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MULTIPLE POSITIVE SOLUTIONS FOR A NONLINEAR CHOQUARD EQUATION WITH NONHOMOGENEOUS

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Abstract. In this paper, we study the existence of multiple positive solutions for the following equation:

$$-\Delta u+u=(K_\alpha(x)*|u|^p)|u|^{p-2}u+\lambda f(x),\ \ x\in\mathbb{R}^N,$$

where $N\geqslant 3$, $\alpha\in(0,N)$, $p\in(1+\alpha/N,(N+\alpha)/(N-2))$, $K_\alpha(x)$ is the Riesz potential, and $f(x)\in H^{-1}(\mathbb{R}^N)$, $f(x)\geqslant 0$, $f(x)\not\equiv 0$. We prove that there exists a constant $\lambda^*>0$ such that the equation above possesses at least two positive solutions for all $\lambda\in(0,\lambda^*)$. Furthermore, we can obtain the existence of the ground state solution.

1. Introduction and main result

Given $N\geqslant 3$, $\alpha\in(0,N)$ and $p\in(1+\alpha/N,(N+\alpha)/(N-2))$, we consider the problem

$$\begin{cases}-\Delta u+u=(K_\alpha(x)*|u|^p)|u|^{p-2}u+\lambda f(x),\\ u\in H^1(\mathbb{R}^N),\end{cases}\tag{1.1}$$

where $K_\alpha:\mathbb{R}^N\to\mathbb{R}$ is the Riesz potential defined for every $x\in\mathbb{R}^N\setminus\{0\}$ by

$$K_\alpha(x)=\frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}},$$

and Γ is the Gamma function, the notation $*$ denotes the convolution operator, and f satisfies the conditions:

$$\begin{cases}f\in H^{-1}(\mathbb{R}^N)\setminus\{0\},\\ f(x)\geqslant 0\ \text{ for all } x\in\mathbb{R}^N\ \text{ and }\ f(x)\not\equiv 0.\end{cases}\tag{1.2}$$

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A special case of equation (1.1), relevant in physical applications, is the Choquard equation

$$-\Delta u+u=\left(\frac{1}{|x|}*|u|^{2}\right)u,\quad x\in\mathbb{R}^{3}, \tag{1.3}$$

as a model of an electron trapped in its own hole, and was proposed by Choquaerd in 1976 as an approximation to Hartee-Fock theory of a one-component plasma [1]. In 1996, Penrose proposed equation (1.3) as a model of self-gravitating matter, in a programme in which quantum state reduction was understood as a gravitational phenomenon [5]. In 1976, Lieb [2] proved the existence and uniqueness of the minimizer solution of the ground state to equation (1.3). Lions [3] obtained the existence of many radically symmetric solutions for equation (1.3) by using variational methods, and further results for related problems may be found in [8, 9, 10, 13] and the references therein. In these contexts, equation (1.1) is usually called the nonlinear Schrödinger-Newton equation. If u solves equation (1.1), the function ψ defined by $\psi(t,x)=e^{it}u(x)$ is a solitary wave of the focusing time-dependent Hartree equation

$$i\psi_t=-\Delta\psi-(K_\alpha*|\psi|^p)|\psi|^{p-2}\psi+\lambda f.$$

In 2010, Ma and Zhao [12] considered the generalized Choquard equation

$$-\Delta u+Vu=\left(K_\alpha(x)*|u|^p\right)|u|^{p-2}u, \tag{1.4}$$

where V is an electric potential. When $V\equiv 1$, [12] proved that every positive solution of problem (1.4) is radially symmetric and monotone decreasing about some point under some assumptions on N,α,p. Especially, the positive solution is uniquely determined up to translations as $\alpha=p=2$ and $N=3$. Moroz and Van Schaftingen [16] obtained the existence of ground state solutions for problem (1.4), and got various qualitative properties of ground state solutions such as the regularity, positivity, radial symmetry and decay asymptotics. For V is a non-constant case, Moroz and Van Schaftingen [22] proved the existence of ground state solutions for problem (1.4) with Hardy-Littlewood-Sobolev critical exponent growth. Clapp and Salazar [15] and Lü [19, 21] obtained positive, sign changing and ground states solutions for problem (1.4) under different potential conditions.

