)|u|^{\gamma-2}u|v|^{\delta} & \text{in } \Omega, \\ -\Delta_q v = \lambda \beta a(x)|u|^{\alpha}|v|^{\beta-2}v + \delta b(x)|u|^{\gamma}|v|^{\delta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a bounded open domain with a smooth boundary $\partial\Omega$, $1 < p,q < N, \alpha \geqslant 0, \beta \geqslant 0, \gamma \geqslant 0, \delta \geqslant 0$ satisfy that $1 < \alpha + \beta < \min\{p,q\}$, $\max\{p,q\} < \gamma + \delta < \min\{p^*,q^*\}$, where $p^* = \frac{Np}{N-p}$ and $q^* = \frac{Nq}{N-q}$ are the critical Sobolev exponents of p,q, respectively. The functions $a(x), b(x) \in C(\bar{\Omega})$ are somewhere positive and may change sign on $\bar{\Omega}$.

The Nehari manifold and the fibering method were introduced by Pohozaev in [7] and were widely used to study the existence of multiple solutions for elliptic equations (see [1], [2], [5], [8] and references therein) and elliptic systems (see [3], [4], [6], [9], [10], [11], [12] and references therein). Especially, Brown and Wu in [5] discussed the existence of at least two positive solutions for the semilinear elliptic equation with the concave-convex nonlinearities. Ramos Quoirin in [8] investigated the existence and multiplicity of non-negative solutions for the the following concave-convex type equation

$$-\Delta_p u + V(x)u^{p-1} = \lambda a(x)|u|^{r-1} + b(x)|u|^{q-1} \quad u \in W_0^{1,p}(\Omega),$$

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where $1 < r < p < q < p^*$. Bozhkov and Mitidieri in [4] proved the existence of multiple solutions for the following (p,q)-Laplacian system

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u + (\alpha+1)c(x)|u|^{\alpha-1}u|v|^{\beta+1} & \text{in } \Omega, \\ -\Delta_q v = \mu b(x)|v|^{q-2}v + (\beta+1)c(x)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where α, β satisfy the following conditions:

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1, \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1. \tag{1.2}$$

Adriouch and El Hamidi in [1] considered the existence and multiplicity results of positive solutions for the following system

$$\begin{cases} -\Delta_p u = \lambda |u|^{p_1-2}u + (\alpha+1)|u|^{\alpha-1}u|v|^{\beta+1} & \text{in } \Omega, \\ -\Delta_q v = \mu |v|^{q-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p_1 < p < N$, and α, β satisfy (1.2) and $\frac{\beta+1}{q} < 1$. For the following quasilinear elliptic boundary value problem

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-1}u|v|^{\beta+1} + \frac{\mu(x)}{(\alpha+1)(\delta+1)}|u|^{\gamma-1}u|v|^{\delta+1} & \text{in } \Omega, \\ -\Delta_q v = \lambda a(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v + \frac{\mu(x)}{(\beta+1)(\gamma+1)}|u|^{\gamma+1}|v|^{\delta-1}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha, \beta \geqslant 0$ satisfy $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$, and $p < \gamma+1$(or $q < \delta+1$) and $\frac{\gamma+1}{p^*} + \frac{\delta+1}{q^*} < 1$, applying the Nehari manifold, Zhang, Liu and Liu in [12] proved that there is at least a nonnegative nonsemitrivial solution for every $\lambda \in (0, \lambda_1)$, where λ_1 is principal eigenvalue for the unperturbed system.

Influenced by these finds, in this paper, we will study the existence of multiple positive solutions for system (1.1) with the concave-convex nonlinearities by using the Nehari manifold and the fibering maps.

Let $W_0^{1,p}(\Omega)$ be the usual Banach space endowed with the norm

$$\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p dx \right)^{1/p} \quad \text{for any } u \in W_0^{1,p}(\Omega).$$

Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\theta(\Omega)$ is continuous and compact for any $\theta \in [1, p^*)$, we define

$$S_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|_{1,p}^{\gamma+\delta}}{|u|_{\gamma+\delta}^{\gamma+\delta}}, \quad s_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|_{1,p}^{\alpha+\beta}}{|u|_{\alpha+\beta}^{\alpha+\beta}},$$

$$S_2 = \inf_{v \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\|v\|_{1,q}^{\gamma+\delta}}{|v|_{\gamma+\delta}^{\gamma+\delta}}, \quad s_2 = \inf_{v \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\|v\|_{1,q}^{\alpha+\beta}}{|v|_{\alpha+\beta}^{\alpha+\beta}},$$

where $|\cdot|_p$ denotes the norm of $L^p(\Omega)$, and $s_1, s_2, S_1, S_2 > 0$. Let $W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ be the product space with the norm

$$\|(u,v)\| = \|u\|_{1,p} + \|v\|_{1,q} \text{ for any } (u,v) \in W.$$

DEFINITION 1. (weak solution) We say that $(u,v) \in W$ is the weak solution of system (1.1), if $(u,v) \in W$, one has

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla z dx = \lambda\alpha \int_\Omega a(x)|u|^{\alpha-2}u|v|^\beta \cdot z dx + \gamma \int_\Omega b(x)|u|^{\gamma-2}u|v|^\delta \cdot z dx,$$
$$\int_\Omega |\nabla v|^{q-2}\nabla v \cdot \nabla w dx = \lambda\beta \int_\Omega a(x)|u|^{\alpha}|v|^{\beta-2}v \cdot w dx + \delta \int_\Omega b(x)|u|^{\gamma}|v|^{\delta-2}v \cdot w dx$$

for any $(z,w) \in W$.

