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A CONNECTION BETWEEN REGULARITY AND DIRICHLET PROBLEMS FOR NON–DIVERGENCE ELLIPTIC EQUATIONS

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Abstract. We observe that a version of Poincaré's inequality for positive solutions to second order linear non-divergence form equations vanishing on a portion of the boundary, implies a natural connection between L^p Dirichlet and L^q Regularity problems for this type of equations.

1. Introduction

After describing how to obtain the Poincaré-type inequality in the first section, in Section 2 we give the simple argument proving that the solvability of an L^q Dirichlet regularity problem, for certain $1 < q \leqslant 2$, implies the solvability of the L^p Dirichlet problem for certain $1 < p < \infty$. A stronger result has already been established for divergence form elliptic equations in [17, Theorem 5.4], where in fact it is proved that $1/p + 1/q = 1$, and where the self-adjointness of the differential operator and an estimate relating Green's functions and elliptic-harmonic measure are employed. The argument we provide here avoids the use of both of these features, but still adapts ideas from [17]. This indicates the possibility of applying these ideas to parabolic equations on non-cylindrical domains. We will address this problem in a separate paper.

2. Preliminary definitions and a Poincaré-type inequality

The elliptic equation

Adopting the notation $\partial_{i,j}u = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$, we consider operators of the form

$$Lu = \sum_{i,j=1}^{n} a_{i,j}(x)\partial_{i,j}u(x), \tag{2.1}$$

where $x \in \mathbb{R}^n$, and $A(x) = (a_{i,j}(x))$ is a symmetric matrix of measurable real-valued functions satisfying the ellipticity condition of the form

$$\lambda_1|\xi|^2 < \langle A(x)\xi,\xi\rangle < \lambda_2|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n \text{ and every } x \in \mathbb{R}^n, \tag{2.2}$$

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and where the uniform constants $\lambda_1, \lambda_2 > 0$ are referred to as *ellipticity constants* of L. Here $\langle x, y\rangle$ denotes the usual dot product on $\mathbb{R}^n$, and from now on for $x \in \mathbb{R}^n$ we denote by $|x|$ its euclidean norm. For non-divergence elliptic operators L as in (2.1), with the features just described, the solutions to $Lu = 0$ are always understood as *strong solutions*.

We work in the setting of a *starlike Lipschitz domain (centered at the origin)* $D \subset \mathbb{R}^n$. This means that D is an open set $D \subset \mathbb{R}^n$ for which there exists positive constants M and δ and a function $\widetilde{\psi} : S^{n-1} \longrightarrow \mathbb{R}$ satisfying $|\widetilde{\psi}(t) - \widetilde{\psi}(s)| \leqslant M|t-s|$ and $\widetilde{\psi}(t) \geqslant \delta > 0$, $s,t \in S^{n-1}$, and such that in spherical coordinates $D = \big\{(\rho, s) : 0 \leqslant \rho \leqslant \widetilde{\psi}(s),\, s \in S^{n-1}\big\}$. Here $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. The pair of constants (M, δ) are referred to as the *Lipschitz character of* D.

We assume that the coefficients $A(x)$ satisfy a Hölder continuity, so that we can use results from [26], which in particular imply that the Green's function associated to L on D or any dilation of it, has a continuous representative. However, the Hölder continuity of the coefficients is only used in the proof of a version of a Caccioppoli-type inequality on the boundary (see (2.13) below), where we will clarify how this continuity is used. For solutions to $Lu = 0$, these Caccioppoli-type inequalities have the generic form

$$\int |\nabla u(Y)|^2 \varphi^2(Y) v(Y)\, dY \lesssim \int u^2(Y) \left[\left|\nabla^2 \varphi^2(Y)\right| + |\nabla \varphi(Y)|^2\right] v(Y)\, dY,$$

where $\varphi \in C_0^2$ is suitable supported within D, and v is an *adjoint solution to* L (as we will shortly define) on a domain containing D. Also, $\left|\nabla^2 w\right|$ denotes the magnitude of the vector of second order derivatives of w.