For semiclassical cases,

$$-\varepsilon^2\nabla u_\varepsilon+Vu_\varepsilon=\varepsilon^{-\alpha}(K_\alpha(x)*|u_\varepsilon|^p)|u_\varepsilon|^{p-2}u_\varepsilon, \tag{1.5}$$

the existence of semiclassical ground state solutions for problem (1.5) has been considered in [18]. Under the assumptions on the decay of potential V, [23] proved the existence of positive solutions by using variational methods and nonlocal penalization technique. Moroz and Van Schaftingen in [17] obtained the nonexistence and optimal decay of supersolutions of the Choquard equations. The existence of multiple semiclassical solutions was also considered in [14]. Cingolani, Clapp and Secchi in [11] considered the existence of semiclassical regime of standing wave solutions of a Schrödinger equation in presence of nonconstant electric and magnetic potentials. Cingolani and Secchi [20] studied the semiclassical limit for the pseudorelativistic Hartree equation.

In 2003, Küpper, Zhang and Xia [7] obtained positive solutions and bifurcation point for the following equation,

$$-\Delta u+u=\left(\frac{1}{|x|}*|u|^2\right)u+\lambda f(x),\ \ x\in\mathbb{R}^3. \tag{1.6}$$

Motivated by the works we mentioned above, we study the existence of positive solutions for problem (1.1) in the present paper. The main idea of our paper is related to the nonhomogeneous for semilinear elliptic equations [3, 6, 7]. The methods in these papers are dependent on the local character of the equation. We use a different method and some special estimates to obtain our results. Likewise, we generalize the results in [7] at the same conditions.

In particular, we study the dependence of solutions on the parameter λ, and work out multiple positive solutions and the ground state solution. Such problems are often referred to being nonlocal because of the appearance of the term $\int_{\mathbb{R}^N}(K_\alpha(x)*|u|^p)|u|^p dx$, which implies that problem (1.1) is no longer a pointwise identity. The main difficulties dealing with this problem lie in the presence of the nonlocal term and the lack of compactness due to the unboundedness of the domain $\mathbb{R}^N$.

Now, we state the main result of this paper.

THEOREM 1. *Assume that* $N\geqslant 3$, $\alpha\in(0,N)$, $p\in(1+\alpha/N,(N+\alpha)/(N-2))$, *and* f *satisfies* (1.2)*, then there exists* $\lambda^*>0$ *such that equation* (1.1) *has at least two positive solutions and a ground state solution for all* $\lambda\in(0,\lambda^*)$.

The rest of this paper is organized as follows. In section 2, we introduce some notations, preliminary results and lemmas for equation (1.1). In section 3, we prove Theorem 1.

2. Some notations and preliminary results

From now on, we will use the following notations.

- $C_1,C_2,C_3,\ldots$ denote various positive constants whose exact values are not important.
- $\rightarrow$ (respectively $\rightharpoonup$) denotes strong (respectively weal) convergence.
- $o_n(1)$ denotes $o_n(1)\rightarrow 0$ as $n\rightarrow\infty$.
- $S_r=\{u\in H^1(\mathbb{R}^N):\|u\|=r\}$, $B_r=\{u\in H^1(\mathbb{R}^N):\|u\|<r\}$ and $\overline{B}_r=\{u\in H^1(\mathbb{R}^N):\|u\|\leqslant r\}$.

Let $H^1(\mathbb{R}^N)$ be the Hilbert space equipped with the inner product and norm

$$\langle u,v\rangle=\int_{\mathbb{R}^N}(\nabla u\cdot\nabla v+uv)dx,\ \ \|u\|^2=\langle u,u\rangle.$$

It follows from the Sobolev inequality that the embedding $H^1(\mathbb{R}^N)\hookrightarrow L^q(\mathbb{R}^{\mathrm{N}})$ is continuous. Thus for each $q\in[2,2^*]$, there exists $b_q>0$ such that

$$|u|_q\leqslant b_q\|u\|\ \text{ for all }\ u\in H^1(\mathbb{R}^N), \tag{2.1}$$

where $2^* = 2N/(N-2)$ and $|u|_q = (\int_{\mathbb{R}^N} |u|^q dx)^{1/q}$ is the usual norm in $L^q(\mathbb{R}^N)$.

