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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF SCHRÖDINGER TYPE STATIONARY EQUATIONS WITH VERY SINGULAR POTENTIALS WITHOUT PRESCRIBING BOUNDARY CONDITIONS AND SOME APPLICATIONS

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Abstract. Motivated mainly by the localization over an open bounded set Ω of $\mathbb{R}^n$ of solutions of the Schrödinger equations, we consider the Schrödinger equation over Ω with a very singular potential $V(x) \geqslant Cd(x,\partial\Omega)^{-r}$ with $r \geqslant 2$ and a convective flow $\vec{U}$. We prove the existence and uniqueness of a very weak solution of the equation, when the right hand side datum $f(x)$ is in $L^1(\Omega, d(\cdot,\partial\Omega))$, even if no boundary condition is a priori prescribed. We prove that, in fact, the solution necessarily satisfies (in a suitable way) the Dirichlet condition $u = 0$ on $\partial\Omega$. These results improve some of the results of the previous paper by the authors in collaboration with Roger Temam. In addition, we prove some new results dealing with the m-accretivity in $L^1(\Omega, d(\cdot,\partial\Omega)^\alpha)$, where $\alpha \in [0,1]$, of the associated operator, the corresponding parabolic problem and the study of the complex evolution Schrödinger equation in $\mathbb{R}^n$.

1. Introduction

The main goal of this paper is to improve some of the results of a previous paper by the authors in collaboration with R. Temam [15], as well as some of the recent researches presented in [25], concerning the Schrödinger type stationary equations with a very singular potentials and/or a possibly unbounded convective flow

$$-\Delta u + \vec{U}(x) \cdot \nabla u + V(x)u = f(x) \text{ in } \Omega, \tag{1}$$

where Ω is an open subset of $\mathbb{R}^n$ and $f \in L^1(\Omega, \delta)$, with

$$\delta(x) := d(x, \partial\Omega). \tag{2}$$

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We assume given a convective flow $\vec{U} \in L^n(\Omega)^n$ such that

$$\begin{cases} \operatorname{div} \vec{U} = 0 & \Omega, \\ \vec{U} \cdot \vec{\nu} = 0 & \partial\Omega, \end{cases} \tag{3}$$

with $\vec{\nu}$ the unit exterior normal vector to $\partial\Omega$ and a potential $V(x)$ in the general class of functions satisfying $V \in L^1_{loc}(\Omega), V \geqslant 0$ a.e. on Ω. Our main motivation is to deal with "very singular potentials" in the sense that they satisfy

$$V(x) \geqslant \frac{C}{\delta(x)^r} \text{ for some } r \geqslant 2, \text{ near } \partial\Omega. \tag{4}$$

but many results are obtained merely for $V \geqslant 0$ when f behaves suitably near $\partial\Omega$. We send the reader to [15] for considerations and references concerning the case of "moderate singular" potentials corresponding to $r \in (0,2)$. Notice that our purpose, as already indicated in the title of the paper, is to prove the existence and uniqueness of a suitable class of solutions of (1) without prescribing any boundary condition in an explicit way. Nevertheless, we shall demand the solutions to have a certain integrability condition which implicitly assumes some behaviour on $\partial\Omega$: we shall enter into details later.

In our previous paper [15] we offered a set of relevant applications leading to the consideration of problem (1). In the special case of $\vec{U} = \vec{0}$ some of those motivations where: linearization of singular and /or degenerate nonlinear equations, shape optimization in Chemical Engineering and, very specially, the study of ground solutions $\psi(t,x) = e^{-iEt}u(x)$ of the Schrödinger equation

$$\begin{cases} i\dfrac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi & \text{in } (0,\infty) \times \mathbb{R}^n \\ \psi(0,x) = \psi_0(x) & \text{on } \mathbb{R}^n \end{cases} \tag{5}$$

for very singular potentials (i.e., satisfying (4)) which try to confine the wave function ψ of the particle in the domain Ω of $\mathbb{R}^n$. A very interesting source of concrete singular potentials examples was described in the long paper [11] where only asymptotic technics were sketched for the treatment of the problems. We recall that the confinement takes place once that we prove that the solutions of (1) are, in fact, "flat solutions" (in the sense that $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$).

Concerning the case $\vec{U} \neq \vec{0}$ the main motivation mentioned in [15] was the study of the vorticity equation in Fluid Mechanics. Schrödinger equations involving also a flux term, motivated by some questions in Control Theory, were already considered also by several authors when proving the "unique continuation property" (see, e.g. [20] and its references). Notice that the existence of flat solutions to this equation implies the failure of the "unique continuation property" for such very singular class of potentials.

So, roughly speaking, the aim of this paper is to study the problem

$$Au = f \text{ in } \Omega, \tag{6a}$$

$$u = 0 \text{ on } \partial\Omega, \tag{6b}$$

where

$$Au = -\Delta u + \vec{U} \cdot \nabla u + Vu. \tag{7}$$

The content of this paper is organized as follows: after a short presentation of notations, definitions and previous results (in Section 2), we list in Section 3 some of the main new results in this paper. The equivalence between two different notions of very weak solutions of the equation under considerating is proved in Section 4 by means of a sharper approximation argument applied to the test functions. Section 5 contains the proof of the new existence and regularity regularity results, while the uniqueness of such solutions is considered in Section 6. Here the main tool is a new "local type Kato inequality" in which no use is made on possible boundary conditions (in the standard sense). The analysis of the solution when the right hand side datum f is in $L^1(\Omega;\delta^\alpha)$ with $\alpha \in [0,1]$ is made in Section 7. Finally, Section 8 collects several applications. In Section 8.1 we prove the m-accretiveness of the operator in $L^1(\Omega, d(\cdot,\partial\Omega)^\alpha)$ (and in $L^p(\Omega, d(\cdot,\partial\Omega)^\alpha)$ when $\vec{U} = 0$ or $\alpha = 0$). Some consequences in terms of the associated parabolic problem are presented. Section 8.2 deals with the evolution (complex) Schrödinger problem in $\mathbb{R}^n$ associated to the very singular potential. We prove the localization of the solution in the sense that if $\operatorname{supp}\psi_0 \subset \overline{\Omega}$ then $\operatorname{supp}\psi(t,\cdot) \subset \overline{\Omega}$, for all $t \geqslant 0$.

2. Notations, definitions and previous results

We shall adopt the same notations as in our previous paper [15]. We set

$$L^0(\Omega) = \Big\{ v : \Omega \to \mathbb{R} \text{ Lebesgue measurable} \Big\}$$

and we denote by $L^p(\Omega)$ the usual Lebesgue space $1 \leqslant p \leqslant +\infty$. Although it is not too often used, we shall use the notation

$$W^{1,p}(\Omega) = W^1L^p(\Omega)$$

for the associated Sobolev space. We need the following definitions:

DEFINITION 2.1. (of the distribution function and monotone rearrangement) Let $u \in L^0(\Omega)$. The distribution function of u is the decreasing function

$$\begin{aligned} m = m_u : \mathbb{R} &\to [0,|\Omega|] \\ t &\mapsto \text{measure}\{x : u(x) > t\} = |\{u > t\}|. \end{aligned}$$

The generalized inverse u_* of m is defined by, for $s \in [0,|\Omega|[$,

$$u_*(s) = \inf\Big\{ t : |\{u > t\}| \leqslant s \Big\},$$

and is called the decreasing rearrangement of u. We shall set $\Omega_* =]0,|\Omega|[$.

DEFINITION 2.2. Let $1 \leqslant p \leqslant +\infty$, $0 < q \leqslant +\infty$:

- If $q<+\infty$, one defines the following norm for $u\in L^0(\Omega)$

$$\|u\|_{p,q}=\|u\|_{L^{p,q}}:=\left[\int_{\Omega_*}\left[t^{\frac{1}{p}}|u|_{**}(t)\right]^q\frac{dt}{t}\right]^{\frac{1}{q}} \text{ where } |u|_{**}(t)=\frac{1}{t}\int_0^t|u|_*(\sigma)d\sigma.$$

- If $q=+\infty$,

$$\|u\|_{p,\infty}=\sup_{0<t\leqslant|\Omega|}t^{\frac{1}{p}}|u|_{**}(t).$$

The space

$$L^{p,q}(\Omega)=\left\{u\in L^0(\Omega):\|u\|_{p,q}<+\infty\right\} \tag{8}$$

is called a **Lorentz space**.

- If $p=q=+\infty$, $L^{\infty,\infty}(\Omega)=L^\infty(\Omega)$.
- The dual of $L^{1,1}(\Omega)$ is called $L_{\exp}(\Omega)$

REMARK 1. We recall that $L^{p,q}(\Omega)\subset L^{p,p}(\Omega)=L^p(\Omega)$ for any $p>1$, $q\geqslant 1$.

DEFINITION 2.3. If X is a Banach space in $L^0(\Omega)$, we shall denote the Sobolev space associated to X by

$$W^1X=\left\{\varphi\in L^1(\Omega):\nabla\varphi\in X^n\right\}$$

or more generally for $m\geqslant 1$,

$$W^mX=\left\{\varphi\in W^1X,\ \forall\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n,\ |\alpha|=\alpha_1+\ldots+\alpha_n\leqslant m,\ D^{|\alpha|}\varphi\in X\right\}.$$

We also set

$$W_0^1X=W^1X\cap W_0^{1,1}(\Omega).$$

We shall often use the principal eigenvalue $\varphi_1\in W_2$ of the homogeneous Dirichlet problem

$$\begin{cases}-\Delta\varphi_1=\lambda_1\varphi_1 & \text{in } \Omega,\\ \varphi_1=0 & \text{on } \partial\Omega,\end{cases} \tag{9}$$

where

$$W_2=\left\{\varphi\in C^2(\bar{\Omega}):\varphi=0 \text{ in } \partial\Omega\right\}. \tag{10}$$

We also need to recall the Hardy's inequality in $L^{n',\infty}$ saying that

$$\int_\Omega\frac{|u|}{\delta}\leqslant C\|\nabla u\|_{L^{n',\infty}}\qquad\forall u\in W_0^1L^{n',\infty}(\Omega), \tag{11}$$

with $n'=\frac{n}{n-1}$. This inequality can be obtained from the results of [23] (see also [16]) since $W_0^1L^{n',\infty}(\Omega)\subset W_0^1(\Omega;1+|\log\delta|)$.