An *adjoint solution for* L *on a domain* $\Omega \subset \mathbb{R}^n$ is defined (see e.g. [1, p. 154]) as a locally integrable function v such that

$$\int_\Omega v(Y) L\varphi(Y)\, dY = 0 \qquad \text{for} \quad \varphi \in C_0^\infty(\Omega). \tag{2.3}$$

For shortness' sake we simply write and say that $L^* v = 0$ *on* Ω. This way of defining adjoint solutions leads, for instance, to the inclusion of an adjoint solution as a weight in the integrals of the aforementioned Caccioppoli-type estimate (see [10] and the argumentation leading to (2.12) below).

Local geometric definitions

It is known (see e.g. [28]) that the boundary of a starlike Lipschitz domain can be covered with a finite number of patches of the same size, in such a way that after rotating each patch, the portion corresponding to ∂D is given by the graph of certain Lipschitz function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$. We refer to this function as the *local functions describing* ∂D. Of course these functions are related with the original $\widetilde{\psi} : S^{n-1} \to \mathbb{R}$ from the definition of D.

Calling $0 < r_0 < \operatorname{diam} D$ the diameter of any of these local patches, the definitions below will make geometric sense if we restrict the size of the following local geometric objects to be smaller than r_0. There will be no loss of generality assuming this, since in fact our main theorems will require a property that must hold for all *small* scales.

Given $Q \in \partial D$, for $0 < r < r_0$ we define the *local tube at Q with radius r*, denoted as $\tau(Q,r)$, as the rotation and translation of the tube $\tau_0(Q,r) \equiv B(\vec{0},r) \times \mathbb{R}$ so that $B(x,r) \times \{0\} = Q$ and the main positive axis of $\tau_0(Q,r)$ has overlap with the oriented segment line $\overrightarrow{Q0}$.

We also define $\Gamma_\alpha(Q)$ as the cone with vertex at $Q \in \partial D$ and aperture α, whose principal axis follows the radial direction $\overrightarrow{Q0}$, and that is truncated in such a way that the upper portion of it reaches the origin. The aperture $\alpha > 0$ is chosen so that $\Gamma_\alpha(Q) \subset D$ for all $Q \in \partial D$, and since it can be fixed from now on, we may drop it from the notation.

Another geometric constant $R_0 > 0$ is chosen, depending only on α and the constant M of the Lipschitz character of D, and such that

$$\partial B(Q,R_0 r) \cap \tau(Q,r) \cap D \subset \Gamma(Q) \quad \text{for every } Q \in \partial D. \tag{2.4}$$

This constant may be chosen, by picking r_0 smaller if necessary, because of the Lipschitz property of the local function ψ describing D.

With this constant at hand, and $0 < r < r_0$, we define truncated versions of $\Gamma_\alpha(Q)$ as $\Gamma_\alpha^r(Q) = \Gamma_\alpha(Q) \cap B(Q,r)$. Also, the *Carleson regions* based on $Q \in \partial D$ are defined as

$$\Psi_r(Q) = \Psi(\Delta_r(Q)) = B(Q,R_0 r) \cap \tau(Q,r) \cap D \tag{2.5}$$

The *surface ball* $\Delta_r(Q)$ may be defined as $\Delta_r(Q) \equiv \overline{\Psi_r(Q)} \cap \partial D$. And if

$$Q = (\varphi(s_0), s_0) \in \partial D \text{ and } 0 < r < r_0,$$

we write $A_r(Q) = (\varphi(s_0) - R_0 r/2, s_0)$. The *extended Carleson regions* are defined by $\widetilde{\Psi}_r(Q) = \widetilde{\Psi}(\Delta_r(Q)) = B(Q,R_0 r) \cap \tau(Q,r)$.

Note in particular that the notations $\Psi(\Delta)$, $A(\Delta)$ or $\widetilde{\Psi}(\Delta)$ make sense for any surface ball $\Delta \subset \partial D$, even without referring to the center or the radius of Δ. However, if the radius r of the surface ball Δ is determined, one also refers to $\Psi(\Delta)$ as a *Carleson region of radius r*.

For $X \in \mathbb{R}^n$ one defines $\delta(X) = \operatorname{dist}(X, \partial D)$, and for $X \in D$ it is useful to set $B(X) = B(X, k_0 \delta(X))$, where $0 < k_0 < 1$ is chosen and fixed so that $B(X) \subset D$ for every $X \in D$. Associated to $B(X)$ we define Carleson regions $\Psi(X)$ defined as the smallest Carleson region $\Psi(\Delta)$ that contains $B(X)$, with Δ centered at the radial projection Q_X of X on ∂D.

