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MULTIPLE POSITIVE SOLUTIONS FOR NONLOCAL BOUNDARY VALUE PROBLEMS WITH p–LAPLACIAN OPERATOR

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Abstract. The main goal of this article is to establish an existence result for the following multi-point boundary value problem:

$$(\phi_p(u'))' + q(t)f(t,u(t),u'(t)) = 0,\ \ t \in (0,1),$$

$$u(0) = \Sigma_{i=1}^{m-2} a_i u(\xi_i),\ \phi_p(u'(1)) = \Sigma_{i=1}^{m-2} b_i \phi_p(u'(\xi_i)),$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. By means of fixed point theorem due to Avery and Peterson, we study the existence of at least three positive solutions to our problem and get some information about these solutions under some sufficient conditions posed.

1. Introduction

In this paper, we consider the existence of multiple positive solutions to the following nonlinear boundary value problem

$$(\phi_p(u'(t)))' + q(t)f(t,u(t),u'(t)) = 0,\ \ t \in (0,1), \tag{1.1}$$

$$u(0) = \Sigma_{i=1}^{m-2} a_i u(\xi_i),\ \phi_p(u'(1)) = \Sigma_{i=1}^{m-2} b_i \phi_p(u'(\xi_i)), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and a_i, b_i, f, q satisfy

(H_1) $a_i, b_i \in [0,1)$ satisfies $\Sigma_{i=1}^{m-2} a_i < 1$ and $\Sigma_{i=1}^{m-2} b_i < 1$;

(H_2) $f \in C([0,1] \times [0,\infty) \times \mathbb{R}, [0,\infty))$, $q \in L^1[0,1]$ is nonnegative on $(0,1)$ and q is not identically to zero on any subinterval of $(0,1)$.

Multi-point boundary value problems of ordinary differential equations arise in a variety of areas of applied mathematics and physics. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem (see [1]). As the linear case initiated by Il'in and Moiseev [2], this interesting topic recently still engages many

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researchers and has been studied extensively. Karakostas [3] proved the existence of positive solutions for

$$x''(t) - sign(1-\alpha)q(t)f(x,x')x' = 0, \ \ t \in (0,1),$$

with one of the following sets of boundary conditions:

$$x(0) = 0, \quad x'(1) = \alpha x'(0),$$

or

$$x(1) = 0, \quad x'(1) = \alpha x'(0),$$

where $\alpha > 0$, $\alpha \neq 1$. By using indices of convergence of the nonlinearities at 0 and at 1, the author provide a priori upper and lower bounds for the slope of the solutions. Ma [4] proved the existence of positive solutions for the multi-point boundary value problem

$$x''(t) + q(t)f(x,x')x' = 0, \ \ t \in (0,1),$$

$$x(0) = \Sigma_{i=1}^{n-2} b_i x(\xi_i), \quad x'(1) = \alpha x'(0),$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $b_i \in (0,1)$, $\alpha > 1$. They provided sufficient conditions for the existence of multiple positive solutions to the above boundary value problem by applying the fixed point theorem in cones. We refer the readers to several excellent works, for example, [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15], even with the so-called p-Laplicain operator (see [16, 17, 18, 19, 20, 21]). However, to the best knowledge of the authors, no work has been done for (1.1), (1.2). The aim of this paper is to fill this gap in the relevant literature.

Motivated by these results, our purpose of this paper is to show the existence of at least three positive solutions to multi-point BVP (1.1) and (1.2). Other organization of this article contains, in section 3, to establish an existence theorem of multiple positive solutions for the problem (1.1), (1.2), that is, Theorem 2, under some sufficient conditions provided. An example is offered for the application of our main theorem in the last section.

2. Preliminaries

Before starting this section, we probably describe the challenges when considering (1.1), (1.2). In fact, there are lots of schemes to deal with the existence of various boundary value problems, such as different fixed point theorems, iteration method, Leray-Schauder continuation theorem, barrier method. This paper uses the fixes point theorems in cones due to Avery and Peterson [22, 23]. The difficulty may occur on finding suitable cone. Another key point is that the nonlinear boundary condition (1.2) will result in finding corresponding operator. We try to overcome them and obtain the desired results as follows.