DEFINITION 2.4. In the weak setting, by (3) we will mean

$$\int_\Omega \varphi \nabla\phi \cdot \vec{U} = -\int_\Omega \phi \nabla\varphi \cdot \vec{U} \qquad \forall \phi, \varphi \in W_2. \tag{12}$$

In fact we will consider one of the following general assumptions (independently of the singularity of V):

$$\begin{cases} V \in L^1_{loc}(\Omega), V \geqslant 0, \\ \vec{U} \in L^{p,1}(\Omega)^n, \text{ for some } p > n, \text{ and such that (12) holds.} \end{cases} \tag{H$_1$}$$

or

$$\begin{cases} V \in L^1_{loc}(\Omega), V \geqslant 0, \\ \vec{U} \in L^{n,1}(\Omega)^n, \text{ with small norm (as in Theorem 4.1 in [15]), and such that (12) holds.} \end{cases} \tag{H$_2$}$$

Most frequently we will assume that

$$\text{either (H}_1\text{) or (H}_2\text{) holds.} \tag{H}$$

DEFINITION 2.5. Under assumption (H), the local very weak formulation of (6a) results

$$\int_\Omega u(-\Delta\phi - \vec{U}\cdot\nabla\phi + V\phi) = \int_\Omega f\phi \qquad \forall \varphi \in \mathscr{C}^2_c(\Omega). \tag{13}$$

For $V \in L^1_{loc}(\Omega)$, we say that u is a "very weak solution in the sense of Brezis" of (6) if

$$\begin{cases} Vu\delta \in L^1(\Omega) \text{ and} \\ \displaystyle\int_\Omega u(-\Delta\phi - \vec{U}\cdot\nabla\phi + V\phi) = \int_\Omega f\phi \qquad \forall \varphi \in W_2. \end{cases} \tag{14a}$$

We will say that u is a "very weak distributional solution" of (6) if

$$\begin{cases} Vu\delta \in L^1(\Omega) \text{ and} \\ \displaystyle\int_\Omega u(-\Delta\phi - \vec{U}\cdot\nabla\phi + V\phi) = \int_\Omega f\phi \qquad \forall \varphi \in \mathscr{C}^2_c(\Omega). \end{cases} \tag{14b}$$

When $f \in L^1(\Omega,\delta)$ the natural setting for both types of solutions is

$$u \in L^{n',\infty}(\Omega). \tag{15}$$

In our previous paper [15] we proved that:

THEOREM 2.1. ([15]) *Let $f \in L^1(\Omega,\delta)$ and (H) hold. Then, there exists $u \in L^{n',\infty}(\Omega)$ such that (14b) holds. Furthermore if $V \in L^1(\Omega,\delta)$ then (14a) is satisfied.*

Moreover, even without "usual" boundary conditions (see Remark 9 in [15] for some comments on problem of different nature leading to uniqueness without boundary conditions), we also proved the following uniqueness result:

THEOREM 2.2. ([15]) *There exists, at most, one solution u of (14b) such that $\frac{u}{\delta^r} \in L^1(\Omega)$, for some $r > 1$.*

One of the main aims of this paper is to show that this exponent $r > 1$ is not optimal in Theorem 2.2 because, in fact, $r = 1$ suffices. That improves a remark (following different arguments) pointed out by H. Brezis to the second author concerning the case $\vec{U} = \overrightarrow{0}$ (see [19]). Moreover, we shall present here a numerous of other improvements with respect to our previous paper [15], as, for instance, the study of the associated eigenvalue problem, the consideration of flat solutions, the accretiveness in $L^1(\Omega, \delta^\alpha)$ of the operator when $\alpha \in [0,1)$, the consideration of the associated evolution problem, the confinement for the solution of the complex Schrödinger problem, etc.

3. Statement of new existence, uniqueness and regularity results

First, we show the equivalence of the Brezis and distributional formulations, in the space $L^1(\Omega, \delta^{-1})$.

LEMMA 3.1. (equivalence of (14a) and (14b)) *Assume that $f \in L^1(\Omega, \delta)$, (H) and let $u \in L^{n',\infty}(\Omega) \cap L^1(\Omega, \delta^{-1})$. Then (14a) is equivalent to (14b).*

First we prove an existence result in $L^{n',\infty}$ with additional estimates

THEOREM 3.1. (General existence result) *Assume that $f \in L^1(\Omega, \delta)$ and (H). Then there exists $u \in L^{n',\infty}(\Omega)$ such that (14a) holds. Furthermore, if $f \geqslant 0$, then $u \geqslant 0$. Besides*

$$\int_\Omega V|u|\delta \leqslant C_u \int_\Omega |f|\delta. \tag{16}$$

where C_u does not depend on V and f.

Then we will extend our uniqueness result

THEOREM 3.2. (Uniqueness in $L^1(\Omega, \delta^{-1})$) *Assume that $f \in L^1(\Omega, \delta)$ and (H). Then, there exists at most one $u \in L^{n',\infty}(\Omega) \cap L^1(\Omega, \delta^{-1})$ such that (14a) holds.*

From this, several existence and uniqueness results follow. If the potential is "very singular", the condition $Vu\delta \in L^1$ acts as boundary condition.

THEOREM 3.3. *Assume that $f \in L^1(\Omega, \delta)$, (H) and $V \geqslant C\delta^{-2}$ for some $C > 0$. Then there exists a unique $u \in L^{n',\infty}(\Omega) \cap L^1(\Omega, \delta^{-1})$ such that (14a) holds.*

Better integrability of the data improves the differentiability of the solution and, in particular, the (unique) solution satisfies the Dirichlet condition in the sense that $u \in W_0^1 L^{n',\infty}(\Omega)$.

THEOREM 3.4. *Assume that $f \in L^1(\Omega)$ and (H). Then, there exists exactly one $u \in L^{n',\infty}(\Omega) \cap L^1(\Omega;\delta^{-1})$ such that (14a). Furthermore, $u \in W_0^1 L^{n',\infty}(\Omega)$ and*

$$\int_\Omega V|u| \leqslant C \int_\Omega |f|, \tag{17}$$

$$\int_\Omega V|u|\delta \leqslant c_\Omega (1 + \|\vec{U}\|_{L^{n,1}}) \int_\Omega |f|\delta, \tag{18}$$

$$\|\nabla u\|_{L^{n',\infty}} \leqslant C \int_\Omega |f|. \tag{19}$$

The intermediate cases of integrability of the datum f given by the inclusions, for $\alpha \in (0,1)$,

$$L^1(\Omega) \subset L^1(\Omega;\delta^\alpha) \subset L^1(\Omega;\delta(1+|\log\delta|)) \subset L^1(\Omega;\delta) \tag{20}$$

can also be considered. In fact, in [23] it was shown that the condition $\frac{u}{\delta} \in L^1(\Omega)$ is equivalent to the data been in $L^1(\Omega;\delta(1+|\log\delta|))$.

THEOREM 3.5. *Assume that $f \in L^1(\Omega;\delta(1+|\log\delta|))$ and (H_1). Then there exists a unique $u \in L^{n',\infty}(\Omega)$ such that (14a). Furthermore, it is in $L^1(\Omega;\delta^{-1})$.*

When we improve the integrability of f near $\partial\Omega$ we can relax the conditions on $\vec{U}$.

THEOREM 3.6. *Let $0 < \alpha < 1$. Assume that (H_1), $f \in L^1(\Omega,\delta^\alpha)$ and $\vec{U} \in L^{\frac{n}{1-\alpha}}(\Omega)$. Then, there exists a unique solution $u \in L^{n',\infty}(\Omega)$ of (14a). Moreover, it is in $L^1(\Omega;\delta^{-1})$. Furthermore, $u \in W_0^1 L^{\frac{n}{n+1+\alpha}}$ and*

$$\int_\Omega V|u|\delta^\alpha \leqslant \int_\Omega |f|\delta^\alpha. \tag{21}$$

4. Proof of the Lemma 3.1

The proof is based on the following lemma, which improves [15]. The idea is how well we can approximate a test function $\phi \in W_2$ by functions $\phi_j \in \mathscr{C}_c^\infty$. In [15] our approximation was that, for $r > 1$, we can have the convergence of derivatives: $\delta^r D^\alpha \phi_j \to \delta^r D^\alpha \phi$ in L^∞ for $|\alpha| \leqslant 2$ (although this idea is older, see, e.g., Theorem 9.17 in [4]). Our improvement here is that, for $r = 1$, we can obtain the same approximation in L^∞-weak-$\star$.

LEMMA 4.1. (Approximation of test functions in W_2) *Let $\phi \in W_2$. Then, there exists a sequence $\phi_j \in C_c^\infty(\Omega)$ such that*

1. *There exists* $C > 0$ *such that* $\|\nabla\phi_j\|_{L^\infty} \leqslant C$ *for all* $j \geqslant 1$.
2. $\|\phi_j - \phi\|_{L^\infty} + \|\nabla\phi_j - \nabla\phi\|_{L^1} \to 0$.
3. $\delta\Delta\phi_j \rightharpoonup \delta\Delta\phi$ *in* L^∞*-weak-*$\star$.
4. $\frac{\phi_j}{\delta} \rightharpoonup \frac{\phi}{\delta}$ *in* L^∞*-weak-*$\star$.

Proof. Following [15], we shall consider $h \in C^\infty(\mathbb{R})$ such that

$$h(t) = \begin{cases} 1 & \text{if } t \geqslant 2, \\ 0 & \text{if } t \leqslant 1, \end{cases}$$

for $j \in \mathbb{N}^*$ set $\varepsilon = \frac{1}{j}$ and let $h_j(x) = h\left(\frac{\delta(x)-\varepsilon}{\varepsilon}\right)$, $x \in \Omega$. Setting

$$E_j = \left\{x \in \Omega : \frac{2}{j} \leqslant \delta(x) \leqslant \frac{3}{j}\right\}, \qquad E_j^c = \Omega \backslash E_j.$$

One has the following properties of h_j:

1. $\Delta h_j(x) = |\nabla h_j(x)| = 0$ for $x \in E_j^c$,
2. $h_j(x) \to 1$ as $j \to +\infty$, for any $x \in \Omega$ (since $h_j(x) = 1$ if $\delta(x) \geqslant \frac{3}{j}$),
3. $\|\delta h_j - \delta\|_\infty = \max_{x\in\overline{\Omega}} |\delta(x)h_j(x) - \delta(x)| \leqslant 3(1 + \|h'\|_\infty)\varepsilon$,
4. $\delta(x)|\nabla h_j(x)| \leqslant 3\|h'\|_\infty$ and $\delta^2(x)|\Delta h_j(x)| \leqslant c_h$ on Ω, where c_h is constant (depending only on h and Ω).