REMARK 1. We call a solution (u,v) of system (1.1) is nontrivial, if $u \neq 0$ and $v \neq 0$, a solution (u,v) is positive if $u > 0$ and $v > 0$, and semitrivial if it is of the form $(u,0)$ with $u \neq 0$ or $(0,v)$ with $v \neq 0$. It is easy to prove that if $(u,v) \neq (0,0)$ is a solution of system (1.1), then it is nontrivial.

The main result can be described as follows:

THEOREM 1. *If* $1 < \alpha+\beta < \min\{p,q\}$, $\max\{p,q\} < \gamma+\delta < \min\{p^*,q^*\}$, *then there exists* $\Lambda_0 > 0$ *such that when* $0 < \lambda < \Lambda_0$, *system* (1.1) *has at least two nontrivial solutions.*

REMARK 2. (1) From $\alpha+\beta < \min\{p,q\}$, we have

$$\frac{\alpha}{p} + \frac{\beta}{q} \leqslant \frac{\alpha}{\min\{p,q\}} + \frac{\beta}{\min\{p,q\}} < 1,$$

which implies that the nonlinearity $|u|^\alpha|v|^\beta$ is the concave term. As far as we know, there is no paper to consider the existence of weak solution for the (p,q)-elliptic systems with the concave nonlinearity $|u|^\alpha|v|^\beta$, hence our result is new. And from $\max\{p,q\} < \gamma+\delta < \min\{p^*,q^*\}$, we obtain

$$\frac{\gamma}{p^*} + \frac{\delta}{q^*} \leqslant \frac{\gamma}{\min\{p^*,q^*\}} + \frac{\delta}{\min\{p^*,q^*\}} < 1,$$

and

$$\frac{\gamma}{p} + \frac{\delta}{q} \geqslant \frac{\gamma}{\max\{p,q\}} + \frac{\delta}{\max\{p,q\}} > 1,$$

which implies that the nonlinearity $|u|^\gamma|v|^\delta$ is subcritical and convex.

(2) Theorem 1 extends the results of [5, 12] from the semilinear elliptic equation to the (p,q)-elliptic systems and is complement for the ones of [1, 4, 12].

2. Proof of the Theorem

From a variational point of view, the weak solutions of system (1.1) correspond to the critical points of the functional $J_\lambda : W \to R$ given by

$$J_\lambda(u,v) = \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|v\|_{1,q}^q - \lambda\int_\Omega a(x)|u|^\alpha|v|^\beta dx - \int_\Omega b(x)|u|^\gamma|v|^\delta dx. \tag{2.1}$$

It is not difficult to prove that J_λ is unbounded below on W. In order to get rid of the unboundedness of the functional J_λ, we will consider the functional J_λ on the Nehari manifold:

$$M_\lambda(\Omega) = \{(u,v) \in W\setminus\{(0,0)\} | \langle J_\lambda'(u,v),(u,v)\rangle = 0\}.$$

The Nehari manifold $M_\lambda(\Omega)$ is closely linked to the behavior of the functions of the form $\psi_{(u,v)} : t \to J_\lambda(tu,tv)$ for $t \geqslant 0$ defined by

$$\psi_{(u,v)}(t) = \frac{t^p}{p}\|u\|_{1,p}^p + \frac{t^q}{q}\|v\|_{1,q}^q - \lambda t^{\alpha+\beta}\int_\Omega a(x)|u|^\alpha|v|^\beta dx - t^{\gamma+\delta}\int_\Omega b(x)|u|^\gamma|v|^\delta dx.$$

By a simple computation, we have

$$\psi_{(u,v)}'(t) = t^{p-1}||u||_{1,p}^p + t^{q-1}||v||_{1,q}^q - \lambda(\alpha+\beta)t^{\alpha+\beta-1}\int_\Omega a(x)|u|^\alpha|v|^\beta dx$$
$$-(\gamma+\delta)t^{\gamma+\delta-1}\int_\Omega b(x)|u|^\gamma|v|^\delta dx,$$

$$\psi_{(u,v)}''(t) = (p-1)t^{p-2}||u||_{1,p}^p + (q-1)t^{q-2}||v||_{1,q}^q - \lambda(\alpha+\beta)(\alpha+\beta-1)$$
$$\times t^{\alpha+\beta-2}\int_\Omega a(x)|u|^\alpha|v|^\beta dx - (\gamma+\delta)(\gamma+\delta-1)t^{\gamma+\delta-2}\int_\Omega b(x)|u|^\gamma|v|^\delta dx.$$