In order to control nonlocal term in problem (1.1), we need the following Hardy-Littlewood-Sobolev inequality,

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{f(x)g(x)}{|x-y|^\alpha} dxdy \leqslant C_{m,\lambda,N}|f|_r|g|_m \ \text{ for all } \ f\in L^r(\mathbb{R}^N),\ \ g\in L^m(\mathbb{R}^N),$$

where $0<\alpha<N$, $1<r,m<\infty$ and $1/r+1/m+\alpha/N=2$, which states that if $s\in(1,N/\alpha)$ and every $\phi\in L^s(\mathbb{R}^N)$, then $K_\alpha*\phi\in L^{Ns/(N-\alpha s)}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |K_\alpha*\phi|^{\frac{Ns}{N-\alpha s}} dx \leqslant C_{N,\alpha,s}\left(\int_{\mathbb{R}^N}|\phi|^s dx\right)^{\frac{N}{N-\alpha s}}, \tag{2.2}$$

where $C_{N,\alpha,s}>0$ depends on N,α,s. Set

$$\mathscr{K}(u) = \int_{\mathbb{R}^N} (K_\alpha(x)*|u|^p)|u|^p dx = \int\int_{\mathbb{R}^N\times\mathbb{R}^N} K_\alpha(x-y)|u(x)|^p|u(y)|^p dxdy.$$

From (2.2), we get

$$\mathscr{K}(u) \leqslant C_{N,\alpha,p}\left(\int_{\mathbb{R}^N}|u|^{\frac{2Np}{N+\alpha}}dx\right)^{\frac{N+\alpha}{N}} = C_{N,\alpha,p}|u|_{pr}^{2p}, \tag{2.3}$$

for every $u\in H^1(\mathbb{R}^N)$, where $C_{N,\alpha,p}$ is a positive constant and $r=2N/(N+\alpha)$. By the Sobolev embedding, $H^1(\mathbb{R}^N)\hookrightarrow L^{2Np/(N+\alpha)}(\mathbb{R}^N)$ if and only if

$$\frac{2Np}{N+\alpha}\in(2,2^*),\ \text{that is,}\ p\in\left(\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}\right).$$

By (2.3), we know that $\mathscr{K}(u)$ is well defined in $H^1(\mathbb{R}^N)$ for $p\in(1+\alpha/N,(N+\alpha)/(N-2))$. Furthermore, $\mathscr{K}(u)\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$.

Let $u^+=\max\{u,0\}$, $u^-=\min\{u,0\}$, $u=u^++u^-$. One can verify that the weak solution of equation (1.1) is equivalent to the non-zero critical point of the functional

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)dx - \frac{1}{2p}\int_{\mathbb{R}^N}(K_\alpha(x)*(u^+)^p)(u^+)^p dx - \lambda\int_{\mathbb{R}^N} fudx.$$

Moreover, the functional I_λ is well defined and $I_\lambda\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$ from (1.2) and (2.3). Define

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)dx - \frac{1}{2p}\int_{\mathbb{R}^N}(K_\alpha(x)*(u^+)^p)(u^+)^p dx, \tag{2.4}$$

which is the functional of the equation

$$-\Delta u+u = (K_\alpha(x)*|u|^p)|u|^{p-2}u,\ \ u\in H^1(\mathbb{R}^N). \tag{2.5}$$

We consider the Nehari manifold

$$\begin{aligned}\mathcal{N} =&\{u \in H^1(\mathbb{R}^N)\setminus\{0\} : \langle I'(u), u\rangle = 0\}\\ =&\left\{u \in H^1(\mathbb{R}^N)\setminus\{0\} : \|u\|^2 = \int_{\mathbb{R}^N}(K_\alpha(x) * (u^+)^p)(u^+)^p\,\mathrm{d}x\right\}.\end{aligned}$$

Denote

$$m = \inf\{I(u) : u \in \mathcal{N}\}, \tag{2.6}$$

from [16, Theorem 1], we know that m is achieved by a function positive $w \in H^1(\mathbb{R}^N)$, which is a critical point of the functional equation I.

In order to prove our results, we need some lemmas as follows.