Let $\phi \in W_2$, the sequence $\varphi_j = h_j\phi$ is in $C_c^2(\Omega)$ and enjoy the following property,

$$\text{there is a constant } c > 0 \text{ such } \|\nabla\varphi_j\|_\infty \leqslant c\|\nabla\phi\|_\infty. \tag{22}$$

Indeed

$$|\nabla\varphi_j(x)| \leqslant 3\|h'\|_\infty\|\nabla\phi\|_\infty + \|h\|_\infty\|\nabla\phi\|_\infty.$$

Moreover, one has

$$\|h_j\phi - \phi\|_\infty \leqslant c\varepsilon\|\nabla\phi\|_\infty, \tag{23}$$

$$\int_\Omega |\nabla\varphi_j(x) - \nabla\phi(x)|dx \leqslant c\,\text{meas}\left\{x \in \Omega : \delta(x) \leqslant \frac{3}{j}\right\} \xrightarrow[j\to+\infty]{} 0, \tag{24}$$

$$|\delta(x)\Delta\varphi_j(x) - \delta(x)\Delta\phi(x)| \leqslant \|\delta h_j - \delta\|_\infty|\Delta\phi(x)| \text{ for } x \in E_j^c. \tag{25}$$

For $x \in E_j$, we have

$$\begin{aligned} |\delta(x)\Delta\varphi_j(x) - \delta\Delta\phi(x)| \leqslant{} & \|\delta h_j - \delta\|_\infty|\Delta\phi(x)| + \delta^2(x)\|\nabla\phi\|_\infty|\Delta h_j(x)| \\ & + 2\delta(x)|\nabla h_j(x)|\|\nabla\phi\|_\infty. \end{aligned} \tag{26}$$

The statements (25) and (26) are obtained with a straightforward computation. From those statements, we deduce that there is a constant $c_\phi > 0$ such that

$$\|\delta\Delta\varphi_j - \delta\Delta\phi\|_\infty \leqslant c_\phi. \tag{27}$$

Since

$$\text{meas}\,(E_j) \xrightarrow[j\to+\infty]{} 0 \text{ and } \|\delta h_j - \delta\|_\infty \xrightarrow[j\to+\infty]{} 0$$

we have

$$\int_\Omega |\delta(x)\Delta\varphi_j(x) - \delta(x)\Delta\phi(x)|dx \leqslant \int_{E_j^c} |\delta(x)\Delta\varphi_j(x) - \delta(x)\Delta\phi(x)|dx + c_\phi \text{meas}\,(E_j)$$
$$\leqslant \|\delta h_j - \delta\|_\infty \|\Delta\phi\|_\infty + c_\phi \text{meas}\,(E_j) \xrightarrow[j\to+\infty]{} 0. \tag{28}$$

One deduces from relations (27) and (28) that

$$\delta\Delta\varphi_j \rightharpoonup \delta\Delta\phi \quad \text{weakly-}\star \text{ in } L^\infty(\Omega).$$

Since $C_c^\infty(\Omega)$ is dense in $C_c^2(\Omega)$, we obtain the desired result.

With this technique we can now move the proof of the equivalence.

Proof. [Proof of Lemma 3.1] Let ϕ be in W_2. Then, we have a sequence $\phi_j \in C_c^\infty(\Omega)$ with the convergence stated in Theorem 4.1 such that

$$\int_\Omega u\Big[-\Delta\phi_j + \vec{U}\cdot\nabla\phi_j + V\phi_j\Big]dx = \int_\Omega f\phi_j dx. \tag{29}$$

Therefore, we have

$$\lim_{j\to+\infty}\int_\Omega u\Delta\phi_j dx = \lim_j \int_\Omega \frac{u}{\delta}(\delta\Delta\phi_j)dx = \int_\Omega u\Delta\phi\, dx, \tag{30}$$

since $\dfrac{u}{\delta} \in L^1(\Omega)$ and $\delta\Delta\phi_j \rightharpoonup \delta\Delta\phi$ in $L^\infty(\Omega)$-weak-$\star$ as $j\to\infty$.

For the same reason, one has:

$$\lim_j \int_\Omega u\vec{U}\cdot\nabla\phi_j dx = \int_\Omega u\vec{U}\cdot\nabla\phi dx$$

since $u\vec{U} \in L^1$ and $\nabla\phi_j \rightharpoonup \nabla\phi$ in L^∞-weak-$\star$. Moreover,

$$\lim \int_\Omega uV\phi_j dx = \int_\Omega uV\phi dx$$

(since $Vu\delta \in L^1(\Omega)$ and $\frac{\phi_j}{\delta} \rightharpoonup \frac{\phi}{\delta}$ in $L^\infty(\Omega)$-weak-$\star$). We easily pass to the limit in equation (29) and thus u satisfies (14a).

5. Proof of the existence and regularity results

We will consider the approximating sequence

$$\begin{cases} -\Delta u_j + \vec{U}_j \cdot \nabla u_j + V_j u_j = f_j \\ u_j \in W_0^{1,1}(\Omega) \cap W^2 L^{p,1}(\Omega) \end{cases} \tag{31}$$

i.e.

$$\int_\Omega u_j(-\Delta\varphi - \vec{U}_j \cdot \nabla\varphi + V_j\varphi) = \int_\Omega f_j\varphi \qquad \forall\varphi \in W_2. \tag{32}$$

where

$$V_j(x) = \min(V(x), j), \tag{33}$$

$$f_j(x) = \operatorname{sign}(f(x)) \min(|f(x)|, j) \tag{34}$$

and $\vec{U}_j \in \mathscr{C}_c^\infty(\Omega)^n$, such that (3) and

$$\vec{U}_j \to \vec{U} \text{ in } L^{p,1}(\Omega)^n. \tag{35}$$

First we recall our result in [15] about the approximation of solutions

THEOREM 5.1. (existence and approximation of solutions when $f \in L^1(\Omega;\delta)$ *Assume* $f \in L^1(\Omega,\delta)$ *and* (H). *Then, there is a unique solution* $u_j \in W_0^{1,1}(\Omega) \cap W^2 L^{p,1}(\Omega)$ *of* (32) *and there exists* u *such that:*

1. u *is a solution of* (14b),
2. $u_j \to u$ *a.e. in* Ω,
3. $u_j \rightharpoonup u$ *in* $L^{n',\infty}$*-weak-*$\star$ *and* $W^{1,q}(\Omega,\delta)$*-weak, for* $q < n'$,
4. $u_j \to u$ *in* $L^r(\Omega)$ *for* $r < n'$,
5. $u_j\vec{U}_j \to u\vec{U}$ *in* $L^1(\Omega)^n$,
6. $\int_\Omega V_j|u_j|\delta dx \leqslant c(1 + \|\vec{U}_j\|_{L^{n,1}}) \int_\Omega |f_j|\delta dx$,
7. $V_j u_j \delta \rightharpoonup V u\delta$ *weakly in* $L^1_{loc}(\Omega)$.

We can make some additional estimates if we restrict the set of datum f to $L^1(\Omega)$:

PROPOSITION 5.2. existence of solutions when $f \in L^1(\Omega)$ *Assume that* $f \in L^1(\Omega)$ *and* (H). *Then, the sequence* u_j *satisfies*

$$\|\nabla u_j\|_{L^{n',\infty}} \leqslant C \int_\Omega |f_j|, \tag{36}$$

$$\int_\Omega V_j|u_j| \leqslant C \int_\Omega |f_j|. \tag{37}$$

Hence

$$u_j \rightharpoonup u \text{ in } W_0^1 L^{n',\infty}(\Omega), \tag{38}$$

and the equations (36) *and* (37) *hold for* u, V *and* f.

Proof. Let $k > 0$. Then the sequence given in Theorem 5.1 satisfies

$$\int_\Omega \vec{U}_j \cdot \nabla u_j T_k(u_j)dx = 0 \text{ and } \int_\Omega V_j u_j T_k(u_j)dx \geqslant 0. \tag{39}$$

Therefore, we can use $T_k(u_j)$ as a test function in equation (31) and derive

$$\int_\Omega |\nabla T_k(u_j)|^2 dx \leqslant k \int_\Omega |f_j|dx \leqslant k \int_\Omega |f(x)|dx. \tag{40}$$

From relation (40), we deduce (see [3] or [21]) that

$$\|\nabla u_j\|_{L^{n',\infty}} \leqslant c|f|_{L^1(\Omega)}. \tag{41}$$

While to obtain relation (37), we choose as a test function for $t > 0$,

$$\Phi(t;u_j) = (|u_j| - t)_+ \operatorname{sign}(u_j).$$

Knowing as before that

$$\int_\Omega \vec{U}_j \cdot \nabla u_j \Phi(t;u_j)dx = 0 \tag{42}$$

one obtains from equation (31) that

$$\int_{|u_j|>t} |\nabla u_j|^2 dx + \int_\Omega V_j u_j \Phi(t;u_j)dx = \int_\Omega f_j \Phi(t,u_j)dx. \tag{43}$$

We derive with respect to t this equation

$$-\frac{d}{dt}\int_{|u_j|>t} |\nabla u_j|^2 dx + \int_{|u_j|>t} V_j|u_j|dx = \int_{|u_j|>t} f(x)\operatorname{sign}(u_j)dx. \tag{44}$$

Since the first term is non negative, we conclude from relation (44) that, for all $t > 0$,

$$\int_{|u_j|>t} V_j|u_j|dx \leqslant \int_{|u_j|>t} |f(x)|dx. \tag{45}$$

Letting $t \to 0$, we get the desired relation (37). Since $V_j u_j \to Vu$ a.e. in Ω, Fatou's lemma yields

$$\int_\Omega V|u|dx \leqslant \int_\Omega |f(x)|dx. \tag{46}$$

Given that $\nabla u_j \rightharpoonup \nabla u$ in $L^{n',\infty}$-weak-$\star$, we derive

$$\|\nabla u\|_{L^{n',\infty}} \leqslant c|f|_{L^1(\Omega)}. \tag{47}$$

That (14a) is satisfied is a consequence of Lemma 3.1, since, by the Hardy's inequality, we have

$$\left|\frac{u}{\delta}\right|_{L^1(\Omega)} \leqslant c\|\nabla u\|_{L^{n',\infty}} < +\infty. \tag{48}$$

This concludes the proof.

With this we proceed

Proof. [Proof of Theorem 3.4] According to Theorem 5.2, the sequence u_j belongs to a bounded set of $W_0^1 L^{n',\infty}(\Omega)$ and since the sequence converges to a solution u of the equation (14b) given in Theorem 2.1, we deduce that this solution u is in $W_0^1 L^{n',\infty}(\Omega)$ and satisfies the same kind of estimates as u_j. Moreover, $\frac{u}{\delta} \in L^1(\Omega)$ according to relation (48). Now we may appeal Theorem 3.2 to conclude that u is unique.