It is easy to see that $(u,v) \in M_\lambda(\Omega)$ if and only if $\psi_{(u,v)}'(1) = 0$, i.e., the elements in $M_\lambda(\Omega)$ are stationary points of the fibering maps $\psi_{(u,v)}(t)$. Naturally, $M_\lambda(\Omega)$ can be subdivided into three parts: local minima, local maxima and points of inflection respectively, that is

$$M_\lambda^+(\Omega) = \{(u,v) \in M_\lambda(\Omega) |\ \psi_{(u,v)}''(1) > 0\},$$
$$M_\lambda^-(\Omega) = \{(u,v) \in M_\lambda(\Omega) |\ \psi_{(u,v)}''(1) < 0\},$$
$$M_\lambda^0(\Omega) = \{(u,v) \in M_\lambda(\Omega) |\ \psi_{(u,v)}''(1) = 0\}.$$

Next, we describe the nature of the fibering map which is determined by the signs of the functions $\int_\Omega a(x)|u|^\alpha|v|^\beta dx$ and $\int_\Omega b(x)|u|^\gamma|v|^\delta dx$.

Case 1. If $\int_\Omega a(x)|u|^\alpha|v|^\beta dx \leqslant 0$ and $\int_\Omega b(x)|u|^\gamma|v|^\delta dx \leqslant 0$, $\psi_{(u,v)}(t)$ increases strictly for $t > 0$ and no multiple of (u,v) lies in $M_\lambda(\Omega)$.

Case 2. If $\int_\Omega a(x)|u|^\alpha|v|^\beta dx \geqslant 0$ and $\int_\Omega b(x)|u|^\gamma|v|^\delta dx \leqslant 0$, $\psi_{(u,v)}(t)$ decreases firstly and then increases. In this case, $\psi_{(u,v)}(t)$ has a local minimum at $t = t(u,v)$ and $t(u,v)(u,v) \in M_\lambda^+(\Omega)$.

Case 3. If $\int_\Omega a(x)|u|^\alpha|v|^\beta dx < 0$ and $\int_\Omega b(x)|u|^\gamma|v|^\delta dx > 0$, $\psi_{(u,v)}(t)$ increases and then decreases and there is a maximum of $\psi_{(u,v)}(t)$ at $t = t(u,v)$ and $t(u,v)(u,v) \in M_\lambda^-(\Omega)$.

Case 4. If $\int_\Omega a(x)|u|^\alpha|v|^\beta dx > 0$ and $\int_\Omega b(x)|u|^\gamma|v|^\delta dx > 0$, $\psi_{(u,v)}(t)$ decreases and then increases and finally decreases. Hence, $\psi_{(u,v)}(t)$ has a local maximum at $t = t_1(u,v)$ and a local minimum at $t = t_2(u,v)$ with $t_1(u,v)(u,v) \in M_\lambda^-(\Omega)$ and $t_2(u,v)(u,v) \in M_\lambda^+(\Omega)$.

In the following, we will prove a series of lemmas to finish the proof of Theorem 1.

LEMMA 1. *J_λ is coercive and bounded below on $M_\lambda(\Omega)$.*

Proof. If $(u,v) \in M_\lambda(\Omega)$, we have $\psi'_{(u,v)}(1) = 0$, that is

$$\|u\|_{1,p}^p + \|v\|_{1,q}^q - \lambda(\alpha+\beta)\int_\Omega a(x)|u|^\alpha|v|^\beta dx - (\gamma+\delta)\int_\Omega b(x)|u|^\gamma|v|^\delta dx = 0. \quad (2.2)$$

Hence, from (2.1), (2.2) and the Young's inequality, we obtain

$$\begin{aligned} J_\lambda(u,v) = & \left(\frac{1}{p} - \frac{1}{\gamma+\delta}\right)\|u\|_{1,p}^p + \left(\frac{1}{q} - \frac{1}{\gamma+\delta}\right)\|v\|_{1,q}^q \\ & -\lambda\left(1 - \frac{\alpha+\beta}{\gamma+\delta}\right)\int_\Omega a(x)|u|^\alpha|v|^\beta dx \\ \geqslant & \left(\frac{1}{p} - \frac{1}{\gamma+\delta}\right)\|u\|_{1,p}^p + \left(\frac{1}{q} - \frac{1}{\gamma+\delta}\right)\|v\|_{1,q}^q \\ & -\lambda\left(1 - \frac{\alpha+\beta}{\gamma+\delta}\right)\|a\|_\infty\left(\frac{\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha+\beta}dx + \frac{\beta}{\alpha+\beta}\int_\Omega |v|^{\alpha+\beta}dx\right) \\ \geqslant & \frac{\gamma+\delta-p}{(\gamma+\delta)p}\|u\|_{1,p}^p + \frac{\gamma+\delta-q}{(\gamma+\delta)q}\|v\|_{1,q}^q \\ & -\lambda\|a\|_\infty\frac{\gamma+\delta-\alpha-\beta}{(\gamma+\delta)(\alpha+\beta)}(\alpha s_1^{-1}\|u\|_{1,p}^{\alpha+\beta} + \beta s_2^{-1}\|v\|_{1,q}^{\alpha+\beta}), \end{aligned}$$

which shows that the functional J_λ is coercive from $1 < \alpha+\beta < \min\{p,q\}$. □