LEMMA 1. *Let $\alpha \in (0,N)$, $p \in \big[1,(N+\alpha)/(N-2)\big)$. If $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a bounded sequence in $L^{2Np/(N+\alpha)}(\mathbb{R}^N)$ such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$. Then we have:*
(i) $\mathcal{K}(u) - \mathcal{K}(u_n - u) \to \mathcal{K}(u)$ *as* $n \to \infty$,
(ii) $\mathcal{K}'(u_n) - \mathcal{K}'(u_n - u) \to \mathcal{K}'(u)$ *in* $H^{-1}(\mathbb{R}^N)$ *as* $n \to \infty$.

The proof is analogous to that of Lemma 3.3 in [19], we omit it here.

LEMMA 2. *Assume that $N \geqslant 3$, $\alpha \in (0,N)$, $p \in (1+\alpha/N,(N+\alpha)/(N-2))$, f satisfies (1.2) and $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a $(PS)_c$ sequence for I_λ, then there exists $u \in H^1(\mathbb{R}^N)$ such that $u_n \to u$ in $H^1(\mathbb{R}^N)$, $I_\lambda(u) = c$, or $c \geqslant I_\lambda(u) + m$.*

Proof. $\{u_n\}$ is a $(PS)_c$ sequence of I_λ in $H^1(\mathbb{R}^N)$, that is

$$I_\lambda(u_n) \to c \in \mathbb{R}, \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty.$$

We shall claim that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. It follows from (2.1) and (2.3) that

$$\begin{aligned}1 + c + o(1) \geqslant& I_\lambda(u_n) - \frac{1}{2p}\langle I'_\lambda(u_n), u_n\rangle\\ =&\frac{1}{2}\|u_n\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(K_\alpha(x) * (u_n^+)^p)(u_n^+)^p\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} f u_n dx\\ &-\frac{1}{2p}\|u_n\|^2 + \frac{1}{2p}\int_{\mathbb{R}^N}(K_\alpha(x) * (u_n^+)^p)(u_n^+)^p\,\mathrm{d}x + \frac{\lambda}{2p}\int_{\mathbb{R}^N} f u_n dx\\ \geqslant&\left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|^2 - \left(1 - \frac{1}{2p}\right)\lambda C_f\|u_n\|,\end{aligned}$$

for $p > 1$, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Of course, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, we set $v_n = u_n - u$, then $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N$, $v_n \to 0$ a.e. on $\mathbb{R}^N$.

If $v_n \to 0$ strongly in $H^1(\mathbb{R}^N)$, which means $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$ and $I_\lambda(u) = \lim_{n\to\infty} I_\lambda(u_n) = c$. When v_n does not strongly converge to zero in $H^1(\mathbb{R}^N)$, we may assume that

$$\|v_n\| \to \eta > 0.$$

It follows from Brézis-Lieb's Lemma and Lemma 1 that

$$\begin{aligned}
I_\lambda(u_n) =& I_\lambda(v_n+u)\\
=&\frac{1}{2}\|v_n+u\|^2-\lambda\int_{\mathbb{R}^N} f\cdot(v_n+u)dx-\frac{1}{2p}\int_{\mathbb{R}^N}(K_\alpha(x)*((v_n+u)^+)^p)((v_n+u)^+)^p\,dx\\
=&\frac{1}{2}\|v_n\|^2+\frac{1}{2}\|u\|^2+o_n(1)-\lambda\int_{\mathbb{R}^N} f\cdot(v_n+u)dx\\
&-\frac{1}{2p}\int_{\mathbb{R}^N}(K_\alpha(x)*(v_n^+)^p)(v_n^+)^p\,dx-\frac{1}{2p}\int_{\mathbb{R}^N}(K_\alpha(x)*(u^+)^p)(u^+)^p\,dx\\
=&I_\lambda(u)+I(v_n)-\lambda\int_{\mathbb{R}^N} f v_n dx+o_n(1).
\end{aligned}$$

Passing to the limit as $n\to\infty$, one gets

$$c=I_\lambda(u)+\lim_{n\to\infty}I(v_n). \tag{2.7}$$

We can easily obtain $\{v_n\}$ is a $(PS)_{c_1}$ sequence for I from Lemma 2.6 in [21],

$$I(v_n)\to c_1 \text{ and } I'(v_n)\to 0.$$

We choose a sequence $\{t_n\}$, such that

$$t_n{}^{2p-2}=\frac{\|v_n\|^2}{\int_{\mathbb{R}^N}(K_\alpha(x)*(v_n^+)^p)(v_n^+)^p\,dx}.$$