From now on, $A \lesssim B$ means that there is a constant $k > 0$ (depending at most on the dimension n, the constants in (2.2), or geometric features of the domain described above) such that $A \leqslant kB$. Likewise, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$ hold simultaneously.

The Poincaré-type inequality

PROPOSITION 1. (Poincaré-type inequality) *Let $\Delta \subset \partial D$ be any surface ball of radius $0 < r < r_0/4$, and for $Q \in \Delta$, $\Gamma = \Gamma^r(Q)$ take any $X_0 \in \Gamma$. Let u be a positive* *strong solution to* $Lu = 0$ *on* D*, with* $u|_{2\Delta} = 0$ *continuously. For any* $0 < \varepsilon < 1$ *set*

$$\Psi^{\varepsilon}(X_0) = \Psi(X_0) \cap \{Y \in D : \delta(Y) \geqslant \varepsilon\delta(X_0)\}.$$

Then

$$\frac{1}{\delta(X_0)^n}\int_{B(X_0)} |u(X)|^2 dX \lesssim \frac{1}{\delta(X_0)^{n-2}}\int_{\Psi^{\varepsilon}(X_0)} |\nabla u(X)|^2 dX. \tag{2.6}$$

Proof. The proof follows the lines of [17], using the appropriate estimates for solutions to non-divergence elliptic equations, contained for instance in the very complete expository survey article [14] and references therein. Thus in the next paragraphs we just follow through the argumentation from [17, p. 468-469] and indicate the appropriate references applied for solutions to solutions of non-divergence elliptic equations.

The first step is to observe that *for any* $0 < \theta < 1$ and any Carleson region Ψ of radius $0 < \rho < r_0/4$, centered anywhere within Δ one has

$$\int_{\Psi} u^2(X)\delta(X)^{\theta} dX \leqslant C\rho^2 \int_{\Psi} |\nabla u(X)|^2 \delta(X)^{\theta} dX. \tag{2.7}$$

with a constant $C > 0$ depending on θ but independent of ρ and the smoothness of u. This is essentially [17, Lemma 5.5], with a slight adjustment in the regions of integration (since in [17] the domain D is the unit ball). We provide some details of these adjustments.

Assume that $\Psi = \Psi_{\rho}(Q)$ and that $Q = (0', q)$, after applying a rigid motion, where $0'$ denotes the origin in $\mathbb{R}^{n-1}$ and with no loss of generality $q > 0$. We will use a fact already mentioned above, that any starlike Lipschitz domain has boundary given locally by graphs of Lipschitz functions $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$. Denoting by Q_{Δ} the projection of $\Delta_{\rho}(Q) \subset \partial D$ on $\mathbb{R}^{n-1}$ and setting $R = R_0\rho$, we can assume that

$$\widetilde{\Psi}_{\rho} \equiv \{(\psi(x,t), x, t) : (x,t) \in Q_{\Delta}\} \subset \mathscr{B}_{\Delta} \equiv (-R, R) \times Q_{\Delta}. \tag{2.8}$$

By assumption u vanishes continuously on 2Δ, and so we can extend u with zero value for points in $\mathscr{B}_{\Delta} \setminus \Omega$. With this convention we have

$$\begin{aligned}\int_{-R}^{R} u^2(x_0, x)dx_0 &= \int_{-R}^{R}\left[\int_{-R}^{x_0} \frac{\partial}{\partial z} u(z,x)\, dz\right]^2 dx_0 \\ &\leqslant \int_{-R}^{R}\left[\int_{-R}^{x_0} |\nabla u(z,x)|^2 \delta_{\psi}(z,x)^{\theta}\, dz\right]\left[\int_{-R}^{x_0} \frac{1}{\delta_{\psi}(z,x)^{\theta}} dz\right] dx_0\end{aligned}$$

where $\delta_{\psi}(y,x)$ denotes the vertical distance to $\widetilde{\Psi}_{\rho}$ from points $(y,x,t) \in \mathscr{B}_{\Delta}$ where $u \neq 0$. In particular for such points we must have $y > \psi(x)$. Since ψ is Lipschitz, $\delta_{\psi}(y,x) = y - \psi(x)$, and by (2.8)

$$\int_{-R}^{x_0} \frac{1}{\delta_{\psi}(z,x)^{\theta}} dz \lesssim \int_{0}^{2R} \frac{1}{y^{\theta}} dy \lesssim \frac{R^{1-\theta}}{1-\theta},$$

and again, since ψ is of Lipschitz type, then $\delta_\psi(y,x) \approx \delta(y,x)$ uniformly on x. Therefore

$$\int_{-R}^{R} u^2(x_0,x)dx_0 \lesssim \frac{R^{2-\theta}}{1-\theta}\left[\int_{-R}^{R} |\nabla u(z,x)|^2 \delta(z,x)^\theta \, dz\right].$$