LEMMA 2. *Assume that (u,v) is a local maximizer or local minimizer for J_λ on $M_\lambda(\Omega)\setminus M_\lambda^0(\Omega)$, then (u,v) is a critical point of J_λ.*

Proof. Suppose that (u,v) is a local minimizer for J_λ on $M_\lambda(\Omega)\setminus M_\lambda^0(\Omega)$ (if (u,v) is a local maximizer for J_λ on $M_\lambda(\Omega)\setminus M_\lambda^0(\Omega)$, we can consider the functional $-J_\lambda$), by the theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $J'_\lambda(u,v) = \mu I'(u,v)$, where

$$I(u,v) = \|u\|_{1,p}^p + \|v\|_{1,q}^q - \lambda(\alpha+\beta)\int_\Omega a(x)|u|^\alpha|v|^\beta dx - (\gamma+\delta)\int_\Omega b(x)|u|^\gamma|v|^\delta dx.$$

Hence, we obtain

$$\langle J'_\lambda(u,v),(u,v)\rangle = \mu\langle I'(u,v),(u,v)\rangle.$$

From $(u,v)\in M_\lambda(\Omega)$, it follows that $\langle J'_\lambda(u,v),(u,v)\rangle = \psi'_{(u,v)}(1)=0$, and moreover from $(u,v)\notin M^0_\lambda(\Omega)$, we have

$$\begin{aligned}
&\langle I'(u,v),(u,v)\rangle \\
&= p\|u\|^p_{1,p}+q\|v\|^q_{1,q}-\lambda(\alpha+\beta)^2\int_\Omega a(x)|u|^\alpha|v|^\beta dx-(\gamma+\delta)^2\int_\Omega b(x)|u|^\gamma|v|^\delta dx.\\
&= \psi''_{(u,v)}(1)+\psi'_{(u,v)}(1)\\
&= \psi''_{(u,v)}(1)\neq 0.
\end{aligned}$$

Therefore, we obtain $\mu=0$. Thus, the proof is completed. □

LEMMA 3. *There is* $\Lambda_1>0$ *such that for any* $\lambda\in(0,\Lambda_1)$, $M^0_\lambda(\Omega)=\emptyset$.

Proof. Suppose by contradiction that for any $\Lambda_1>0$, there is $\lambda\in(0,\Lambda_1)$ such that $M^0_\lambda(\Omega)\neq\emptyset$. Let $(u,v)\in M^0_\lambda(\Omega)$, we have $\psi'_{(u,v)}(1)=0$ and $\psi''_{(u,v)}(1)=0$, i.e., (2.2) and the following equality hold:

$$(p-1)\|u\|^p_{1,p}+(q-1)\|v\|^q_{1,q}-\lambda(\alpha+\beta)(\alpha+\beta-1)\int_\Omega a(x)|u|^\alpha|v|^\beta dx \\ -(\gamma+\delta)(\gamma+\delta-1)\int_\Omega b(x)|u|^\gamma|v|^\delta dx=0. \quad (2.3)$$

By (2.2), (2.3) and the Young's inequality, we have

$$\begin{aligned}
&(\min\{p,q\}-\alpha-\beta)(\|u\|^p_{1,p}+\|v\|^q_{1,q})\\
&\leqslant [p-(\alpha+\beta)]\|u\|^p_{1,p}+[q-(\alpha+\beta)]\|v\|^q_{1,q}\\
&= (\gamma+\delta)[(\gamma+\delta)-(\alpha+\beta)]\int_\Omega b(x)|u|^\gamma|v|^\delta dx\\
&\leqslant [(\gamma+\delta)-(\alpha+\beta)]\|b\|_\infty\left(\gamma\int_\Omega |u|^{\gamma+\delta}dx+\delta\int_\Omega |v|^{\gamma+\delta}dx\right)\\
&\leqslant [(\gamma+\delta)-(\alpha+\beta)]\|b\|_\infty \max\{\gamma S_1^{-1},\delta S_2^{-1}\}(\|u\|^{\gamma+\delta}_{1,p}+\|v\|^{\gamma+\delta}_{1,q}),
\end{aligned}$$

let $C_1=(\min\{p,q\}-\alpha-\beta)^{-1}(\gamma+\delta-\alpha-\beta)\|b\|_\infty\max\{\gamma S_1^{-1},\delta S_2^{-1}\}$, hence we get

$$\|u\|^p_{1,p}+\|v\|^q_{1,q}\leqslant C_1(\|u\|^{\gamma+\delta}_{1,p}+\|v\|^{\gamma+\delta}_{1,q}). \quad (2.4)$$

Similarly, we obtain

$$\|u\|^p_{1,p}+\|v\|^q_{1,q}\leqslant \lambda C_2(\|u\|^{\alpha+\beta}_{1,p}+\|v\|^{\alpha+\beta}_{1,q}), \quad (2.5)$$

where $C_2=(\gamma+\delta-\max\{p,q\})^{-1}(\gamma+\delta-\alpha-\beta)\|a\|_\infty\max\{\alpha s_1^{-1},\beta s_2^{-1}\}$.