For $\langle I'(v_n),v_n\rangle\to 0,$ we can easily obtain

$$t_n{}^{2p-2}=\frac{\|v_n\|^2}{\|v_n\|^2-\langle I'(v_n),v_n\rangle}=\frac{1}{1-\frac{\langle I'(v_n),v_n\rangle}{\|v_n\|^2}}\to 1.$$

Then there exists a sequence $\{t_n\}$ such that $\langle I'(t_nv_n),t_nv_n\rangle=0.$ So $t_nv_n\in\mathcal{N}$. From (2.6), we have

$$\begin{aligned}
m&\leqslant \lim_{n\to\infty}I(t_nv_n)\\
&=\lim_{n\to\infty}\big(I(t_nv_n)-I(v_n)+I(v_n)\big)\\
&=\lim_{n\to\infty}\left((t_n{}^2-1)\|v_n\|^2+(t_n{}^{2p}-1)\int_{\mathbb{R}^N}(K_\alpha(x)*(v_n^+)^p)(v_n^+)^p\,dx+I(v_n)\right)\\
&=\lim_{n\to\infty}I(v_n).
\end{aligned}$$

For (2.7) and the above inequality, we get

$$c\geqslant I_\lambda(u)+m.$$

This completes the proof of Lemma 2. □

LEMMA 3. *There exist positive constants λ^*, r and ρ such that for every $\lambda \in (0,\lambda^*)$, we have*

$$I_\lambda|_{u\in S_r}(u) \geqslant \rho > 0 \text{ and } \inf_{u\in B_r} I_\lambda(u) < 0.$$

Proof. By (2.3), we have

$$\begin{aligned}
I_\lambda(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(K_\alpha(x)*(u^+)^p)(u^+)^p\,dx - \lambda\int_{\mathbb{R}^N} fudx \\
\geqslant& \frac{1}{2}\|u\|^2 - \frac{C_{N,\alpha,p}}{2p}\|u\|_{pr}^{2p} - C_f\lambda\|u\| \\
\geqslant& \frac{1}{2}\|u\|^2 - C_1\|u\|^{2p} - C_f\lambda\|u\|,
\end{aligned}$$

set $g(t) = \frac{1}{2}t^2 - C_1 t^{2p} - C_f\lambda t$ for $t > 0$, letting $r = (\frac{1}{4C_1})^{\frac{1}{2p-2}} > 0$, $\lambda^* = \frac{r}{8C_f}$, then $g(r) \geqslant \frac{1}{4}r^2 - C_f\lambda r \geqslant \frac{1}{8}r^2$ for every $\lambda \in (0,\lambda^*)$. It follows that there exists a constant $\rho > 0$ such that $I_\lambda|_{u\in S_r}(u) \geqslant \rho > 0$ for every $\lambda \in (0,\lambda^*)$. Choosing $u \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} fudx > 0$, for any $\lambda \in (0,\lambda^*)$, there exists $t > 0$ such that $tu \in B_r$ and

$$I_\lambda(tu) = \frac{1}{2}t^2\|u\|^2 - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N}(K_\alpha(x)*(u^+)^p)(u^+)^p\,dx - t\lambda\int_{\mathbb{R}^N} fudx < 0 \text{ as } t \to 0.$$

This completes the proof of Lemma 3. □

LEMMA 4. *Assume that $N \geqslant 3$, $\alpha \in (0,N)$, $p \in (1+\alpha/N, (N+\alpha)/(N-2))$ and f satisfies (1.2). Then there exists $\lambda^* > 0$ such that equation (1.1) exists a positive solution for all $\lambda \in (0,\lambda^*)$, which is a local minimizer of the function I_λ.*

Proof. Applying Ekeland's variational principle [4, Theorem 4.1] in $\overline{B_r}$, there is a minimizing sequence $\{u_n\} \subset \overline{B_r}$ such that

$$I_\lambda(u_n) \leqslant \inf_{u\in\overline{B_r},} I_\lambda(u) + \frac{1}{n}, \quad I_\lambda(\varphi) \geqslant I_\lambda(u_n) - \frac{1}{n}\|\varphi - u_n\|, \quad \varphi \in \overline{B_r}.$$