Finally we can prove

Proof. [Proof of Theorem 3.1] Let f be in $L^1(\Omega;\delta)$ and consider

$$f_j = \text{sign}\big(f(\cdot)\big)\min\big(|f|;j\big),\ j \geqslant 0.$$

Then according to the above result Theorem 3.4, there exists a unique $\widetilde{u}_j \in W_0^1 L^{n',\infty}(\Omega)$ satisfying

$$\int_\Omega \widetilde{u}_j\big[-\Delta\phi - \vec{U}\cdot\nabla\phi + V\phi\big]dx = \int_\Omega f_j\phi dx, \quad \forall\phi \in W_2. \qquad (14a)_j$$

Since $f_j - f_k \in L^1(\Omega)$ for k and j in $\mathbb{N}$, by the same corollary 1 of Theorem 3.2 and Theorem 5.2, we deduce that $\widetilde{u}_j - \widetilde{u}_k$ is the unique solution of

$$\int_\Omega (\widetilde{u}_j - \widetilde{u}_k)\big[-\Delta\phi - \vec{U}\cdot\nabla\phi + V\phi\big]dx = \int_\Omega (f_j - f_k)\phi dx, \quad \forall\phi \in W_2,$$

then it satisfies

$$\int_\Omega V|\widetilde{u}_j - \widetilde{u}_k|\delta\, dx \leqslant c_u \int_\Omega |f_j - f_k|\delta\, dx$$

and

$$\|\widetilde{u}_j - \widetilde{u}_k\|_{L^{n',\infty}} \leqslant c_u \int_\Omega |f_j - f_k|\delta\, dx. \qquad (49)$$

Thus $(\widetilde{u}_j)_j$ is a Cauchy sequence in $L^{n',\infty}(\Omega)$ and $(V\widetilde{u}_j)_j$ is also a Cauchy one in $L^1(\Omega;\delta)$. Therefore one has easily $\widetilde{u} \in L^{n',\infty}(\Omega)$ with $V\widetilde{u} \in L^1(\Omega;\delta)$ such that $\widetilde{u}$ satisfies equation (14a). Moreover, $\int_\Omega V|\widetilde{u}|\delta\, dx \leqslant c\int_\Omega f\delta\, dx$ and if $f \geqslant 0$ then $f_j \geqslant 0$ therefore $\widetilde{u}_j \geqslant 0$ which yields that $\widetilde{u} \geqslant 0$.

6. Proof of the uniqueness results

To complete the proof of the results above we only need to prove the uniqueness of the solutions of the equations. Once we complete the proof of Theorem 3.2 the rest of the proofs will follow as a corollary. The main tool in this proof will be a Kato type inequality up to the boundary.

6.1. Kato's inequality

Notice that, in the following result no Sobolev space is included, and hence no trace is involved. We do not consider boundary conditions in the usual way.

THEOREM 6.1. (Variant of Kato's inequality) *Let $\overline{u}$ be in $W^{1,1}_{loc}(\Omega)\cap L^{n',\infty}(\Omega)$ with $\frac{\overline{u}}{\delta}\in L^1(\Omega)$ and $\vec{U}\in L^{n,1}(\Omega)^n$ with $\operatorname{div}(\vec{U})=0$ in $\mathscr{D}'(\Omega)$, $\vec{U}\cdot\vec{\nu}=0$ on $\partial\Omega$. Assume that $L\overline{u}=-\Delta\overline{u}+\operatorname{div}(\vec{U}\,\overline{u})\in L^1(\Omega;\delta)$. Then, for all $\phi\in W_2$, $\phi\geqslant 0$ one has*

1. $\displaystyle\int_\Omega \overline{u}_+ L^*\phi\,dx \leqslant \int_\Omega \phi\,\operatorname{sign}_+(\overline{u})\,L\overline{u}\,dx,$
2. $\displaystyle\int_\Omega |\overline{u}| L^*\phi\,dx \leqslant \int_\Omega \phi\,\operatorname{sign}(\overline{u})L\overline{u}\,dx,$

where $L^*\phi=-\Delta\phi-\vec{U}\cdot\nabla\phi=-\Delta\phi-\operatorname{div}(\vec{U}\,\phi)$,

$$\operatorname{sign}_+(\sigma)=\begin{cases}1 & \text{if } \sigma>0,\\ 0 & \text{otherwise},\end{cases}\qquad \text{and}\qquad \operatorname{sign}(\sigma)=\begin{cases}1 & \text{if } \sigma>0,\\ 0 & \text{if } \sigma=0,\\ -1 & \text{if } \sigma<0.\end{cases}$$

The proofs of both theorem (Theorem 3.2 above and Theorem 6.1 below) follow the same argument as we did in [15] (Corollary 4 Theorem 10, Theorem 8). The only difference is the use of the new approximation Theorem 4.1. For the convenience of the reader we sketch here those proofs :

Proof. [Sketch of the proof of Theorem 6.1] Let $\phi\geqslant 0$, $\phi\in W_2$. Then according to Theorem 4.1 one has a sequence $\phi_j\in C_c^\infty(\Omega)$ such that $\delta\Delta\phi_j\rightharpoonup\delta\Delta\phi$ in $L^\infty(\Omega)$-weak-$\star$. This implies, together with the hypothesis that $\frac{u_+}{\delta}\in L^1(\Omega)$, that

$$\lim_{j\to+\infty}\int_\Omega \overline{u}_+\Delta\phi_j dx=\int_\Omega \overline{u}_+\Delta\phi dx. \tag{50}$$

For the same reason

$$\lim_{j\to+\infty}\int_\Omega \vec{U}\cdot\nabla\phi_j\overline{u}_+ dx=\int_\Omega \vec{U}\cdot\nabla\phi\overline{u}_+ dx. \tag{51}$$

We conclude as in [15], knowing that the local Kato's inequality (Theorem 10 in [15]) holds true.

One of the consequence of the Kato's inequality is the following maximum principle.

COROLLARY 6.2. (of Theorem 6.1) *Under the same hypothesis as for Theorem 6.1, assume that $L\overline{u}=f(x)-G(x;\overline{u})\in L^1(\Omega;\delta)$, with $G:\Omega\times\mathbb{R}\to\mathbb{R}$ a Caratheodory function (i.e for a.e x, $\sigma\to G(x;\sigma)$ is continuous, and $x\to G(x;\sigma)$ is measurable $\forall x$), satisfying the sign-function condition*

$$\operatorname{sign}(\sigma)G(x;\sigma)\geqslant 0\quad \forall\sigma\in\mathbb{R}\ a.e\ x\in\Omega.$$

Then, if $f\leqslant 0$ one has $\overline{u}\leqslant 0$.

Proof. Let $\phi \in W_2$ be such that $\phi \geqslant 0$. Then

$$\int_\Omega \overline{u}_+ L^* \phi \, dx \leqslant \int_\Omega \phi \ \mathrm{sign}_+(\overline{u}) f(x) dx - \int_\Omega \phi G(x;\overline{u}_+) dx, \tag{52}$$

since $G(x;0) = 0$ and $\mathrm{sign}_+(\sigma)G(x;\sigma) = G(x;\sigma_+) \geqslant 0$. Therefore, from this last inequality (52), knowing that

$$-\phi G(x;\overline{u}_+) \leqslant 0, \quad f(x)\,\mathrm{sign}_+(\overline{u}) \leqslant 0,$$

we deduce that

$$\forall \phi \geqslant 0,\ \phi \in W_2 : \int_\Omega \overline{u}_+ L^* \phi dx \leqslant 0. \tag{53}$$

Since $\overline{u} \in L^{n',\infty}(\Omega)$ and $L^*\phi = -\Delta\phi - \vec{U} \cdot \nabla\phi$ is in $L^{n,1}(\Omega)$ for $\phi \in W^2L^{n,1}(\Omega) \cap H_0^1(\Omega)$, thus a density argument leads from equation (53) to

$$\int_\Omega \overline{u}_+ L^* \phi dx \leqslant 0 \qquad \forall \phi \in W^2L^{n,1}(\Omega) \cap H_0^1(\Omega),\ \phi \geqslant 0. \tag{54}$$

Thus, we get:

$$\overline{u}_+ = 0.$$

This completes the proof.

6.2. Proof of the uniqueness results

Proof. [Proof of Theorem 3.2] Let $\overline{u} = u_1 - u_2$ where u_i are in $L^{n',\infty}(\Omega) \cap L^1(\Omega;\delta^{-1})$ and are two solutions of equation (14a) (or (14b), these formulations are equivalent due to Lemma 3.1 since $u_i \in L^1(\Omega;\delta^{-1})$). Then

$$L\overline{u} = -V\overline{u} \in L^1(\Omega;\delta).$$

From Theorem 6.1 one has, for a test function $\phi \in W_2$ such that $\phi \geqslant 0$,

$$\int_\Omega |\overline{u}| L^* \phi dx \leqslant -\int_\Omega \phi \,\mathrm{sign}(u) V\overline{u} = -\int_\Omega \phi V |\overline{u}| dx \leqslant 0. \tag{55}$$

As before one has:

$$\int_\Omega |\overline{u}| L^* \phi dx \leqslant 0 \qquad \forall \phi \in W^2L^{n,1}(\Omega) \cap H_0^1(\Omega),\ \ \phi \geqslant 0. \tag{56}$$

Considering $\overline{\phi}_0 \in W^2L^{n,1}(\Omega) \cap H_0^1(\Omega)$, $\overline{\phi}_0 \geqslant 0$ solution of $L^\star \overline{\phi}_0 = 1$, we deduce

$$\int_\Omega |\overline{u}| dx \leqslant 0$$

thus $\overline{u} = 0$.