From (2.4), at least one of the following two inequalities holds:

$$\|u\|_{1,p}^{p} \leqslant C_1\|u\|_{1,p}^{\gamma+\delta} \quad \text{or} \quad \|v\|_{1,q}^{q} \leqslant C_1\|v\|_{1,q}^{\gamma+\delta}.$$

Without loss of generality, we assume that $\|u\|_{1,p}^{p} \leqslant C_1\|u\|_{1,p}^{\gamma+\delta}$, therefore we obtain $\|u\|_{1,p} \geqslant C_1^{-1/(\gamma+\delta-p)}$. From (2.5), it follows that

$$C_1^{-p/(\gamma+\delta-p)} \leqslant \|u\|_{1,p}^{p} + \|v\|_{1,q}^{q} \leqslant \lambda C_2(\|u\|_{1,p}^{\alpha+\beta} + \|v\|_{1,q}^{\alpha+\beta}),$$

which is a contradiction for λ sufficiently small. Hence, the conclusion is proved. □

LEMMA 4. *There exists $\Lambda_2 > 0$ such that, when $0 < \lambda < \Lambda_2$, $\psi_{(u,v)}(t)$ can get positive values for any $(u,v) \in W \setminus \{(0,0)\}$.*

Proof. If $\int_\Omega b(x)|u|^\gamma|v|^\delta dx \leqslant 0$, it is easy to see that $\psi_{(u,v)}(t) > 0$ for t large enough. Assume that $\int_\Omega b(x)|u|^\gamma|v|^\delta dx > 0$, from (2.1) and the Young's inequality, it follows that

$$\begin{aligned}
\psi_{(u,v)}(t) &= \frac{t^p}{p}\|u\|_{1,p}^{p} + \frac{t^q}{q}\|v\|_{1,q}^{q} - \lambda\int_\Omega a(x)|tu|^\alpha|tv|^\beta dx - \int_\Omega b(x)|tu|^\gamma|tv|^\delta dx \\
&\geqslant \frac{t^p}{p}\|u\|_{1,p}^{p} - \lambda\frac{\alpha\|a\|_\infty}{\alpha+\beta}t^{\alpha+\beta}\int_\Omega |u|^{\alpha+\beta}dx - \frac{\gamma\|b\|_\infty}{\gamma+\delta}t^{\gamma+\delta}\int_\Omega |u|^{\gamma+\delta}dx \\
&\quad + \frac{t^q}{q}\|v\|_{1,q}^{q} - \lambda\frac{\beta\|a\|_\infty}{\alpha+\beta}t^{\alpha+\beta}\int_\Omega |v|^{\alpha+\beta}dx - \frac{\delta\|b\|_\infty}{\gamma+\delta}t^{\gamma+\delta}\int_\Omega |v|^{\gamma+\delta}dx \\
&\geqslant \frac{t^p}{p}\|u\|_{1,p}^{p} - \lambda\frac{\alpha\|a\|_\infty}{\alpha+\beta}\frac{\|u\|_{1,p}^{\alpha+\beta}}{s_1}t^{\alpha+\beta} - \frac{\gamma\|b\|_\infty}{\gamma+\delta}\frac{\|u\|_{1,p}^{\gamma+\delta}}{S_1}t^{\gamma+\delta} \\
&\quad + \frac{t^q}{q}\|v\|_{1,q}^{q} - \lambda\frac{\beta\|a\|_\infty}{\alpha+\beta}\frac{\|v\|_{1,q}^{\alpha+\beta}}{s_2}t^{\alpha+\beta} - \frac{\delta\|b\|_\infty}{\gamma+\delta}\frac{\|v\|_{1,q}^{\gamma+\delta}}{S_2}t^{\gamma+\delta}. \qquad (2.6)
\end{aligned}$$

Let

$$f(t) = \frac{\|u\|_{1,p}^{p}}{p}t^p - \lambda\frac{\alpha\|a\|_\infty}{\alpha+\beta}\frac{\|u\|_{1,p}^{\alpha+\beta}}{s_1}t^{\alpha+\beta} - \frac{\gamma\|b\|_\infty}{\gamma+\delta}\frac{\|u\|_{1,p}^{\gamma+\delta}}{S_1}t^{\gamma+\delta},$$

$$g(t) = \frac{\|v\|_{1,q}^{q}}{q}t^q - \lambda\frac{\beta\|a\|_\infty}{\alpha+\beta}\frac{\|v\|_{1,q}^{\alpha+\beta}}{s_2}t^{\alpha+\beta} - \frac{\delta\|b\|_\infty}{\gamma+\delta}\frac{\|v\|_{1,q}^{\gamma+\delta}}{S_2}t^{\gamma+\delta},$$

$$f_1(t) = \frac{\|u\|_{1,p}^{p}}{p}t^p - \frac{\gamma\|b\|_\infty}{\gamma+\delta}\frac{\|u\|_{1,p}^{\gamma+\delta}}{S_1}t^{\gamma+\delta}, \quad g_1(t) = \frac{\|v\|_{1,q}^{q}}{q}t^q - \frac{\delta\|b\|_\infty}{\gamma+\delta}\frac{\|v\|_{1,q}^{\gamma+\delta}}{S_2}t^{\gamma+\delta}.$$