We first provide some background material from the theory of cones in Banach spaces.

DEFINITION 1. Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone if

1. $au+bv \in P$ for all $u,v \in P$ and all $a \geqslant 0$, $b \geqslant 0$;
2. $u,-u \in P$ implies $u=0$.

DEFINITION 2. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0,\infty)$ is continuous and for all $x,y \in P$ and $0 \leqslant t \leqslant 1$,

$$\alpha(tx+(1-t)y) \geqslant t\alpha(x)+(1-t)\alpha(y).$$

Similarly, we say a map γ is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\gamma : P \to [0,\infty)$ is continuous and for all $x,y \in P$ and $0 \leqslant t \leqslant 1$,

$$\gamma(tx+(1-t)y) \leqslant t\gamma(x)+(1-t)\gamma(y).$$

Let γ and θ be nonnegative continuous convex functionals on a cone P, α be a nonnegative continuous concave functional on a cone P, and ψ be a nonnegative continuous functional on a cone P. For given positive real numbers a,b,c,d, we define the following sets:

$$P(\gamma,d) = \{u \in P \mid \gamma(u) < d\}, \tag{2.1}$$

$$P(\gamma,\alpha,b,d) = \{u \in P \mid b \leqslant \alpha(u), \gamma(u) \leqslant d\}, \tag{2.2}$$

$$P(\gamma,\theta,\alpha,b,c,d) = \{u \in P \mid b \leqslant \alpha(u), \theta(u) \leqslant c, \gamma(u) \leqslant d\}, \tag{2.3}$$

$$R(\gamma,\psi,a,d) = \{u \in P \mid a \leqslant \psi(u), \gamma(u) \leqslant d\}. \tag{2.4}$$

In order to obtain our main result, the following fixed point theorem due to Avery and Peterson [22, 23] is essential.

THEOREM 1. *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda u) \leqslant \lambda\psi(u)$ for $0 \leqslant \lambda \leqslant 1$, such that for some positive numbers K and d,*

$$\alpha(u) \leqslant \psi(u) \text{ and } ||u|| \leqslant K\gamma(u)$$

for all $u \in \overline{P(\gamma,d)}$. Suppose $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$ is completely continuous and there exist positive numbers a, b and c with $a < b$ such that

(S_1) $\{u \in P(\gamma,\theta,\alpha,b,c,d) \mid \alpha(u) > b\} \neq \emptyset$ and $\alpha(Tu) > b$ for $u \in P(\gamma,\theta,\alpha,b,c,d)$;
(S_2) $\alpha(Tu) > b$ for $u \in P(\gamma,\alpha,b,d)$ with $\theta(Tu) > c$;
(S_3) $0 \notin R(\gamma,\psi,a,d)$ and $\psi(Tu) < a$ for $u \in R(\gamma,\psi,a,d)$ with $\psi(u) = a$.

Then T has at least three fixed points u_1, u_2 $u_3 \in \overline{P(\gamma,d)}$ such that $\gamma(u_i) \leqslant d$ for $i = 1,2,3$, $b < \alpha(u_1)$, $a < \psi(u_2)$, with $\alpha(u_2) < b$, $\psi(u_3) < a$.

Next, we consider the Banach space $E := C^1[0,1]$ equipped with the norm

$$||u|| = \max\{\max_{t\in[0,1]}|u(t)|, \max_{t\in[0,1]}|u'(t)|\}.$$

Define the cone $P \subset E$ by

$$P = \{u \in E \mid u(t) \text{ is nonnegative, concave and nondecreasing on } [0,1]\}. \tag{2.5}$$

One can immediately obtain some important lemmas which will be applied to conclude our main result in next section.