Integrating with respect to the remaining variables x to cover $\Psi_\rho(Q)$, and using

$$\delta(x_0,x)/R < 1$$

we get

$$\begin{aligned}\int_{\Psi_\rho(Q)} u^2(x_0,x)\delta(x_0,x)^\theta \, dx_0 dx &\leqslant R^\theta \int_{\Psi_\rho(Q)} u^2(x_0,x)\, dx_0 dx \\ &\leqslant C\rho^2 \int_{\Psi_\rho(Q)} |\nabla u(z,x)|^2 \delta(z,x)^\theta \, dz dx\end{aligned}$$

with a constant $C = C(n,M,r_0,R_0,\theta) > 0$, since $R = R_0\rho$. This is precisely (2.7),

Moving on, for $X_0 \in \Gamma^r(Q)$, set $\rho = \delta(X_0)$. Note that by boundary Harnack principle and Hölder continuity at the boundary (see e.g. [1, Lemmata 2.3 and 2.4] or [14, Lemmata 5.3 and 5.5]) we have

$$\begin{aligned}\frac{1}{\delta(X_0)^n}\int_{B(X_0)} u^2(X)dX &\lesssim \frac{1}{\delta(X_0)^{n+\gamma}}\int_{B(X_0)} \delta(X)^\gamma u^2(X)\, dX \\ &\leqslant \frac{1}{\delta(X_0)^{n+\gamma}}\int_{\Psi(X_0)} \delta(X)^\gamma u^2(X)\, dX. \end{aligned} \tag{2.9}$$

Now for $0 < \varepsilon < 1/4$, $0 < \rho < r_0/2$ and any Carleson region Ψ of radius ρ we define its ε*-upper portion* as $\Psi^\varepsilon_\rho \equiv \Psi_\rho \cap \{X : \varepsilon\rho \leqslant \delta(X)\}$. And again, the notation Ψ^ε makes sense, as long as the Carleson region Ψ is well defined. By (2.7) and using that $\delta(X) \lesssim \rho$ for $X \in \Psi^\varepsilon(X_0)$ and $\delta(X)/\rho < \varepsilon$ for $X \in \Psi(X_0)\setminus\Psi^\varepsilon(X_0)$, we obtain

$$\begin{aligned}\frac{1}{\rho^\gamma}\int_{\Psi(X_0)} \delta(X)^\gamma u^2(X)\, dX &\lesssim \frac{\rho^2}{\rho^\gamma}\int_{\Psi(X_0)} \delta(X)^\gamma |\nabla u(X)|^2\, dX \\ &\lesssim \rho^2 \int_{\Psi^\varepsilon(X_0)} |\nabla u(X)|^2\, dX + \\ &\quad + \varepsilon^\gamma\rho^2 \int_{\Psi(X_0)\setminus\Psi^\varepsilon(X_0)} |\nabla u(X)|^2\, dX \end{aligned} \tag{2.10}$$

In order to estimate the second integral in (2.10), we require a Caccioppoli-type inequality on the boundary of D, and in order to establish it we follow the idea in [10, p. 280].

We start recalling well known formulae for the operator L applied on products of functions: if $Lu = 0$ on any region $\Omega \subset \mathbb{R}^n$, $\varphi \in C_0^2(\Omega)$ then

$$L(u^2) = 2\langle A\nabla u, \nabla u\rangle, \qquad L(u^2\varphi^2) = u^2 L(\varphi^2) + \varphi^2 L(u^2) + 8u\varphi\langle A\nabla u, \nabla\varphi\rangle.$$