By a simple calculation, we know that $f_1(t)$ takes on a maximum value of

$$\frac{\gamma+\delta-p}{(\gamma+\delta)p}\left(\frac{S_1}{\gamma\|b\|_\infty}\right)^{p/(\gamma+\delta-p)} \text{ at } \bar{t}_{max} = \left(\frac{S_1}{\gamma\|b\|_\infty}\right)^{1/(\gamma+\delta-p)}\frac{1}{\|u\|_{1,p}},$$

and $g_1(t)$ obtains a maximum value at $\widetilde{t}_{max} = (S_2/\delta\|b\|_\infty)^{1/(\gamma+\delta-q)}/\|v\|_{1,q}$.

Without loss of generality, we assume $\bar{t}_{max} \leqslant \widetilde{t}_{max}$. Then, we have

$$\begin{aligned} f(\bar{t}_{max}) &= f_1(\bar{t}_{max}) - \lambda \frac{\alpha\|a\|_\infty}{\alpha+\beta} \frac{\|u\|_{1,p}^{\alpha+\beta}}{s_1} \bar{t}_{max}^{\alpha+\beta} \\ &= \frac{\gamma+\delta-p}{(\gamma+\delta)p} \left(\frac{S_1}{\gamma\|b\|_\infty} \right)^{\frac{p}{\gamma+\delta-p}} - \lambda \frac{\alpha\|a\|_\infty}{(\alpha+\beta)s_1} \left(\frac{S_1}{\gamma\|b\|_\infty} \right)^{\frac{\alpha+\beta}{\gamma+\delta-p}}. \end{aligned} \tag{2.7}$$

From (2.7), it is obvious that there exist $\lambda_1 > 0$ and $c_1 > 0$, independent of u, such that $f(\bar{t}_{max}) \geqslant c_1$ for any $0 < \lambda < \lambda_1$.

On the other hand, from $\bar{t}_{max} \leqslant \widetilde{t}_{max}$, $g_1(0) = 0$ and $g_1(t)$ is increasing in $t \in [0, \bar{t}_{max}]$, $g_1(\bar{t}_{max}) > 0$. Noting that

$$\begin{aligned} 0 \leqslant \lambda \frac{\beta\|a\|_\infty}{\alpha+\beta} \frac{\|v\|_{1,q}^{\alpha+\beta}}{s_2} \bar{t}_{max}^{\alpha+\beta} &\leqslant \lambda \frac{\beta\|a\|_\infty}{\alpha+\beta} \frac{\|v\|_{1,q}^{\alpha+\beta}}{s_2} \widetilde{t}_{max}^{\alpha+\beta} \\ &= \lambda \frac{\beta\|a\|_\infty}{(\alpha+\beta)s_2} \left(\frac{S_2}{\delta\|b\|_\infty} \right)^{\frac{\alpha+\beta}{\gamma+\delta-q}} \to 0^+ \end{aligned}$$

as $\lambda \to 0^+$, then there exists $\lambda_2 > 0$ sufficiently small, independent of u, v, such that

$$g(\bar{t}_{max}) \geqslant g_1(\bar{t}_{max}) - \lambda \frac{\beta\|a\|_\infty}{\alpha+\beta} \frac{\|v\|_{1,q}^{\alpha+\beta}}{s_2} \widetilde{t}_{max}^{\alpha+\beta} \geqslant -\frac{c_1}{2} \quad \text{for any } 0 < \lambda < \lambda_2. \tag{2.8}$$

From (2.6), (2.7) and (2.8), for any $0 < \lambda < \Lambda_2 = \min\{\lambda_1, \lambda_2\}$, we have

$$\psi_{(u,v)}(\bar{t}_{max}) \geqslant f(\bar{t}_{max}) + g(\bar{t}_{max}) \geqslant \frac{c_1}{2} > 0. \quad \square$$

COROLLARY 1. *Assume that* $0 < \lambda < \Lambda_2$, *there exists* $\nu_1 > 0$ *such that* $J_\lambda(u,v) \geqslant \nu_1$ *for all* $(u,v) \in M_\lambda^-(\Omega)$.

LEMMA 5. *Assume that* $0 < \lambda < \Lambda_0 = \min\{\Lambda_1, \Lambda_2\}$, *there is a minimum of* $J_\lambda(u,v)$ *on* $M_\lambda^+(\Omega)$.