Proof. [Proof of Theorem 3.5] First let us assume that $f \geqslant 0$. Since f is a non-negative function in $L^1(\Omega;\delta)$, the existence of a solution $u \geqslant 0$ is a consequence of

Theorem 3.1. To prove the uniqueness result, let us show that exists a $c>0$ independent of u, f and V such that

$$\int_\Omega \frac{u}{\delta}dx + \int_\Omega V u\delta(1+|\log\delta|)dx \leqslant c\int_\Omega f(x)(1+|\log\delta|)\delta dx. \tag{57}$$

For this, we use the argument introduced in [23] by choosing as a test function

$$\phi = \varphi_1 \log(\varphi_1+\varepsilon),\ \varepsilon>0,$$

where φ_1 the first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary condition.
One obtains

$$\begin{aligned} -\int_\Omega u\Delta(\varphi_1\log(\varphi_1+\varepsilon))dx - \int_\Omega \vec{U}u\cdot\nabla(\varphi_1\log(\varphi_1+\varepsilon))dx \\ + \int_\Omega Vu\varphi_1\log(\varphi_1+\varepsilon)dx = \int_\Omega f\varphi_1\log(\varphi_1+\varepsilon)dx. \end{aligned} \tag{58}$$

We develop each term in relation (58) as we did in [23] knowing that φ_1 is equivalent to the distance function (say $\exists c_0>0,\ c_1>0,\ c_0\delta \leqslant \varphi_1 \leqslant c_1\delta$). We derive

$$\begin{aligned} \int_\Omega |\nabla\varphi_1|^2 \frac{u}{\varphi_1+\varepsilon}dx - \int_\Omega V u\varphi_1\log(\varphi_1+\varepsilon)dx \\ \leqslant c\left[\int_\Omega u(x)dx + \int_\Omega f(x)(1+|\log\delta|)\delta dx\right] \\ + c\int_\Omega \|\vec{U}\|\,|\log\delta| u dx + c\int_\Omega \|\vec{U}\|(x)u(x)dx. \end{aligned} \tag{59}$$

Since $\vec{U}\in L^{p,1}(\Omega),\ p>1$ then $\|\vec{U}\|\log\delta \in L^{n,1}(\Omega)$ and there exists a constant $c>0$.

$$\left\| |\vec{U}|\log\delta \right\|_{L^{n,1}} \leqslant c\|\vec{U}\|_{L^{p,1}(\Omega)}.$$

Therefore, we have

$$c\int_\Omega \|\vec{U}\|\,|\log\delta| u\,dx + c\int_\Omega \|\vec{U}\|(x)u(x)dx \leqslant c_U\|u\|_{L^{n',\infty}} \leqslant c\int_\Omega f(x)\delta(x)dx. \tag{60}$$

From relations (59) and (60), we deduce

$$\int_\Omega |\nabla\varphi_1|^2 \frac{u}{\varphi_1+\varepsilon}dx - \int_\Omega V u\varphi_1\log(\varphi_1+\varepsilon)dx \leqslant c\int_\Omega f(x)(1+|\log\delta|)\delta dx. \tag{61}$$

As in [23] we write

$$\int_\Omega V u\varphi_1|\log(\varphi_1+\varepsilon)|dx = -\int_\Omega V u\varphi_1\log(\varphi_1+\varepsilon)dx + 2\int_{\varphi_1+\varepsilon>1} V u\varphi_1\log(\varphi_1+\varepsilon)dx. \tag{62}$$

Combining these two last relations, we get

$$\int_\Omega |\nabla\varphi_1|^2 \frac{u}{\varphi_1+\varepsilon}dx + \int_\Omega V u\varphi_1 |\log(\varphi_1+\varepsilon)|dx$$
$$\leqslant c\int_\Omega f(x)(1+|\log\delta|)\delta dx + c\int_\Omega V u\delta dx. \quad (63)$$

Noticing that in a neighborhood of the boundary $\partial\Omega\subset U\subset\overline{\Omega}$ one has $\inf_{x\in U}|\nabla\varphi_1|^2(x) > 0$, we derive from relation (63) the inequality (57).

Let f be in $L^1\big(\Omega;\delta(1+|\log\delta|)\big)$, we decompose $f=f_+-f_-$ where $f_+,f_-\geqslant 0$. Due to the first part of the proof, we have u_1 (resp. u_2) a nonnegative solution of (14a) associated to f_+ (resp. f_-). One has according to relation (57) for $i=1,2$

$$\int_\Omega \frac{u_i}{\delta}dx + \int_\Omega V u_i\delta(1+|\log\delta|)dx \leqslant c\int_\Omega |f|(1+|\log\delta|)\delta dx. \quad (64)$$

By linearity we deduce that $\widetilde{u}=u_1-u_2$ is a solution of equation (14b) and satisfies $\frac{\widetilde{u}}{\delta}\in L^1(\Omega)$. We conclude with Theorem 3.2 to obtain the result.

7. Estimates when the datum f is $L^1(\Omega;\delta^\alpha)$, $0\leqslant\alpha\leqslant 1$

LEMMA 7.1. *Under the same assumptions as for Theorem 3.5, if furthermore $f\in L^1(\Omega;\delta^\alpha)$, $0\leqslant\alpha<1$ then the function $\widetilde{u}$ solution of equation (14a) verifies*

$$\int_\Omega (V|\widetilde{u}|\delta^\alpha)(x)dx \leqslant c_\alpha\int_\Omega |f(x)|\delta^\alpha(x)dx.$$

Proof. For $k\geqslant 0$, let us consider $V_k=\min(V;k)$ and define the linear operator T_k on $L^1(\Omega;\delta)$ by setting $T_kf=V_k\widetilde{u}_{kf}$, where $\widetilde{u}_{kf}$ is the unique solution of

$$\int_\Omega \widetilde{u}_{kf}\left[-\Delta\phi+\vec{U}\cdot\nabla\phi+V_k\phi\right]dx = \int_\Omega f\phi dx \qquad \forall\phi\in W_2. \quad (65)$$

The existence and uniqueness follows from Theorem 7 in [15].

According to Corollary 3.4 of Theorem 3.2 and Theorem 5.2. T_k maps $L^1(\Omega)$ into itself with

$$\|T_kf\|_{L^1(\Omega)} = \int_\Omega V_k|\widetilde{u}_{kf}|dx \leqslant \|f\|_{L^1(\Omega)}, \quad (66)$$

and T_k maps $L^1(\Omega;\delta)$ into itself with

$$\|T_kf\|_{L^1(\Omega;\delta)} \leqslant c(1+\|\vec{U}\|_{L^{n,1}})\|f\|_{L^1(\Omega;\delta)}. \quad (67)$$

Since $L^1(\Omega;\delta^\alpha)$ is the interpolation space in the sense of Peetre between $L^1(\Omega;\delta)$ and $L^1(\Omega)$, that is

$$L^1(\Omega,\delta^\alpha) = \Big(L^1(\Omega;\delta),L^1(\Omega)\Big)_{\alpha,1},$$

we derive from Marcinkewicz's interpolation theorem (see [2, 21]) that T_k maps $L^1(\Omega;\delta^\alpha)$ into itself and

$$\|T_kf\|_{L^1(\Omega;\delta^\alpha)} \leqslant c^\alpha(1+\|\vec{U}\|_{L^{n},1})^\alpha\|f\|_{L^1(\Omega,\delta^\alpha)}, \quad \forall f\in L^1(\Omega;\delta^\alpha).$$

Considering the unique solution $\widetilde{u}_{kj}$ for j fixed in $\mathbb{N}$, of the equation

$$\int_{\Omega} \widetilde{u}_{kj}\Big[-\Delta\phi - \vec{U}\cdot\nabla\phi + V_k\phi\Big]dx = \int_{\Omega} f_j\phi dx, \quad \forall\varphi \in W_2, \qquad (14a)_{kj}$$

where $f_j = \operatorname{sign}(f)\min(|f|, j)$, applying Theorem 5.1 with the sequence $(\widetilde{u}_{kj})_k$, and due to the uniqueness result we deduce that, when $k \to +\infty$, $\widetilde{u}_{kj} \to \widetilde{u}_j$ in $L^{n',\infty}(\Omega)$ and $\widetilde{u}_j$ is the solution of $(14a)_j$. Therefore, one has

$$\int_{\Omega} V|\widetilde{u}_j|\delta^{\alpha}dx \leqslant \lim_{k\to+\infty} |T_k f_j|_{L^1(\Omega;\delta^{\alpha})} \leqslant c_{\alpha}|f_j|_{L^1(\Omega;\delta^{\alpha})}. \qquad (68)$$

As we have shown in the proof of Theorem 3.1, $\widetilde{u}_j$ converges to $\widetilde{u}$ as $j \to +\infty$; we deduce the desired inequality.

The proof of Theorem 3.6 needs the following lemma given in Theorem 13 of [15].

LEMMA 7.2. *Let* $0 < \alpha < 1$, $g \in L^1(\Omega;\delta^{\alpha})$, $\vec{U}$ *in* $L^{\frac{n}{1+\alpha}}(\Omega)^n$, (3). *Then, there exists a unique solution* $\overline{u} \in L^{n',\infty}(\Omega)$ *satisfying*

$$\int_{\Omega} \overline{u}\Big[-\Delta\phi - \vec{U}\cdot\nabla\phi\Big]dx = \int_{\Omega} g\phi dx \qquad \forall\phi \in W_2. \qquad (69)$$

Moreover, there exists a constant $K(\alpha;\Omega) > 0$ *such that*

$$\|\overline{u}\|_{W_0^1 L^{\frac{n}{n-1+\alpha}}(\Omega)} \leqslant K(\alpha;\Omega)\left(1 + \|\vec{U}\|_{L^{\frac{n}{1-\alpha}}}\right)|g|_{L^1(\Omega;\delta^{\alpha})}. \qquad (70)$$

Proof. [Proof of Theorem 3.6] Let u be the unique solution (2) given by Theorem 3.5 when $f \in L^1(\Omega;\delta^{\alpha})$, $0 < \alpha < 1$. We set $g = Vu - f$. Then following Lemma 7.1, one has $g \in L^1(\Omega;\delta^{\alpha})$ and u satisfies the same type equation (69). Therefore, we can apply Lemma 7.2 to conclude.

8. Some consequences: principal eigenvalue and eigenfunction of $-\Delta + \vec{U}\cdot\nabla$ and of the operator A, the m -accretivity of A and the complex Schrödinger problem in the whole space

8.1. Principal eigenvalue and eigenfunction for $-\Delta + \vec{U}\cdot\nabla$ and the m-accretivity of $-\Delta + \vec{U}\cdot\nabla + V$

Let us start by recalling a well-known result (see, e.g., [12])

THEOREM 8.1. (Krein-Rutman's theorem) *Let X be an ordered Banach space, the interior positive cone K of which $\mathring{K}$ is non void, $T : X \to X$ a compact linear operator which is strongly positive, i.e $Tf > 0$ if $f > 0$. Then, the spectral radius of T, $r(T) > 0$ and is a simple eigenvalue with an eigenvector $\psi_1 \in \mathring{K}$.*

We recall the following definition of an m-accretive operator.