Next, we prove an integral estimate involving u, φ, having an adjoint solution v as a weight. By the definition of the adjoint operator and adjoint solutions we obtain:

$$
\begin{aligned}
0 &= \int L\left[u^2(Y)\varphi^2(Y)\right] v(Y)\,dY \\
&= \int \left[u^2(Y)L(\varphi^2)(Y) + 2\varphi^2(Y)|\nabla u(Y)|^2 + 8u(Y)\langle\nabla u,\nabla\varphi\rangle\right] v(Y)\,dY,
\end{aligned}
\tag{2.11}
$$

where v is any adjoint solution to L on any domain $\Omega' \supset \Omega$ (i.e. $L^* v = 0$ on Ω'). This implies

$$
\begin{aligned}
\int |\nabla u(Y)|^2\varphi^2(Y)v(Y)\,dY \lesssim \int u^2(Y)\left|D^2\varphi^2(Y)\right| v(Y)\,dY + \\
+ \int |\langle\nabla u(Y), u(Y)\nabla\varphi(Y)\rangle|\, v(Y)\,dY,
\end{aligned}
$$

which by the elementary inequality $AB \lesssim \eta A^2 + B^2/\eta$ with $\eta > 0$ small enough, and after hiding a small term in the left side of the above estimate yields

$$
\int |\nabla u(Y)|^2\varphi^2(Y)v(Y)\,dY \lesssim \int u^2(Y)\left[\left|\nabla^2\varphi^2(Y)\right| + |\nabla\varphi(Y)|^2\right] v(Y)\,dY \tag{2.12}
$$

Notice that the weight v in the above integrals appear because of our use in (2.11) of the property $L^* v = 0$ in the sense of (2.3) above.

To describe how to obtain from (2.12) the desired version of boundary Caccioppoli inequality, we set $\Omega = \left\{X \in \widetilde{\Psi}_\rho(Q) : \delta(X) < \varepsilon\rho\right\}$, and pick $\varphi \in C_0^2(\Omega')$, $0 \leqslant \varphi \leqslant 1$, with $\varphi \equiv 1$ on Ω, and such that $|\nabla\varphi| \lesssim 1/\rho$, $|D^2\varphi| \lesssim 1/\rho^2$. Here Ω' is a small dilation of Ω given by $\Omega' = \{X \in \mathbb{R}^n : \delta(X,\Omega) < \varepsilon\rho/16\}$.

Define for $t > 0$ the *dilation* tD of the Lipschitz domain D by a factor of t, namely $tD = \{(\rho,s) : 0 \leqslant \rho \leqslant t\varphi(s),\ s \in S^{n-1}\}$. Also set $v(Y) = G(Y) \equiv G_0(X_0,Y)$, where G_0 denotes the Green's function on $10D$ with $X_0 \in \partial(9D)$. We will exploit the fact that $G(Y)$ is an adjoint solution of L on $10D$, and since $X_0 \notin D$, $v(Y)$ is a continuous function by results from [26]. Moreover, v vanishes only at points in $\partial(10D)$, and so is bounded below uniformly away from 0 on D. Hence obtain from (2.12) the following *boundary Caccioppoli inequality*

$$
\rho^2 \int_\Omega |\nabla u(Y)|^2\varphi^2(Y)\,dY \lesssim \int_{\Omega'} u^2(Y)\,dY. \tag{2.13}
$$

In this way, the second term in the right of (2.10) is estimated as follows:

$$
\varepsilon^\gamma\rho^2 \int_{\Psi(X_0)\setminus\Psi^\varepsilon(X_0)} |\nabla u(X)|^2\,dX \lesssim \varepsilon^\gamma \int_{\Omega'\cap D} |u(X)|^2\,dX
$$

which back in (2.9), and using (2.10), yields

$$
\frac{1}{\delta(X_0)^n}\int_{B(X_0)} u^2(X)dX \lesssim \rho^{2-n}\int_{\Psi^\varepsilon(X_0)} |\nabla u(X)|^2\,dX + \rho^{-n}\varepsilon^\gamma\int_{\Omega'\cap D} |u(X)|^2\,dX. \tag{2.14}
$$

Now we observe that applying Carleson-type estimate (see [1, Lemma 2.4]) and Harnack's inequality (twice) we obtain

$$\int_{\Omega'\cap D} |u(X)|^2\,dX \lesssim u^2(A(X_0))|\Omega'\cap D| \lesssim u^2(X_0)|B(X_0)| \lesssim \int_{B(X_0)} |u(X)|^2 dX. \quad (2.15)$$