LEMMA 1. *Assume that* (H_1), (H_2) *hold. For any* $x \in C^+[0,1] := \{x \in C^1[0,1] | x(t) \geqslant 0\}$, *the problem*

$$(\phi_p(u'(t)))' + q(t)f(t,x(t),x'(t)) = 0, \ \ t \in (0,1), \tag{2.6}$$

$$u(0) = \Sigma_{i=1}^{m-2} a_i u(\xi_i), \ \phi_p(u'(1)) = \Sigma_{i=1}^{m-2} b_i \phi_p(u'(\xi_i)), \tag{2.7}$$

has the unique solution

$$\begin{aligned} u(t) = &\int_0^t \phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2} b_i}\Sigma_{i=1}^{m-2} b_i \int_{\xi_i}^1 q(\tau)f(\tau,x(\tau),x'(\tau))d\tau \\ &+ \int_s^1 q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds + \frac{1}{1-\Sigma_{i=1}^{m-2} a_i}\Sigma_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_p^{-1} \\ &\times \Big(\frac{1}{1-\Sigma_{i=1}^{m-2} b_i}\Sigma_{i=1}^{m-2} b_i \int_{\xi_i}^1 q(\tau)f(\tau,x(\tau),x'(\tau))d\tau \\ &+ \int_s^1 q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\Big)ds. \end{aligned}$$

Proof. For any $x \in C^+[0,1]$, suppose u is a solution of (2.6), (2.7). By direct integration of (2.6) and the boundary condition (2.7), we can find out the explicit form as above. Conversely, it is easy to verify that if u is of the above form, then u is a solution of (2.6), (2.7). $\square$

LEMMA 2. *Suppose that* (H_1), (H_2) *hold. If* $x \in C^+[0,1]$, *then the unique solution* $u(t)$ *of* (2.6), (2.7) *is concave and* $u(t) \geqslant 0$, $u'(t) \geqslant 0$, *for* $t \in [0,1]$.

Proof. Since u is the solution of (2.6), (2.7), it follows from that $(\phi_p(u'(t)))' = -q(t)f(t,x(t),x'(t))$ that $u'(t)$ is nonincreasing on $(0,1)$, which implies that $u(t)$ is concave. Moreover, the other conclusions of this lemma can be obtained via direct computation of the explicit form of $u(t)$ stated in Lemma 1. $\square$

LEMMA 3. *Set* $\overline{M} = 1 + \frac{\Sigma_{i=1}^{m-2} a_i \xi_i}{\Sigma_{i=1}^{m-2} a_i}$. *If* $u \in P$, *then*

$$\max_{t\in[0,1]}|u(t)| \leqslant \overline{M} \max_{t\in[0,1]}|u'(t)|. \tag{2.8}$$

Proof. For $u \in P$, since u is concave and nondecreasing, one have

$$u(1) - u(0) \leqslant u'(0) = \max_{t\in[0,1]} |u'(t)|.$$

Moreover, it follows from

$$\begin{aligned}(1 - \Sigma_{i=1}^{m-2} a_i)u(0) &= u(0) - \Sigma_{i=1}^{m-2} a_i u(0) = \Sigma_{i=1}^{m-2} a_i u(\xi_i) - \Sigma_{i=1}^{m-2} a_i u(0) \\ &= \Sigma_{i=1}^{m-2} a_i(u(\xi_i) - u(0)) = \Sigma_{i=1}^{m-2} a_i \xi_i u'(\eta_i),\end{aligned}$$

where $\eta_i \in (0, \xi_i)$, that

$$u(0) = \frac{\Sigma_{i=1}^{m-2} a_i \xi_i u'(\eta_i)}{1 - \Sigma_{i=1}^{m-2} a_i} \leqslant \frac{\Sigma_{i=1}^{m-2} a_i \xi_i}{1 - \Sigma_{i=1}^{m-2} a_i} \max_{t\in[0,1]} |u'(t)|. \tag{2.9}$$

Combining (2.8) and (2.9) implies that

$$\max_{t\in[0,1]} |u(t)| = u(1) \leqslant \left(1 + \frac{\Sigma_{i=1}^{m-2} a_i \xi_i}{\Sigma_{i=1}^{m-2} a_i}\right) \max_{t\in[0,1]} |u'(t)| = \overline{M} \max_{t\in[0,1]} |u'(t)|. \quad \square$$