Proof. If $(u,v) \in M_\lambda(\Omega)$, $J_\lambda(u,v)$ is bounded below and so on $M_\lambda^+(\Omega)$. Hence there exists a minimizing sequence $\{(u_n, v_n)\} \subset M_\lambda^+(\Omega)$ such that

$$\lim_{n\to\infty} J_\lambda(u_n, v_n) = \inf_{(u,v)\in M_\lambda^+(\Omega)} J_\lambda(u,v).$$

As $J_\lambda(u,v)$ is coercive, (u_n, v_n) is bounded in W. Therefore there exists a subsequence, still denoted by (u_n, v_n), and $(u_0, v_0) \in W$ such that

$$(u_n, v_n) \rightharpoonup (u_0, v_0) \text{ in } W, \quad \text{and}$$

$$u_n \to u_0 \text{ in } L^\tau(\Omega) \text{ for any } 1 < \tau < p^*, \ \ v_n \to v_0 \text{ in } L^\theta(\Omega) \text{ for any } 1 < \theta < q^*.$$

If $(u,v)\in W$ satisfies $\int_\Omega a(x)|u|^\alpha|v|^\beta dx>0$, $\psi_{(u,v)}(t)$ can only be shown in the case 2 or 4, and there is $t_1=t_1(u,v)>0$ such that $(t_1u,t_1v)\in M^+_\lambda(\Omega)$ and $J_\lambda(t_1u,t_1v)<0$, hence we get $\inf_{(u,v)\in M^+_\lambda(\Omega)}J_\lambda(u,v)<0$. From ((2.1)) and $\{(u_n,v_n)\}\subset M^+_\lambda(\Omega)$, we have

$$\begin{aligned}&\lambda\left(1-\frac{\alpha+\beta}{\gamma+\delta}\right)\int_\Omega a(x)|u_n|^\alpha|v_n|^\beta dx\\ &=-J_\lambda(u_n,v_n)+\left(\frac{1}{p}-\frac{1}{\gamma+\delta}\right)\|u_n\|^p_{1,p}+\left(\frac{1}{q}-\frac{1}{\gamma+\delta}\right)\|v_n\|^q_{1,q}.\end{aligned}$$

Therefore, let $n\to\infty$, we have $\int_\Omega a(x)|u_0|^\alpha|v_0|^\beta dx>0$, which implies that there exists $t_0=t_0(u_0,v_0)>0$ such that $(t_0u_0,t_0v_0)\in M^+_\lambda(\Omega)$ and $\psi_{(u_0,v_0)}(t)$ is decreasing on $(0,t_0)$ and $\psi'_{(u_0,v_0)}(t_0)=0$.

Suppose that $(u_n,v_n)\to(u_0,v_0)$ in W does not hold, we get

$$\int_\Omega|\nabla u_0|^p dx<\liminf_{n\to\infty}\int_\Omega|\nabla u_n|^p dx\ \text{ or }\ \int_\Omega|\nabla v_0|^q dx<\liminf_{n\to\infty}\int_\Omega|\nabla v_n|^q dx.$$

Moreover, noting that

$$\begin{aligned}\psi'_{(u_n,v_n)}(t_0)&=t_0^{p-1}\|u_n\|^p_{1,p}+t_0^{q-1}\|v_n\|^q_{1,q}-\lambda(\alpha+\beta)t_0^{\alpha+\beta-1}\int_\Omega a(x)|u_n|^\alpha|v_n|^\beta dx\\ &\quad-(\gamma+\delta)t_0^{\gamma+\delta-1}\int_\Omega b(x)|u_n|^\gamma|v_n|^\delta dx,\end{aligned}$$

and

$$\begin{aligned}\psi'_{(u_0,v_0)}(t_0)&=t_0^{p-1}\|u_0\|^p_{1,p}+t_0^{q-1}\|v_0\|^q_{1,q}-\lambda(\alpha+\beta)t_0^{\alpha+\beta-1}\int_\Omega a(x)|u_0|^\alpha|v_0|^\beta dx\\ &\quad-(\gamma+\delta)t_0^{\gamma+\delta-1}\int_\Omega b(x)|u_0|^\gamma|v_0|^\delta dx,\end{aligned}$$

and from $\liminf_{n\to\infty}\psi'_{(u_n,v_n)}(t_0)>\psi'_{(u_0,v_0)}(t_0)=0$, we have that $\psi'_{(u_n,v_n)}(t_0)>0$ for n large enough.