Back in (2.14) this yields

$$\frac{1}{\delta(X_0)^n}\int_{B(X_0)} u^2(X)dX \lesssim \delta(X_0)^{2-n}\int_{\Psi^\varepsilon(X_0)} |\nabla u(X)|^2\,dX + \delta(X_0)^{-n}\varepsilon^\gamma \int_{B(X_0)} |u(X)|^2\,dX,$$

by simply noting that $\delta(X) \approx \delta(X_0)$ for $X \in B(X_0)$. By an adequately small choice of ε, the second term may be hidden in the right hand side leading to

$$\frac{1}{\delta(X_0)^n}\int_{B(X_0)} u^2(X)dX \lesssim \frac{\delta(X_0)^2}{\delta(X_0)^n}\int_{\Psi^\varepsilon(X_0)} |\nabla u(X)|^2\,dX.$$

3. A connection between Dirichlet and Regularity problems

The result we describe in this section adapts some ideas from [17, Theorem 5.4], and provides a connection between the L^q regularity problem and L^p Dirichlet problem associated to the non-divergence equation.

Elliptic-harmonic measure

For L as in (2.1) it is known that we can define the *harmonic measure associated to* L, denoted by ω^x for $x \in D$, as the unique Borel measure supported on ∂D such that

$$u_f(x) = \int_{\partial D} f(y)d\omega^x(y) \quad (3.1)$$

is the strong solution of the continuous Dirichlet problem $Lu = 0$ on Ω_T, $u|_{\partial D} = f$ with f continuous and supported on ∂D.

A basic property of harmonic measure is that for any Borel set $E \subset \partial D$, the measure $\omega^x(E)$, as a function of $x \in D$ can be viewed as a solution to $Lu = 0$ with boundary data χ_E. Hence a Harnack principle can be applied, and it implies that ω^x is absolutely continuous with respect to $\omega \equiv \omega^{\vec{0}}$ for every $x \in D$.

We say that $\omega \in A_\infty(\sigma)$ if given any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\omega(E)/\omega(\Delta_r(Q)) < \delta$$

implies $\sigma(E)/\sigma(\Delta_r(Q)) < \varepsilon$ for every Borel set $E \subset \Delta_r(Q) \subset \partial D$. Note that this is a scale invariant version of absolute continuity between elliptic measure and surface measure on D. It is known that $\omega \in A_\infty(\sigma)$ if and only if there exist constants $C > 0$ and $0 < \theta < 1$ such that for every surface ball $\Delta \subset \partial D$ and every Borel measurable set $E \subset \Delta$ the following holds

$$\frac{\omega(E)}{\omega(\Delta)} \leqslant C\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^\theta. \quad (3.2)$$

Dirichlet and Regularity problems

It is said that that the *L^p Dirichlet problem is solvable for L on D* $(1<p<\infty)$ if for f continuous on ∂D, the strong solution to the continuous Dirichlet problem $Lu = 0$, $u|_{\partial D} = f$, satisfies the estimate $\|Nu\|_{L^p(d\sigma)} \leqslant C\|f\|_{L^p(d\sigma)}$, with $C>0$ a constant independent of f. Here the *non-tangential maximal function* is defined as $Nu(P) = \sup\{|u(x)| : x \in \Gamma(P)\}$, where $\Gamma(P)$ is the cone with vertex at $P \in \partial D$ defined in the first section.

Assuming that ω is absolutely continuous with respect to σ, we say that the Radon-Nikodým derivative $k = d\omega/d\sigma$ satifies a *reverse Hölder inequality*, if for some constant C and some $1<q<\infty$

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta k^q d\sigma\right)^{1/q} \leqslant \frac{C}{\sigma(\Delta)}\int_\Delta k d\sigma \tag{3.3}$$

for every surface ball Δ. In this case we abbreviate $k \in RH^q(\partial D, d\sigma)$, and it is known that under this assumption the L^p Dirichlet problem, $1/p + 1/q = 1$, associated to L on D is solvable. It is also known that if $\omega \in A_\infty(\sigma)$ then there exists a $1<p<\infty$ such that $k \in RH^p(\partial D, d\sigma)$.