DEFINITION 8.1. (m-accretive operator) Let X be a Banach space. A linear unbounded operator

$$A : D(A) \subset X \to X$$

is called accretive if

1. $\forall \widetilde{u} \in D(A)$ and $\forall \lambda > 0$ it holds that $\|\widetilde{u}\|_X \leqslant \|\widetilde{u} + \lambda A \widetilde{u}\|_X$.

The operator is called m-accretive if it is accretive and

2. $\forall \lambda > 0$ we have that $\overline{D(A)} \subset R(I + \lambda A)$.

Let us consider $\vec{U} \in L^{p,1}(\Omega)^n$, $p > n$ (or in $L^{n,1}(\Omega)^n$ but with a small norm as in [15]), we define a compact operator

$$T : C(\overline{\Omega}) \to W_0^1 L^{p,1}(\Omega) \hookrightarrow C(\overline{\Omega})$$

by setting

$$Tf = u \text{ if and only if } \begin{cases} -\Delta u - \vec{U} \cdot \nabla u = f \\ u \in W_0^1 L^{p,1}(\Omega), p > n \end{cases}$$

(the existence, uniqueness and regularity of u in given in [15]). Using the Bony's maximum principle or Stapamcchia's argument, we have for $f > 0$ the solution $u > 0$. Since the positive cone $K = C_+(\overline{\Omega}) = \{\varphi \in C(\overline{\Omega}) : \varphi \geqslant 0\}$ has its interior $\mathring{K}$ non void, we may apply the Krein-Rutman's theorem (see Theorem 8.3) to derive the

THEOREM 8.2. *There exist a real* $\lambda_1 > 0$ *and a positive function* $\psi_1 \in W^2 L^{p,1}(\Omega) \cap H_1^0(\Omega)$ *such that*

$$-\Delta \psi_1 - \vec{U} \cdot \nabla \psi_1 = \lambda_1 \psi_1.$$

Moreover, $L^1(\Omega;\delta) \hookrightarrow L^1(\Omega;\psi_1)$ *and if* $\vec{U} \in L^\infty(\Omega)^n$ *then* $\psi_1 \geqslant c\delta$ *so that*

$$L^1(\Omega;\delta) = L^1(\Omega;\psi_1).$$

REMARK 2. The fact that $L^1(\Omega;\delta) \hookrightarrow L^1(\Omega;\psi_1)$ comes from the fact

$$0 < \psi_1(x) \leqslant \delta(x) \|\nabla \psi_1\|_\infty \leqslant c \|\psi_1\|_{W^2 L^{p,1}} \delta(x) < +\infty, \; x \in \Omega.$$

Next, we want to prove Theorem 8.3 concerning the m-accretivity of $A = -\Delta + \vec{U} \cdot \nabla + V$ in the Banach space $L^1(\Omega;\delta^\alpha)$, $0 \leqslant \alpha \leqslant 1$. The argument is similar to the one given in [22].
First, we endow the space $L^1(\Omega;\delta^\alpha)$ with the following equivalent norm

$$\|f\|_\alpha = \int_\Omega |f(x)| \psi_1^\alpha(x) dx,$$

with ψ_1 given in Theorem 8.2. We shall introduce the following definition

DEFINITION 8.2. Let $\overline{u}$ be in $L^1(\Omega,;\delta^\alpha)$ with $V\overline{u} \in L^1(\Omega;\delta^\alpha)$. We will say that $A\overline{u} \in L^1(\Omega;\delta^\alpha)$ if there exists a function $f \in L^1(\Omega;\delta^\alpha)$ such that $A\overline{u} = f$ and

$$\int_\Omega \phi f dx = \int_\Omega \overline{u}\Big[-\Delta\phi - \vec{U}\cdot\nabla\phi + V\phi\Big]dx, \quad \forall \phi \in C_c^2(\Omega). \tag{71}$$

Here, $V \geqslant 0$ locally integrable and $\vec{U}$ is as in Theorem 2.1. When $\vec{U} = 0$ and $0 \leqslant V \in L^\infty(\Omega)$ then we choose $D(A) \subset W_0^{1,1}(\Omega)$. In this setting, in which traces exist, previous results apply (see, e.g., [10]). However, when $V \geqslant c\delta^{-2}$ (our main case of interest due to the Schrödinger equation) we can no longer expect that $D(A) \subset W_0^{1,1}(\Omega)$. Nonetheless, we have shown that $D(A) \subset L^1(\Omega;\delta^{-1})$, a space which *also* acts as having a Dirichlet boundary condition on $\partial\Omega$.

We can define the operator $A : D(A) \subset L^1(\Omega;\delta^\alpha) \to L^1(\Omega;\delta^\alpha)$, where the domain of A is

$$D(A) = \Big\{\overline{u} \in L^{n',\infty}(\Omega) \cap L^1(\Omega;\delta^{-1}) \cap L^1(\Omega;V\delta) : A\overline{u} \in L^1(\Omega;\delta^\alpha)\Big\}.$$

Therefore, we always have $C_c^2(\Omega) \subset D(A) \subset L^1(\Omega;\delta^\alpha)$ this implies that $D(A)$ is dense in $L^1(\Omega;\delta^\alpha)$, $0 \leqslant \alpha \leqslant 1$. Moreover, one has the :

LEMMA 8.1. *Let $V \geqslant 0$, locally integrable, $\vec{U} \in L^\infty(\Omega)$ be such that (3) and $0 \leqslant \alpha < 1$. Then, for all $\lambda > 0$ and $f \in L^1(\Omega;\delta^\alpha)$, there exists a unique function $u \in D(A)$ such that*

$$u + \lambda A u = f.$$

Proof. Indeed, since $L^1(\Omega;\delta^\alpha) \subset L^1\big(\Omega;\delta(1+|\log\delta|)\big)$, we may apply Theorem 3.5 to derive that for all $\lambda > 0$ all $f \in L^1(\Omega;\delta^\alpha)$ we have a unique function $u \in L^{n',\infty}(\Omega)$ with $\dfrac{u}{\delta} \in L^1(\Omega)$, $Vu \in L^1(\Omega;\delta^\alpha)$ and for all $\phi \in W^2L^{n,1}(\Omega) \in H_0^1(\Omega)$,

$$\int_\Omega f\phi dx = \int_\Omega u\big[\phi + \lambda(-\Delta\phi - \vec{U}\cdot\nabla\phi + V\phi)\big]dx. \tag{72}$$

This is equivalent to say that $u + \lambda A u = f$ and $u \in D(A)$.

So for $0 \leqslant \alpha < 1$, it remains to show that for all $\overline{u} \in D(A)$, for all $\lambda > 0$

$$\|\overline{u}\|_\alpha \leqslant \|\overline{u} + \lambda A\overline{u}\|_\alpha. \tag{73}$$

That is to say, setting $f = \overline{u} + \lambda A\overline{u}$,

$$\int_\Omega |\overline{u}|\psi_1^\alpha dx \leqslant \int_\Omega |f|\psi_1^\alpha dx. \tag{74}$$

To prove such inequality, we introduce as in [22] the

LEMMA 8.2. *Let* $\varepsilon > 0$, $0 \leqslant \alpha \leqslant 1$ *and let*

$$\psi_{1\varepsilon} = (\psi + \varepsilon)^{\alpha} - \varepsilon^{\alpha} \in W^2 L^{n,1}(\Omega) \cap H_0^1(\Omega). \tag{75}$$

Then, for all $\overline{u} \in L^{n',\infty}(\Omega)$, $\overline{u} \geqslant 0$, *one has*

$$J_{\varepsilon} = \int_{\Omega} \overline{u} \left[-\Delta \psi_{1\varepsilon} - \vec{U} \cdot \nabla \psi_{1\varepsilon} \right] dx \geqslant 0. \tag{76}$$

Proof. We develop the term $-\Delta \psi_{1\varepsilon} - \vec{U} \cdot \psi_{1\varepsilon}$ to derive the

$$\begin{aligned} J_{\varepsilon} &= \alpha \int_{\Omega} \overline{u} \Big[-\Delta \psi_1 - \vec{U} \cdot \nabla \psi_1 \Big] (\psi_1 + \varepsilon)^{\alpha-1} dx + \alpha(1-\alpha) \int_{\Omega} |\nabla \psi_1|^2 (\psi_1 + \varepsilon)^{\alpha-2} \overline{u} dx \\ &= \alpha \lambda_1 \int_{\Omega} \overline{u} \psi_1 (\psi_1 + \varepsilon)^{\alpha-1} dx + \alpha(1-\alpha) \int_{\Omega} |\nabla \psi_1|^2 (\psi_1 + \varepsilon)^{\alpha-2} \overline{u} dx \geqslant 0. \end{aligned}$$

Let us decompose $f = f_+ - f_-$, $f_+ \in L^1(\Omega; \delta^{\alpha})$, $f_- \in L^1(\Omega; \delta^{\alpha})$. By Theorem 3.5, we know that we have $u_1 \in D(A)$ (resp. $u_2 \in D(A)$ such that

$$u_1 + \lambda A u_1 = f_+ \qquad u_2 + \lambda A u_2 = f_-. \tag{77}$$

So by linearity and uniqueness, one has

$$\overline{u} = u_1 - u_2. \tag{78}$$

Therefore, it suffices to show that the inequality (74) holds for u_1 (resp. u_1). That is to say that is sufficient to prove the inequality for $f \geqslant 0$. But in that case, the unique solution of (72) is non negative : $\overline{u} \geqslant 0$ and we can choose as a test function $\phi = \psi_{1\varepsilon}$ given in Lemma 8.2. We then have

$$\int_{\Omega} f \psi_{1\varepsilon} dx = \int_{\Omega} \overline{u} \psi_{1\varepsilon} dx + \lambda \int_{\Omega} \overline{u} \Big[-\Delta \psi_{1\varepsilon} - \vec{U} \cdot \nabla \psi_{1\varepsilon}] dx + \lambda \int_{\Omega} V \, \psi_{1\varepsilon} \overline{u} dx. \tag{79}$$

According to Lemma 8.2 and the fact that $V u \psi_{1\varepsilon} \geqslant 0$ the two last integrals in relation (79) are non negative. Therefore,

$$\int_{\Omega} f \psi_{1\varepsilon} dx \geqslant \int_{\Omega} \overline{u} \, \psi_{1\varepsilon} dx, \qquad \varepsilon > 0. \tag{80}$$

Letting $\varepsilon \to 0$ in (80), we obtain

$$\int_{\Omega} \overline{u} \, \psi_1^{\alpha} dx \leqslant \int_{\Omega} f \psi_1^{\alpha} dx \tag{81}$$

whenever $f \in \overline{u} + \lambda A \overline{u}$, $\overline{u} \in D(A)$.