Through calculation of the standard, we have

$$\|I_\lambda'(u_n)\| \to 0 \quad \text{and} \quad I_\lambda(u_n) \to c_\lambda \quad \text{as} \quad n \to \infty,$$

where c_λ stands for the infimum of I_λ on $\overline{B}_r$. Since $\{u_n\}$ is bounded and $\overline{B}_r$ is a closed convex set, there exists $w_1 \in \overline{B}_r$, up to subsequences, such that $u_n \rightharpoonup w_1$ in $H^1(\mathbb{R}^N)$ and $u_n \to w_1$ a.e. in $\mathbb{R}^N$. Consequently, one gets

$$\begin{aligned}
\langle I_\lambda'(w_1), v\rangle =& \lim_{n\to\infty}\langle I_\lambda'(u_n), v\rangle \\
=& \lim_{n\to\infty}\left(\langle u_n, v\rangle - \int_{\mathbb{R}^N}(K_\alpha(x)*(u_n^+)^p)(u_n^+)^{p-1}v\,dx - \lambda\int_{\mathbb{R}^N} fvdx\right) \\
=& 0,
\end{aligned}$$

for all $v \in H^1(\mathbb{R}^N)$. Then w_1 is a non-trivial solution of problem (1.1). Taking the test function w_1^-, we have

$$\langle I'_\lambda(w_1), w_1^- \rangle = \|w_1^-\| - \lambda \int_{\mathbb{R}^N} f w_1^- dx = 0,$$

so $\|w_1^-\| = 0$. Thus $w_1 = w_1^+ \geqslant 0$. From the strong maximum principle, we deduce that $w_1 > 0$. Obviously, w_1 is a local minimizer of the function I_λ. This completes the proof of Lemma 4. □

LEMMA 5. *Let $\alpha \in (0,N)$, $p \in (1+\alpha/N, (N+\alpha)/(N-2))$, $\lambda > 0$ and f satisfies* (1.2)*, there exists $\lambda^* > 0$ such that for all $\lambda \in (0,\lambda^*)$, then the functional I_λ satisfies the following conditions:*
(i) *there exist r,ρ such that $I_\lambda \geqslant \rho > 0$ for all $\|u\| = r$,*
(ii) *there exists $e \in H^1(\mathbb{R}^N)$ with $\|e\| > r$ such that $I_\lambda(e) < 0$.*

Proof. (i) It has been proved by Lemma 3.
(ii) In fact, we have

$$\begin{aligned} I_\lambda(w_1+tw) =& \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla(w_1+tw)|^2 + (w_1+tw)^2\right)dx - \lambda \int_{\mathbb{R}^N} f\cdot(w_1+tw)dx \\ &- \frac{1}{2p}\int_{\mathbb{R}^N} \left(K_\alpha(x) * ((w_1+tw)^+)^p\right)((w_1+tw)^+)^p dx \\ <& \frac{1}{2}\|w_1\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} \left(K_\alpha(x) * w_1^p\right) w_1^p dx - \lambda \int_{\mathbb{R}^N} f w_1 dx \\ &+ t\int_{\mathbb{R}^N} (\nabla w_1 \cdot \nabla w + w_1 w)dx - t\int_{\mathbb{R}^N} \left(K_\alpha(x) * w_1^p\right) w_1^{p-1} w dx \\ &- t\lambda \int_{\mathbb{R}^N} f w dx + \frac{t^2}{2}\|w\|^2 - \frac{t^{2p}}{2p}\int_{\mathbb{R}^N} \left(K_\alpha(x) * w^p\right) w^p dx, \end{aligned} \tag{2.8}$$

where $t > 0$ and w is a positive solutions of problem (2.5) by [16, Theorem1]. Hence

$$\lim_{t\to+\infty} \frac{I_\lambda(w_1+tw)}{t^{2p}} = -\frac{1}{2p}\int_{\mathbb{R}^N} (K_\alpha(x) * w^p) w^p\, dx < 0.$$

Taking $t_0 > 0$ large enough such that $I_\lambda(w_1+t_0 w) < 0$. Let $e = w_1 + t_0 w$, obviously $\|w_1 + t_0 w\| > r$. □