To define the L^q regularity problem we introduce the notation

$$\mathbf{T}(Q) = \left\{\vec{T}_1(Q), \dots, \vec{T}_{n-1}(Q)\right\}$$

for the system of unit vectors, tangential to ∂D at the point $Q \in \partial D$. For $1<q<\infty$, the space $L^q_1(\partial D)$ is defined as the class of $f \in L^q(\partial D)$ such that

$$\left(\int_{\partial D} |\nabla_T f|^q d\sigma\right)^{1/q} = \left(\sum_{i=1}^{n-1}\int_{\partial D} |\nabla f \cdot \vec{T}_i(Q)|^q d\sigma\right)^{1/q} < \infty. \tag{3.4}$$

The *regularity problem with datum in $L^q_1(\partial D)$, $1<q<\infty$, is solvable for L*, if the strong solution u to to the Dirichlet problem $Lu=0$, $u|_{\partial D} = f$, with f continuous and in $L^q_1(\partial D)$ is continuous on $\overline{D}$, and it satisfies the estimate $\|\widetilde{N}(\nabla u)\|_{L^q(d\sigma)} \leqslant C\|f\|_{L^q_1(d\sigma)}$, with $C>0$ independent of f. Here, the *modified non-tangential maximal function* is defined as

$$\widetilde{N}(\nabla u) = \sup_{X \in \Gamma(Q)} \left(\frac{1}{Z(B(X))}\int_{B(X)} |\nabla u(Z)|^2 dZ\right)^{1/2},$$

and for shortness sake one simply says that the *L^q regularity problem is solvable for L*.

The main result

It is well known (see [2, 3, 27]) that for the Laplace equation the L^p Dirichlet problem is solvable for $2-\varepsilon < p < \infty$, and the L^q regularity problem is solvable for $1<q<2+\varepsilon$, where in both cases $\varepsilon > 0$ depends on the Lipschitz character of D.

Moreover, it has been established for more general divergence form second order elliptic operators, that the solvability of the L^q regularity problem implies the solvability of the L^p Dirichlet problem, $1/p+1/q=1$ (see [17, Theorem 5.4]). Our result relating L^q regularity problem with L^p Dirichlet problem for non-divergence elliptic operators L as in (2.1) is the following.

THEOREM 1. *Let $D\subset\mathbb{R}^n$ be a starlike Lipschitz domain. Then the solvability of L^q regularity problem, with $1<q<2$, implies the solvability of L^p Dirichlet problem, for certain $1<p<\infty$.*

Proof. We will establish that ω is absolutely continuous with respect to σ, and that (3.2) holds. For this purpose we let $\Delta\subset\partial D$ denote any surface ball of radius $0<r<r_0/2$, and let $\Delta_s\subset\Delta$ be any surface ball of radius $s\in(r/32,r/8)$ centered at some point in Δ. Denote by $\Delta'\subset\partial D\setminus\Delta$ another surface ball of radius r such that it lies at a distance r from 2Δ.

Once we think of s as a fixed parameter, we consider the boundary datum given by a smooth function f satisfying $0\leqslant f\leqslant 1$, $f\equiv 1$ on $\Delta_{2s/3}$, $f\equiv 0$ on $\partial D\setminus\Delta_s$ and $|\nabla f|\lesssim 1/s$. Let u be the solution of the Dirichlet problem $Lu=0$, $u|_{\partial D}=f$. By our assumption, we know that $\|\widetilde{N}(\nabla u)\|_q\lesssim\|\nabla_T f\|_q$, which by the choice of f yields

$$\|\widetilde{N}(\nabla u)\|_q\lesssim\frac{1}{s}\sigma(\Delta_s)^{1/q}. \tag{3.5}$$

On the other hand, by the doubling property of ω (cf. [14, Remark 5.21])

$$\frac{\omega(\Delta_s)}{\omega(\Delta)}\lesssim\frac{1}{\omega(\Delta)}\int f\,d\omega=\frac{u(\vec{0})}{\omega(\Delta)}. \tag{3.6}$$

We observe at this point that with a standard proof, using essentially the maximum principle, the comparison principle (see [1, Theorem 2.1]) and Harnack's inequality (see e.g. the argumentation in [13, Lemma 4.11]) one gets the following