We have shown that the Schrödinger operator $A = -\Delta + \vec{U} \cdot \nabla + V$ is m-accretive in $L^1(\Omega, \delta^{\alpha})$, whenever $0 \leqslant \alpha < 1$, as in the first statement of Theorem 8.3. □

We have a similar result in $L^1(\Omega;\delta)$ provided that $V(x) \geqslant c\delta(x)^{-2}$ in a neighborhood U of the boundary. The argument is similar to the preceding one but we need to replace the use of Theorem 3.5 by Theorem 3.3. Indeed, if $f = f_+ - f_- \in L^1(\Omega;\delta)$ and $\overline{u} \in D(A)$ satisfies $\overline{u} + \lambda A\overline{u} = f$ then, Theorem 3.3 allows us to spleet $\overline{u} = u_2 - u_1$ with $u_i \in D(A)$ and $u_1 + \lambda A u_1 = f_+$ (idem $u_2 + \lambda A u_2 = f_-$). therefore, it suffices to show the inequality

$$\int_\Omega \overline{u}\,\psi_1 dx \leqslant \int_\Omega f\psi_1 dx \text{ for } f \geqslant 0,\ \overline{u} \geqslant 0.$$

To do so, we choose $\phi = \psi_1$ in equation (72) and derive

$$\int_\Omega f\psi_1 dx = (1+\lambda\lambda_1)\int_\Omega \overline{u}\psi_1 dx + \int_\Omega V\overline{u}\,\psi_1 dx. \tag{82}$$

We drop the nonnegative term with V to derive

$$\int_\Omega u\psi_1 dx \leqslant \frac{1}{1+\lambda\lambda_1}\int_\Omega f\psi_1 dx \leqslant \int_\Omega f\psi_1 dx. \tag{83}$$

This show the desired inequality and implies that
$\forall\lambda > 0,\ \forall\overline{u} \in D(A),\ \overline{u} + \lambda A\overline{u} = f \in L^1(\Omega;d)$

$$\int_\Omega |\overline{u}|\psi_1 dx \leqslant \int_\Omega |\overline{u} + \lambda A\overline{u}|\psi_1 dx. \tag{84}$$

$\diamondsuit$

Therefore, we have shown the following theorem :

THEOREM 8.3. *Let $\vec{U} \in L^\infty(\Omega)^n$ such that* (3) *and $V \geqslant 0$ locally integrable. Then the Schrödinger operator*

$$Au = -\Delta u + \vec{U}\cdot\nabla u + Vu, \qquad \text{for } u \in D(A)$$

is m-accretive in $L^1(\Omega;\delta^\alpha)$ for any $0 \leqslant \alpha < 1$. If $\alpha = 1$ and $V(x) \geqslant c\delta(x)^{-2}$ in a neighborhood U of the boundary then the operator A is still m-accretive in $L^1(\Omega;\delta)$.

The operator A is also m-accretive in $L^p(\Omega;\delta^\alpha)$ when $\vec{U} = 0$ for $p \in (1,+\infty]$ and $\alpha \in [0,1]$. The result for the case $\alpha = 0$ was already proved by Brezis and Strauss [7] for bounded potentials.

THEOREM 8.4. *Let $p \in (1,+\infty]$. Assume that*

$$\begin{cases} \alpha \in [0,1], \\ \text{and } \vec{U} = 0 \end{cases} \quad \text{or} \quad \begin{cases} \alpha = 0, \\ \text{and } (H_1). \end{cases} \tag{85}$$

Let $f \in L^p(\Omega,\delta^\alpha)$, $0 \leqslant V \in L^1_{loc}(\Omega)$ and let $u \in D(A)$ be the unique solution of the equation

$$Au + u = f. \tag{86}$$

Then

$$\|u\|_{L^p(\Omega;\delta^\alpha)} \leqslant \|f\|_{L^p(\Omega;\delta^\alpha)}. \tag{87}$$

Proof. As in the proof of Theorem 8.3 we can assume without loss of generality that $f \geqslant 0$ and thus $u \geqslant 0$. By regularity arguments it can be well-justified that we can take as test function the one given $u^{p-1}\psi_{1\varepsilon}(x)$ with $\psi_{1,\varepsilon}$ as in Theorem 8.2 if $\vec{U}=0$ and u^{p-1} if $\alpha=0$. Then, from (14a), and since $V \geqslant 0$, we get that

$$\int_\Omega |u|^p \psi_{1,\varepsilon} + I \leqslant \int_\Omega f u^{p-1}\psi_{1\varepsilon} \leqslant \left(\int_\Omega f^p \psi_{1\varepsilon}\right)^{\frac{1}{p}} \left(\int_\Omega u^p \psi_{1\varepsilon}\right)^{\frac{p-1}{p}},$$

where

$$I = \int_\Omega u(-\Delta(u^{p-1}\psi_{1\varepsilon})) - \int_\Omega u\,\vec{U}\cdot\nabla(u^{p-1}\psi_{1,\varepsilon}).$$

We recall that in $L^p(\Omega,\delta^\alpha)$ we can use as an equivalent norm to the one given by

$$\left(\int_\Omega |u|^p \psi_{1\varepsilon}\right)^{\frac{1}{p}}.$$

Thus, it is enough to prove that $I \geqslant 0$. Assume now that $\vec{U}=0$. Again, we can assume that u is regular and so

$$I = -\int_\Omega \Delta u(u^{p-1}\psi_{1,\varepsilon}) = (p-1)\int_\Omega u^{p-2}|\nabla u|^2\psi_{1,\varepsilon} + \int_\Omega u^{p-1}\nabla u\cdot\nabla\psi_{1,\varepsilon}.$$

The first integral is clearly nonnegative. Moreover

$$\int_\Omega u^{p-1}\nabla u\cdot\nabla\psi_{1,\varepsilon} = \int_\Omega \frac{1}{p}\nabla u^p\cdot\nabla\psi_{1,\varepsilon} = \int_\Omega \frac{u^p}{p}(-\Delta\psi_{1,\varepsilon}),$$

and, from the definition of $\psi_{1,\varepsilon}$, we get that $I \geqslant 0$. This concludes the proof for the case $\vec{U}=0$.
Assume now that $\alpha=0$. Then, by applying Lemma 2.6 in [15], we get that

$$\int_\Omega u\,\vec{U}\cdot\nabla u^{p-1} = 0$$

and again $I \geqslant 0$.

As a first application of Theorem 8.3 and Theorem 8.4 we get the solvability of the associated parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \vec{U}\cdot\nabla u + V(x)u = f(x,t) & \text{in } \Omega\times(0,T), \\ u(x,0) = u_0(x) & \text{on } \partial\Omega, \end{cases} \tag{88}$$

for the class $0 \leqslant V \in L^1_{loc}(\Omega)$ and thus also for very singular potentials. Here is a simple statement in term of "mild solutions" (see, e.g. [1], [9] or Proposition 1.5.14 in [5]).

THEOREM 8.5. *Let $T>0$, $\alpha\in[0,1]$, $\vec{U}\in L^\infty(\Omega)^n$ such that (3) and $V\geqslant 0$ locally integrable (satisfying (4) if $\alpha=1$). Let $u_0\in L^1(\Omega;\delta^\alpha)$ and $f\in L^1(0,T;L^1(\Omega;\delta^\alpha))$.*

Then, there exists a unique $u \in C([0,T] : L^1(\Omega;\delta^\alpha))$ *mild solution of* (88). *Moreover* $V(x)u \in L^1(0,T;L^1(\Omega,\delta^\alpha))$ *and, if* $\hat{u}$ *denotes the solution to data* $\hat{u}_0$ *and* $\hat{f}$ *under the same assumptions, then, for any* $t \in [0,T]$,

$$\|(u(t,\cdot)-\hat{u}(t,\cdot))_+\|_{L^1(\Omega;\delta^\alpha)} \leqslant \|(u_0-\hat{u}_0)_+\|_{L^1(\Omega,\delta^\alpha)} + \int_0^t \|(f(t,\cdot)-\hat{f}(t,\cdot))_+\|_{L^1(\Omega,\delta^\alpha)}. \tag{89}$$

In addition, if $\vec{U}=\vec{0}$, $u_0 \in L^p(\Omega,\delta^\alpha)$ *and* $f \in L^1(0,T;L^p(\Omega,\delta^\alpha))$ *for some* $p \in (1,+\infty]$, *then* $u \in C([0,T];L^p(\Omega,\delta^\alpha))$ *and* (89) *holds replacing the norm of* $L^1(\Omega,\delta^\alpha)$ *by the norm of* $L^p(\Omega,\delta^\alpha)$.

The application of abstract semigroup theory results on the time differentiability of solutions of (88) requires the reflexivity condition on the abstract Banach space. This holds in the case of the second part of Theorem 8.5 when $1<p<+\infty$ (and $\vec{U}=0$ or $\alpha=0$). Nevertheless, a direct approach to this question for problem (88) can be obtained as an application of Proposition 1.3.4 of [5] if $f=0$ and Proposition 1.5.5 if $f \neq 0$. We have

THEOREM 8.6. *Let* $T>0$, $u_0 \in D(A)$, $f \in C([0,T];D(A)) \cup C^1([0,T];L^1(\Omega;\delta^\alpha))$ *for some* $\alpha \in [0,1]$. *Then, there exists a (unique) function satisfying :*

$$\begin{cases} u \in C\Big([0,T];D(A)\Big) \cap C^1\Big([0,T];L^1(\Omega;\delta^\alpha)\Big) \\ \dfrac{du}{dt}(t)+Au(t)=f(t),\ \forall t \in [0,T], \quad u(0)=u_0. \end{cases}$$

REMARK 3. It is possible to obtain several qualitative properties of solutions of the parabolic problem (88). The smoothening effect for bounded potentials can be found, e.g., in [5, 6, 10, 22]. If $V(x)$ is a very singular potential then the Dirichlet condition is verified in $W_0^1L^{n',\infty}(\Omega)$ once we assume that $\alpha \in [0,1)$. In fact, it is not complicated to adapt the techniques of proof of the Section 8.2 of this paper to show that if u_0 and $f(t,\cdot)$ are "flat" data near $\partial\Omega$ then the (unique) solution of (88) is also a "flat solution" in the sense that not only $u=0$ but $\frac{\partial u}{\partial \vec{v}}=0$ on $\partial\Omega$. Notice that this is in contrast with the instantaneous blow-up of solutions which arises when $\vec{U}=0$, $V(x)<0$, $\lambda_1(-\Delta+(1-\varepsilon)V)=-\infty$ for some $\varepsilon>0$ and V is very singular (see [8] and the references therein).