LEMMA 4. *Define an operator* $T : P \to C^1[0,1]$ *by*

$$\begin{aligned}(Tu)(t) = &\int_0^t \phi_p^{-1}\Big(\frac{1}{1 - \Sigma_{i=1}^{m-2} b_i} \Sigma_{i=1}^{m-2} b_i \int_{\xi_i}^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \\ &+ \int_s^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau\Big) ds + \frac{1}{1 - \Sigma_{i=1}^{m-2} a_i} \Sigma_{i=1}^{m-2} a_i \\ &\times \int_0^{\xi_i} \phi_p^{-1}\Big(\frac{1}{1 - \Sigma_{i=1}^{m-2} b_i} \Sigma_{i=1}^{m-2} b_i \int_{\xi_i}^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau \\ &+ \int_s^1 q(\tau) f(\tau, u(\tau), u'(\tau)) d\tau\Big) ds.\end{aligned}$$

Then $T : P \to P$ *is completely continuous.*

Proof. According to Lemma 2, it is easy to show that $T(P) \subset P$. By similar arguments in Lemma 2.4 [19] associated with standard applications of Arzelà-Ascoli's theorem, the complete continuity of T can be obtained. $\square$

3. Main result

Note that from (H_2), there exists $k > \max\{\frac{1}{\xi_1}, \frac{1}{1-\xi_{m-2}}\}$ such that

$$0 < \int_{\frac{1}{k}}^{1-\frac{1}{k}} q(t) dt < \infty. \tag{3.1}$$

Before constructing our main result, we must define other elements. Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals θ, γ and the nonnegative continuous functional ψ be defined on the cone P by

$$\gamma(u) = \max_{t\in[0,1]} |u'(t)|, \quad u \in P, \tag{3.2}$$

$$\psi(u) = \theta(u) = \max_{t\in[0,1]} |u(t)|, \quad u \in P, \tag{3.3}$$

$$\alpha(u) = \min_{t\in[\frac{1}{k},1-\frac{1}{k}]} |u(t)|, \quad u \in P. \tag{3.4}$$

REMARK 1. For $u \in P$,

1. $\psi(\lambda) \leqslant \lambda\psi(u)$, for $0 \leqslant \lambda \leqslant 1$.
2. with Lemma 3 and the concavity, we have

$$\frac{1}{k}\theta(u) \leqslant \alpha(u) \leqslant \theta(u) = \psi(u)$$

and

$$||u|| = \max\{\theta(u), \gamma(u)\} \leqslant \overline{M}\gamma(u).$$

Set several constants appeared in the next theorem as follows:

$$L := \phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1} q(\tau)d\tau + \int_0^1 q(\tau)d\tau\Big),$$

$$\begin{aligned} M := &\int_{\frac{1}{k}}^{1-\frac{1}{k}} \phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1-\frac{1}{k}} q(\tau)d\tau + \int_s^{1-\frac{1}{k}} q(\tau)d\tau\Big)ds \\ &+ \frac{1}{1-\Sigma_{i=1}^{m-2}a_i}\Sigma_{i=1}^{m-2}a_i\int_{\frac{1}{k}}^{\xi_i} \phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1-\frac{1}{k}} q(\tau)d\tau + \int_s^{1-\frac{1}{k}} q(\tau)d\tau\Big)ds, \end{aligned}$$

and

$$\begin{aligned} N := &\int_0^1 \phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1} q(\tau)d\tau + \int_s^1 q(\tau)d\tau\Big)ds \\ &+ \frac{1}{1-\Sigma_{i=1}^{m-2}a_i}\Sigma_{i=1}^{m-2}a_i\int_0^{\xi_i} \phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1} q(\tau)d\tau + \int_s^1 q(\tau)d\tau\Big)ds. \end{aligned}$$

We now are ready to study our main result as follows.