Since $\{(u_n,v_n)\}\subset M^+_\lambda(\Omega)$, from the case 2 or 4, it follows that $\psi'_{(u_n,v_n)}(t)<0$ for any $0<t<1$ and $\psi'_{(u_n,v_n)}(1)=0$, Hence $t_0>1$. But $(t_0u_0,t_0v_0)\in M^+_\lambda(\Omega)$, we have

$$J_\lambda(t_0u_0,t_0v_0)<J_\lambda(u_0,v_0)<\lim_{n\to\infty}J_\lambda(u_n,v_n)=\inf_{(u,v)\in M^+_\lambda(\Omega)}J_\lambda(u,v),$$

which is a contradiction. Hence, we obtain $(u_n,v_n)\to(u_0,v_0)$ in W, and

$$J_\lambda(u_0,v_0)=\lim_{n\to\infty}J_\lambda(u_n,v_n)=\inf_{(u,v)\in M^+_\lambda(\Omega)}J_\lambda(u,v),$$

which implies that (u_0,v_0) is a minimum for $J_\lambda(u,v)$ on $M^+_\lambda(\Omega)$. □

LEMMA 6. *Suppose that $0<\lambda<\Lambda_0$, there is a minimum of $J_\lambda(u,v)$ on $M^-_\lambda(\Omega)$.*

Proof. From Corollary 1, we get $J_\lambda(u,v) \geqslant \nu_1 > 0$ for any $(u,v) \in M_\lambda^-(\Omega)$. Hence $\inf_{(u,v)\in M_\lambda^-(\Omega)} J_\lambda(u,v) \geqslant \nu_1$, and there is a minimizing sequence $\{(u_n,v_n)\} \subset M_\lambda^-(\Omega)$ such that

$$\lim_{n\to\infty} J_\lambda(u_n,v_n) = \inf_{(u,v)\in M_\lambda^-(\Omega)} J_\lambda(u,v) > 0.$$

Since $J_\lambda(u,v)$ is coercive, (u_n,v_n) is bounded in W. Without loss of generality, we can suppose that

$$u_n \rightharpoonup \tilde{u} \text{ in } W_0^{1,p}(\Omega),\ v_n \rightharpoonup \tilde{v} \text{ in } W_0^{1,q}(\Omega), \quad \text{and}$$

$$u_n \to \tilde{u} \text{ in } L^\tau(\Omega) \text{ for any } 1 < \tau < p^*, v_n \to \tilde{v} \text{ in } L^\theta(\Omega) \text{ for any } 1 < \theta < q^*.$$

Similar to the proof of Lemma 5, we can get that there is $\tilde{t} > 0$ such that $(\tilde{t}\tilde{u},\tilde{t}\tilde{v}) \in M_\lambda^-(\Omega)$. If $(u_n,v_n) \to (\tilde{u},\tilde{v})$ does not hold, we know

$$\int_\Omega |\nabla \tilde{u}|^p dx < \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p dx \quad \text{or} \quad \int_\Omega |\nabla \tilde{v}|^q dx < \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^q dx$$

and $J_\lambda(u_n,v_n) \geqslant J_\lambda(tu_n,tv_n)$ for all $t \geqslant 0$. Then we get

$$\begin{aligned}
\psi_{(\tilde{u},\tilde{v})}(\tilde{t}) &= \frac{\tilde{t}^p}{p}\|\tilde{u}\|_{1,p}^p + \frac{\tilde{t}^q}{q}\|\tilde{v}\|_{1,q}^q - \lambda \tilde{t}^{\alpha+\beta}\int_\Omega a(x)|\tilde{u}|^\alpha|\tilde{v}|^\beta dx - \tilde{t}^{\gamma+\delta}\int_\Omega b(x)|\tilde{u}|^\gamma|\tilde{v}|^\delta dx \\
&< \lim_{n\to\infty}\left(\frac{\tilde{t}^p}{p}\|u_n\|_{1,p}^p + \frac{\tilde{t}^q}{q}\|v_n\|_{1,q}^q - \lambda \tilde{t}^{\alpha+\beta}\int_\Omega a(x)|\tilde{u}_n|^\alpha|\tilde{v}_n|^\beta dx \right.\\
&\quad \left. - \tilde{t}^{\gamma+\delta}\int_\Omega b(x)|u_n|^\gamma|v_n|^\delta dx\right) \\
&= \lim_{n\to\infty} J_\lambda(\tilde{t}u_n,\tilde{t}v_n) \\
&\leqslant \lim_{n\to\infty} J_\lambda(u_n,v_n) \\
&= \inf_{(u,v)\in M_\lambda^-(\Omega)} J_\lambda(\tilde{u},\tilde{v}),
\end{aligned}$$

we obtain a contradiction. Then $(u_n,v_n) \to (\tilde{u},\tilde{v})$ as $n \to \infty$. Similar to Lemma 5, the proof can be completed. □

Proof of Theorem 1. From Lemma 5 and Lemma 6, there are $(u^+,v^+) \in M_\lambda^+(\Omega)$, $(u^-,v^-) \in M_\lambda^-(\Omega)$ such that

$$J_\lambda(u^+,v^+) = \inf_{(u,v)\in M_\lambda^+(\Omega)} J_\lambda(u,v), \quad J_\lambda(u^-,v^-) = \inf_{(u,v)\in M_\lambda^-(\Omega)} J_\lambda(u,v).$$

Moreover, $J_\lambda(u^\pm,v^\pm) = J_\lambda(|u^\pm|,|v^\pm|)$, hence, we can assume $u^\pm \geqslant 0, v^\pm \geqslant 0$. From Lemma 2, $(u^\pm,v^\pm)$ are two critical points of the functional J_λ. Thus, the proof is completed. □

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