8.2. Complex Schrödinger problem

Let us apply our previous results to the mathematical treatment of problem (5). In some sense, our main aim now is to show that the solution of this Schrödinger equation is *localized* for any $t>0$, in the sense that if we start with a localized initial wave packet $\boldsymbol{\psi}_0 \in H^1(\mathbb{R}^n : \mathbb{C})$ (here $\mathbb{C}$ denotes the complex numbers), i.e. such that

$$\text{support } \boldsymbol{\psi}_0 \subset \overline{\Omega},$$

then the particle still remains permanently confined in Ω in the sense that

$$\text{support} \boldsymbol{\psi}(t,\cdot) \subset \overline{\Omega} \text{ for any } t>0.$$

As in [14] we start by considering the auxiliary eigenvalue problem

$$DP(V,\lambda,\Omega) \quad \begin{cases} -\Delta u+\vec{U}\cdot\nabla u+V(x)u=\lambda u & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

PROPOSITION 8.7. *Let $0 \leqslant V \in L^1_{loc}(\Omega)$. Then there exists a sequence of eigenvalues $\lambda_m \to +\infty$, $\lambda_1 > \lambda_{1,\Omega}$ (the first eigenvalue for the Dirichlet problem associated to the operator $-\Delta+\vec{U}\cdot\nabla$ on Ω), λ_1 is isolated and $u_1 > 0$ on Ω.*

Proof. We start by arguing as in the proof of Proposition 3 of [15]. We introduce the space

$$W=\left\{\varphi \in H^1_0(\Omega) : V\varphi^2 \in L^1(\Omega)\right\}.$$

For any $h \in L^2(\Omega)$ we define the operator $Th=z \in W$ solution of the linear problem

$$\begin{cases} Az=h & \text{in } \Omega, \\ z=0 & \text{on } \partial\Omega. \end{cases} \tag{90}$$

We recall that the existence and uniqueness of a solution was obtained in Proposition 3 in [15] when $h \in W'$ (the dual space of W) and that, trivially, $L^2(\Omega) \subset W'$. Then the composition with the (compact) embedding $H^1_0(\Omega) \subset L^2(\Omega)$ is a selfadjoint compact linear operator $\widetilde{T}=i\circ T : L^2(\Omega) \to L^2(\Omega)$ for which we obtain in the usual way a sequence of eigenvalues $\lambda_m \to +\infty$. By well-known results (see e.g. [24] or [4]) we know that $\lambda_1 > 0$ (notice that $\lambda_1 = 0$ would imply that $z=0$). In fact, since $V(x) \geqslant 0$, by the comparison principle we know that $\lambda_1 > \lambda_{1,\Omega}$. The positivity of the first eigenfunction u_1 is an easy modification of Proposition 3.2 of [14]. Moreover a variant of the Krein-Rutman can be applied (see [12]) and so we know that λ_1 is isolated.

REMARK 4. When $r=2$ in (4) then, by the Hardy inequality, $W=H^1_0(\Omega)$.

A different, and useful, consequence of Proposition 3 of [15] is the following:

PROPOSITION 8.8. *Let $0 \leqslant V \in L^1_{loc}(\Omega)$. Then the operator $A : D(A)(\subset L^2(\Omega)) \to L^2(\Omega)$ given by $D(A)=W=\left\{\varphi \in H^1_0(\Omega) : V\varphi^2 \in L^1(\Omega)\right\}$ and $Au=-\Delta u+\vec{U}\cdot\nabla u+Vu$ if $u \in D(A)$ is a maximal monotone operator in $L^2(\Omega)$.*

Proof. Given $h \in L^2(\Omega)$, the existence and uniqueness of solution of the equation $Au+u=h$ was obtained in Proposition 3 of [15]. Moreover, thanks to the assumptions on $\vec{U}$, by Lemmas 2.6 and 2.7 of [15] we get that

$$\|u\|_{L^2(\Omega)} \leqslant \|h\|_{L^2(\Omega)}$$

which proves the monotonicity in $L^2(\Omega)$ (i.e. the operator is m-accretive in $L^2(\Omega)$).

Let us prove now that the singularity of the potential implies that all the eigenfunctions u_m of operator A are flat solutions (in the sense that $u=\frac{\partial u}{\partial n}=0$ on $\partial\Omega$).

As usual in Quantum Mechanics we shall pay attention to the associate eigenfunctions with normalized L^2-norm, i.e. such that

$$\|u_m\|_{L^2(\Omega)} = 1. \tag{91}$$

THEOREM 8.9. *Assume* (4) *and let* u_m *be an eigenfunction associated to the eigenvalue* λ_m. *Then* $u_m \in L^\infty(\Omega)$ *and* u_m *is a flat solution of* $DP(V,\lambda_m,\Omega)$. *In fact, there exists* $\overline{K}_m > 0$ *such that*

$$|u_m(x)| \leqslant \overline{K}_m d(x,\partial\Omega)^2 \quad a.e.\ x \in \Omega. \tag{92}$$

Proof. It suffices to repeat all the arguments of Theorem 2.1 of [14] (concerning the case $r = 2$ and $\vec{U} = \vec{0}$) since the the main idea of the proof consists in the use of a Moser-type iterative argument (as in [17]) and take as test functions

$$\varphi(x) = v_{m,M}^{2\kappa+1}(x), \text{ with } v_{m,M}(x) := \min\{|u_m(x)|, M\}\,\mathrm{sign}(u_m(x)), \tag{93}$$

for any arbitrary $M, \kappa > 0$. Then, by using (4) and Lemmas 2.6 and 2.7 of [15] we conclude that $\varphi \in H_0^1(\Omega)$ is an appropriate test function and

$$\begin{aligned}(2\kappa+1)\int_\Omega \left|v_M^{2\kappa}(x)\right| |\nabla u_m|^2\,dx + \int_\Omega \frac{C}{\delta(x)^2}\left|v_M^{2\kappa+1}(x)\right| |u_m|\,dx \\ \leqslant (2\kappa+1)\int_\Omega \left|v_M^{2\kappa}(x)\right| |\nabla u_m|^2\,dx + \int_\Omega V(x)\left|v_M^{2\kappa+1}(x)\right| |u_m|\,dx \\ = \lambda_n \int_\Omega \left|v_M^{2\kappa+1}(x)\right| |u_m|\,dx\end{aligned} \tag{94}$$

where we used the simplified notation $v_M = v_{m,M}$. This is exactly the same starting energy estimate than the one used in the proof of Theorem 2.1 of [14] and thus the rest of the proof (passing to the limit when $M \nearrow +\infty$) applies without any other modification.

REMARK 5. The flatness of the eigenfunctions u_m of operator A can be also proved by using Proposition 2.7 of [25] nevertheless the statement given here supplies some decay estimates on u_m near $\partial\Omega$ which are not given in the mentioned reference.

REMARK 6. The decay estimate (92) is not optimal if $r > 2$ in (4). It seems possible to adapt the formal exposition made in [11] developing asymptotically some Bessel functions to prove that in that case

$$|u_m(x)| \leqslant \overline{K}_m \delta(x)^{r/4} \exp\left(-\frac{\widehat{K}_m}{(r-2)}\delta(x)^{-(r-2)/2}\right) \quad \text{a.e. } x \in \Omega, \tag{95}$$

for some positive constants $\overline{K}_m$ and $\widehat{K}_m$, but we shall not enter into the details here.

REMARK 7. Arguing as in [14] it is easy to get several qualitative properties of solutions of the complex evolution Schrödinger problem

$$\begin{cases} \mathbf{i}\dfrac{\partial \boldsymbol{\psi}}{\partial t} = -\Delta\boldsymbol{\psi} + \vec{U}\cdot\nabla\boldsymbol{\psi} + V(x)\boldsymbol{\psi} & \text{in } (0,\infty)\times\mathbb{R}^n \\ \boldsymbol{\psi}(0,x) = \boldsymbol{\psi}_0(x) & \text{on } \mathbb{R}^n \end{cases} \tag{96}$$

for very singular potentials over Ω which are extended (for instance) in a finite way to the whole space. So, we assume now that there exists $q \in [0,+\infty)$ such that

$$V_{q,\Omega}(x) = \begin{cases} V(x) & \text{if } x \in \Omega, \\ q & \text{if } x \in \mathbb{R}^n \setminus \Omega \end{cases} \tag{97}$$

and that (4) holds. We can study the time evolution of a localized initial wave packet $\psi_0 \in H^1(\mathbb{R}^n : \mathbb{C})$ such that support $\psi_0 \subset \overline{\Omega}$.

Then we can prove that there exists a unique solution $\boldsymbol{\psi} \in C([0,+\infty) : L^2(\mathbb{R}^n : \mathbb{C}))$ with $\boldsymbol{\psi} \in C([0,+\infty) : H^1(\mathbb{R}^n : \mathbb{C}))$ and $V_{q,\Omega}(x)\psi \in L^2(0,T : L^2(\mathbb{R}^n : \mathbb{C}))\}$ for any $T > 0$, and that the Galerkin decomposition

$$\boldsymbol{\psi}_\Omega(t,x) = \sum_{m=1}^{\infty} \mathbf{a}_m e^{-\mathbf{i}\lambda_m t} u_m(x), \tag{98}$$

holds with convergence at least in $L^2(\mathbb{R}^n : \mathbb{C})$ where λ_m and u_m are the eigenvalues and eigenfunctions given in Proposition 8.7 and

$$\mathbf{a}_m = \int_\Omega \boldsymbol{\psi}_0(x) u_m(x)dx.$$

For localizing purposes we assume that

$$\sum_{m=1}^{\infty} |\mathbf{a}_m| \overline{K}_m < +\infty, \tag{99}$$

where $\overline{K}_m > 0$ was given in Theorem 8.9. Thus, we conclude that

$$|\boldsymbol{\psi}(t,x)| \leqslant K d(x,\partial\Omega)^2 \quad \text{for any } t > 0 \text{ and a.e. } x \in \Omega, \tag{100}$$

for some $K > 0$, and in consequence the unique solution of (96) satisfies that support $\boldsymbol{\psi}(t,.) \subset \overline{\Omega}$ for any $t > 0$.

Concerning the existence of solutions, it is enough to apply the Hille-Yosida theorem (see, e.g. [24, 4, 5]). For the Galerkin decomposition we can adapt the arguments given in [5].

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