THEOREM 2. *Let $\overline{M}$ be defined as in Lemma 3. Suppose (H_1), (H_2) hold and there exist positive constants a,b,d such that $0 < a < b \leqslant \frac{2}{k}d$. Moreover, assume that f satisfies*

(A_1) $f(t,u,v) \leqslant \phi_p(\frac{d}{L})$, $(t,u,v) \in [0,1]\times[0,\overline{M}d]\times[0,d]$;

(A_2) $f(t,u,v) \geqslant \phi_p(\frac{kb}{M})$, $(t,u,v) \in [\frac{1}{k}, 1-\frac{1}{k}] \times [b,kb] \times [0,d]$;

(A_3) $f(t,u,v) < \phi_p(\frac{a}{N})$, $(t,u,v) \in [0,1] \times [0,a] \times [0,d]$.

Then the problem (1.1), (1.2) *has at least three positive solutions* u_1, u_2, u_3 *such that* $\max_{0\leqslant t\leqslant 1}|u_i'(t)| \leqslant d$, *for* $i = 1,2,3$, $\min_{\frac{1}{k}\leqslant t\leqslant 1-\frac{1}{k}}|u_1(t)| > b$, $\max_{0\leqslant t\leqslant 1}|u_1(t)| \leqslant \overline{M}d$, $\max_{0\leqslant t\leqslant 1}|u_2(t)| > a$ *with* $\min_{\frac{1}{k}\leqslant t\leqslant 1-\frac{1}{k}}|u_2(t)| < b$, *and* $\max_{0\leqslant t\leqslant 1}|u_3(t)| < a$.

Proof. We know that the problem (1.1), (1.2) has a solution u if and only if u solves $u = Tu$, where T is defined as in Lemma 4. Thus, we will verify that T satisfies the Avery-Peterson theorem, that is, Theorem 1, which shows the existence of three fixed points of T. The proof is separated as four parts.

Part (I) – $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$, *where* $P(\gamma,d)$ *is defined as* (2.1). For $u \in \overline{P(\gamma,d)}$, $\gamma(u) = \max_{t\in[0,1]}|u'(t)| \leqslant d$. By Lemma 3, we have $\max_{0\leqslant t\leqslant 1}|u(t)| \leqslant \overline{M}d$. Hence, the condition (A_1) implies $f(t,u(t),u'(t)) \leqslant \phi_p(\frac{d}{L})$, for $u \in \overline{P(\gamma,d)}$. Furthermore, for $u \in P$, we have $Tu \in P$ by Lemma 4, which leads to Tu is concave and $\max_{t\in[0,1]}|(Tu)'(t)| = (Tu)'(0)$. Thus, one also have

$$\begin{aligned}
\gamma(Tu) &= \max_{t\in[0,1]}|(Tu)'(t)| = (Tu)'(0)\\
&= \phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1}q(\tau)f(\tau,u(\tau),u'(\tau))d\tau+\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)\\
&\leqslant \phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1}q(\tau)\phi_p\Big(\frac{d}{L}\Big)d\tau+\int_0^1 q(\tau)\phi_p\Big(\frac{d}{L}\Big)d\tau\Big)\\
&= \frac{d}{L}\phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1}q(\tau)d\tau+\int_0^1 q(\tau)d\tau\Big)\\
&= \frac{d}{L}L = d.
\end{aligned}$$

Part (II) – The condition (S_1) *in Theorem 1 (Avery-Peterson's) holds.* We take that

$$u_0(t) = -\frac{kb}{4}t^2 + \frac{kb}{2}t + \frac{3}{4}kb, \ \ t \in [0,1].$$

It is easy to see that $u_0(t) \geqslant 0$ and is concave, nondecreasing on $[0,1]$, hence, $u_0 \in P$. Immediately, we get