LEMMA 1. *If $Lu=0$ on D, vanishing continuously on $\partial D\setminus 2\Delta$ then there exists $C>0$ depending at most on the Lipschitz character of D, the ellipticity constants and n, such that*

$$u(X)\leqslant Cu(A(\Delta)),\quad\text{and}\quad u(X)\leqslant Cu(A(\Delta))\omega^X(\Delta)\quad\text{hold for every } X\in D\setminus\Psi(2\Delta).$$

Armed with this result we can continue our estimates, using Harnack inequality, as follows:

$$\frac{u(\vec{0})}{\omega(\Delta)}\lesssim u(A_r)\lesssim u(A_r')\lesssim\left(\frac{1}{X(B(A_r'))}\int_{B(A_r')}u^2\,dX\right)^{1/2}$$

Here we have used the notation $A_r=A(\Delta)$, $A_r'=A(\Delta')$. Now we use Proposition 1 to obtain

$$\left(\frac{1}{X(B(A_r'))}\int_{B(A_r')}u^2\,dX\right)^{1/2}\lesssim r\left(\frac{1}{X(\Psi^\varepsilon)}\int_{\Psi^\varepsilon}|\nabla u|^2\,dX\right)^{1/2}$$

where Ψ^{ε} denotes an ε-upper portion of the Carleson region $\Psi(A_r')$. Back in (3.6) we obtain

$$\frac{\omega(\Delta_s)}{\omega(\Delta)} \lesssim r\widetilde{\mathscr{N}}(\nabla u)(Q) \qquad \text{for } Q \in \Delta'',$$

where Δ'' is a surface ball of radius of the order of r, and where

$$\widetilde{\mathscr{N}}(\nabla u)(Q) \equiv \sup_{X\in\Gamma(Q)} \left(\frac{1}{Z(\Psi^{\varepsilon}(X))} \int_{\Psi^{\varepsilon}(X)} |\nabla u(Z)|^2 dZ \right)^{1/2}.$$

Invoking [17, Lemma 5.12] we know $\|\widetilde{N}(\nabla u)\|_p \approx \|\widetilde{\mathscr{N}}(\nabla u)\|_p$, $1 \leqslant p < \infty$. Integrating over Δ'' with respect to σ, by the estimate (3.5) for the L^q norm of $\widetilde{N}(\nabla u)$, we obtain

$$\sigma(\Delta)^{1/q} \frac{\omega(\Delta_s)}{\omega(\Delta)} \lesssim \sigma(\Delta_s)^{1/q},$$

because $r/s \leqslant 32$ and $\sigma(\Delta'') \approx \sigma(\Delta)$. This is (3.2) with $E = \Delta_s$, an arbitrary surface ball of radius s, centered at a point in Δ.

With a covering lemma of Vitali-type one can get the result for $E \subset \Delta$ any open set replacing Δ_s, because both σ and ω are doubling measures. Finally, by regularity of both ω and σ, and invoking the continuity property of measures, we get (3.2) for any Borel measurable set $E \subset \Delta$, which is what we wanted to prove.

Additional remarks

Some works have been dealt with the L^p-Dirichlet problem for nondivergence equations. For instance, [20] has established the preservation of solvability of this problem under small perturbations of the main coefficients, based on results of [4, 11], adapting and extending techniques from [11]. Further results including sufficients conditions on the oscillation of the main coefficients for the solvability of some L^p Dirichlet problems are contained in [21]. See also [9].

Although the regularity problem does not explicitly involves any square function estimate, the estimate for the non-tangential maximal function of the gradient may be loosely viewed as some sort of square function estimate. This was an initial intuition for the development of the main result herein.

To our knowledge, the connection between square function estimates and solvability of L^p Dirichlet problems was originally developed in [16, 18, 19], where an explicit mention to non-symmetric matrices of coefficients as well as non-divergence form equations is included. See also [7]. An important recent improvement has been obtained in [15] extending a result from [5]. In this regard see also [12] and [24]. Parabolic versions of some of these results are obtained, for instance, in [22, 23, 8].

We have already mentioned that for elliptic divergence-form operators beyond the laplacian, the solvability of L^p Regularity problem implies the solvability of L^q Dirichlet problem, with $1/p + 1/q = 1$ (see [17, §5]). A partial result on the converse implication is explored in [25]. With a different scale of regularity of Hardy-Sobolev type, some results have been obtained in [6]. To our knowledge, no result of this type of connections has been obtained for elliptic non-divergence form operators.

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