Define

$$\Gamma = \left\{\gamma \in C^1([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = w_1, \gamma(1) = w_1 + t_0 w\right\},$$

$$c = \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} I_\lambda(u).$$

We can claim

$$c < I_\lambda(w_1) + m, \tag{2.9}$$

Since w_1 is a positive solution of problem (1.1), we have

$$\int_{\mathbb{R}^N} \left(\nabla w_1 \cdot \nabla w + w_1 w\right) dx = \int_{\mathbb{R}^N} \left(K_\alpha(x) * w_1^p\right) w_1^{p-1} w dx + \lambda \int_{\mathbb{R}^N} f w dx. \tag{2.10}$$

It follows from (2.8) and (2.10) that

$$\sup_{t>0} I_\lambda(w_1 + tw) < I_\lambda(w_1) + m.$$

Then

$$c < I_\lambda(w_1) + m.$$

This completes the proof of our claim.

3. Proof of the main result

In this section, we give the proof of our main result.

Proof of Theorem 1. According to Lemma 4, we have got that w_1 is a local minimizer of functional I_λ.

From Lemma 5, there exists a $(PS)_c$ sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ of I_λ. It is easy to prove that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence there exists $w_2 \in H^1(\mathbb{R}^N)$ such that, up to a subsequences, $u_n \rightharpoonup w_2$ in $H^1(\mathbb{R}^N)$.

If $I_\lambda(w_1) > I_\lambda(w_2)$. One see that w_1, w_2 are two positive solutions of problem (1.1) from Lemmas 2 and 4.

Otherwise, $I_\lambda(w_1) \leqslant I_\lambda(w_2)$, from (2.9), we have

$$c < I_\lambda(w_1) + m \leqslant I_\lambda(w_2) + m.$$

Combining with Lemma 2, one gets

$$I_\lambda(w_2) = \lim_{n\to\infty} I_\lambda(u_n) = c,$$

So w_2 is a non-trivial solution of problem (1.1). Taking the test function w_2^-, we have

$$\langle I_\lambda'(w_2), w_2^- \rangle = \|w_2^-\| - \lambda \int_{\mathbb{R}^N} f w_2^- dx = 0,$$

then $\|w_2^-\| = 0$, $w_2 = w_2^+ \geqslant 0$. By strong maximum principle, we obtained that w_2 is a positive solution of equation (1.1).

To sum up, we obtain two positive solutions of problem (1.1).

Last, we give the proof of the existence of ground state solution. In order to find the ground state solution of equation (1.1), we denote

$$\mathscr{G} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : I_\lambda'(u) = 0\},$$

$$m_1 = \inf\{I_\lambda(u) : u \in H^1(\mathbb{R}^N), u \in \mathscr{G}\}.$$

In the following, we show that there exists $u_* \in \mathscr{G}$ with $I_\lambda(u_*) = m_1$, that is, u_* is a ground state solution of problem (1.1). For every $u \in \mathscr{G}$, we have

$$\begin{aligned} I_\lambda(u) =& I_\lambda(u) - \frac{1}{2p}\langle I'_\lambda(u), u\rangle \\ =& \frac{1}{2}\|u\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(K_\alpha(x)*(u^+)^p)(u^+)^p\,dx - \lambda\int_{\mathbb{R}^N} fudx \\ &- \frac{1}{2p}\|u\|^2 + \frac{1}{2p}\int_{\mathbb{R}^N}(K_\alpha(x)*(u^+)^p)(u^+)^p\,dx + \frac{\lambda}{2p}\int_{\mathbb{R}^N} fudx \\ \geqslant& \left(\frac{1}{2} - \frac{1}{2p}\right)\|u\|^2 + \left(\frac{1}{2p} - 1\right)\lambda C_f\|u\|, \end{aligned}$$

hence $I_\lambda(u) > -\infty$ from $p > 1$. Then I_λ on $\mathscr{G}$ is bounded from below.

Let $\{u_n\} \subset \mathscr{G}$ be a minimizing sequence of I_λ. From Lemma 2, there exists $u_* \in \mathscr{G}$ such that $u_n \to u_*$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Otherwise,

$$m_1 \geqslant I_\lambda(u_*) + m,$$

which is in contradiction with the fact that $m > 0$ for $I_\lambda(u_*) \geqslant m_1$. Therefore u_* is a ground state solution of problem (1.1). $\square$

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