$$\begin{aligned}
\alpha(u_0) &= \min_{t\in[\frac{1}{k},1-\frac{1}{k}]}|u_0(t)| = u_0\Big(\frac{1}{k}\Big) = -\frac{b}{4k} + \frac{b}{2} + \frac{3}{4}kb > b,\\
\theta(u_0) &= \max_{t\in[0,1]}|u_0(t)| = u_0(1) = -\frac{kb}{4} + \frac{kb}{2} + \frac{3}{4}kb = kb,\\
\gamma(u_0) &= \max_{t\in[0,1]}|u_0'(t)| = u_0'(0) = \frac{kb}{2} \leqslant d,
\end{aligned}$$

which imply that $u_0 \in P(\gamma,\theta,\alpha,b,kb,d)$, that is, $\{u \in P(\gamma,\theta,\alpha,b,kb,d) \mid \alpha(u) > b\} \neq \emptyset$, where $P(\gamma,\theta,\alpha,b,kb,d)$ is defined of the form (2.3). Therefore, for $u \in$

$P(\gamma,\theta,\alpha,b,kb,d)$, we conclude for $t\in[\frac{1}{k},1-\frac{1}{k}]$, $b\leqslant u(t)\leqslant kb$, $0\leqslant u'(t)\leqslant d$, and according to (A_2), $f(t,u(t),u'(t))\geqslant \phi_p(\frac{kb}{M})$. By Lemma 2 and the above-mentioned Remark, one arrives that

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[\frac{1}{k},1-\frac{1}{k}]}|(Tu)(t)|\geqslant \frac{1}{k}\max_{t\in[0,1]}|(Tu)(t)| = \frac{1}{k}(Tu)(1)\\
&> \frac{1}{k}\int_{\frac{1}{k}}^{1-\frac{1}{k}}\phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1-\frac{1}{k}}q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\\
&+\int_s^{1-\frac{1}{k}}q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds+\frac{1}{k(1-\Sigma_{i=1}^{m-2}a_i)}\Sigma_{i=1}^{m-2}a_i\\
&\times\int_{\frac{1}{k}}^{\xi_i}\phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1-\frac{1}{k}}q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\\
&+\int_s^{1-\frac{1}{k}}q(\tau)f(\tau,u(\tau),u'(\tau))d\tau\Big)ds\\
&\geqslant \frac{1}{k}\int_{\frac{1}{k}}^{1-\frac{1}{k}}\phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1-\frac{1}{k}}q(\tau)\phi_p\Big(\frac{kb}{M}\Big)d\tau\\
&+\int_s^{1-\frac{1}{k}}q(\tau)\phi_p\Big(\frac{kb}{M}\Big)d\tau\Big)ds+\frac{1}{k(1-\Sigma_{i=1}^{m-2}a_i)}\Sigma_{i=1}^{m-2}a_i\\
&\times\int_{\frac{1}{k}}^{\xi_i}\phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^{1-\frac{1}{k}}q(\tau)\phi_p\Big(\frac{kb}{M}\Big)d\tau+\int_s^{1-\frac{1}{k}}q(\tau)\phi_p\Big(\frac{kb}{M}\Big)d\tau\Big)ds\\
&=\frac{1}{k}\frac{kb}{M}M=b.
\end{aligned}$$

Thus, $\alpha(Tu)>b$ for $u\in P(\gamma,\theta,\alpha,b,kb,d)$.

Part (III) – The condition (S_2) in Theorem 1 (Avery-Peterson's) holds. By using the Remark mentioned above, we obtain, for $u\in P(\gamma,\alpha,b,d)$ with $\theta(Tu)>kb$,

$$\alpha(Tu)\geqslant\frac{1}{k}\theta(Tu)>\frac{1}{k}kb=b,$$

where $P(\gamma,\alpha,b,d)$ is defined as (2.2).

Part (IV) – The condition (S_3) in Theorem 1 (Avery-Peterson's) holds. Since $\psi(0)=0<a$, $0\notin R(\gamma,\psi,a,d)$ defined as (2.4). Suppose $u\in R(\gamma,\psi,a,d)$ with $\psi(u)=a$, then, by means of the condition (A_3),

$$\begin{aligned}
\psi(Tu) &= \max_{t\in[0,1]}|(Tu)(t)| = (Tu)(1)\\
&< \int_0^1\phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^1 q(\tau)\phi_p\Big(\frac{a}{N}\Big)d\tau+\int_s^1 q(\tau)\phi_p\Big(\frac{a}{N}\Big)d\tau\Big)ds\\
&+\frac{1}{1-\Sigma_{i=1}^{m-2}a_i}\Sigma_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_p^{-1}\Big(\frac{1}{1-\Sigma_{i=1}^{m-2}b_i}\Sigma_{i=1}^{m-2}b_i\int_{\xi_i}^1 q(\tau)\phi_p\Big(\frac{a}{N}\Big)d\tau
\end{aligned}$$

$$+\int_s^1 q(\tau)\phi_p\Big(\frac{a}{N}\Big)d\tau\Big)ds$$
$$=\frac{a}{N}N=a.$$

Combining Part (I)–Part (IV), Theorem 1 implies that (1.1), (1.2) has at least three positive solutions satisfying the statement of Theorem 2. □

4. An example

Consider the nonlinear boundary value problem

$$(P)\qquad \begin{cases} (|u'(t)|u'(t))'+f(t,u(t),u'(t))=0,\ \ t\in(0,1),\\ u(0)=\frac{1}{2}u(\frac{1}{3})+\frac{1}{4}u(\frac{2}{3}),\ |u'(1)|u'(1)=\frac{1}{2}|u'(\frac{1}{3})|u'(\frac{1}{3})+\frac{1}{4}|u'(\frac{2}{3})|u'(\frac{2}{3}), \end{cases}$$

where $f(t,u,v)=\begin{cases} 11u^{15}+\frac{\sqrt{v}}{1000}, & u\leqslant 1;\\ 11\sqrt{u}+\frac{\sqrt{v}}{1000}, & u>1. \end{cases}$

Note that if we set $p=3$, $q(t)\equiv 1$ in (1.1), $m=4$, $a_1=b_1=\frac{1}{2}$, $a_2=b_2=\frac{1}{4}$, $\xi_1=\frac{1}{3}$, $\xi_2=\frac{2}{3}$ in (1.2), the problem (1.1), (1.2) can be reduced as (P).

Choose $a=\frac{2}{3}$, $b=1$, $k=6$, $d=30$, we can compute $L=\frac{2}{3}\sqrt{6}$, $M\doteq 1.8843$, $N\doteq 3.54$ and $\overline{M}=\frac{7}{3}$. Consequently, $f(t,u,v)$ satisfies

1. $f(t,u,v)<92.04<\phi_3(\frac{d}{L})=337.5$, for $(t,u,v)\in[0,1]\times[0,70]\times[0,30]$,

2. $f(t,u,v)>11>\phi_3(\frac{kb}{M})\doteq 10.138$, for $(t,u,v)\in[\frac{1}{6},\frac{5}{6}]\times[1,6]\times[0,30]$,

3. $f(t,u,v)<0.0311<\phi_3(\frac{a}{N})\doteq 0.0354$, for $(t,u,v)\in[0,1]\times[0,\frac{2}{3}]\times[0,30]$.

All conditions of our main result, that is, Theorem 2, hold. Hence, we can conclude that the problem (P) has at least three positive solutions u_1, u_2 and u_3, which satisfy

$$\max_{0\leqslant t\leqslant 1}|u_i'(t)|\leqslant 30\ \text{ for }\ i=1,2,3,$$

$$\min_{\frac{1}{6}\leqslant t\leqslant\frac{5}{6}}|u_1(t)|>1,\quad \max_{0\leqslant t\leqslant 1}|u_1(t)|\leqslant 70,$$

$$\max_{0\leqslant t\leqslant 1}|u_2(t)|>\frac{2}{3},\quad \min_{\frac{1}{6}\leqslant t\leqslant\frac{5}{6}}|u_2(t)|<1$$

and

$$\max_{0\leqslant t\leqslant 1}|u_3(t)|<\frac{2}{3}.$$

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