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EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR FRACTIONAL NEUMANN ELLIPTIC EQUATIONS

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Abstract. This article is devoted to study the fractional Neumann elliptic problem

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + u = u^p & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N > 2s$, $0 < s \leqslant s_0 < 1$, $1 < p < (N+2s)/(N-2s)$, $\varepsilon > 0$ and ν is the outer normal to $\partial\Omega$. We show that there exists at least one nonconstant solution u_ε to this problem provided ε is small. Moreover, we prove that $u_\varepsilon \in L^\infty(\Omega)$ by using Moser-Nash iteration.

1. Introduction

In this paper, we study the following Neumann elliptic problem involving fractional Laplacian:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + u = u^p & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N > 2s$, $0 < s < 1$, $1 < p < (N+2s)/(N-2s)$, ε is a positive parameter and ν is the outer normal to $\partial\Omega$. The operator $(-\varepsilon^2\Delta)^s$ is understood as the fractional Laplacian in the bounded domain Ω encoding the homogeneous Neumann boundary condition, that is, the fractional Neumann Laplacian which is defined as follows.

Let ϕ_k $(k \in \mathbb{N}_0)$ be an eigenfunction of $-\Delta$ given by

$$\begin{cases} -\Delta\phi_k = \mu_k\phi_k & \text{in } \Omega, \\ \partial_\nu\phi_k = 0 & \text{on } \partial\Omega, \end{cases}$$

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where μ_k is the corresponding eigenvalue of ϕ_k. The ε-Neumann Laplacian is the operator that acts on an L^2 function

$$u = \sum_{k=1}^{\infty} u_k \phi_k \quad \text{where} \quad u_k = \int_\Omega u\phi_k dx,$$

as

$$-\varepsilon^2 \Delta_N u = \sum_{k=1}^{\infty} (\varepsilon^2 \mu_k) u_k \phi_k$$

in a suitable sense. Then the fractional $(\varepsilon-)$ Neumann Laplacian is given by

$$(-\varepsilon^2 \Delta_N)^s u = \sum_{k=1}^{\infty} (\varepsilon^2 \mu_k)^s u_k \phi_k.$$

This operator can be extended by density for u in the Hilbert space

$$\mathscr{H}_\varepsilon^s(\Omega) \equiv \mathrm{Dom}\left((-\varepsilon^2 \Delta_N)^s\right) = \left\{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} (\varepsilon^2 \mu_k)^s |u_k|^2 < \infty \right\}$$

under the scalar product

$$\langle u, v \rangle_{\mathscr{H}_\varepsilon^s(\Omega)} := \langle u, v \rangle_{L^2(\Omega)} + \sum_{k=1}^{\infty} (\varepsilon^2 \mu_k)^s u_k v_k,$$

so that the norm in $\mathscr{H}_\varepsilon^s(\Omega)$ is given by

$$\|u\|^2_{\mathscr{H}_\varepsilon^s(\Omega)} = \|u\|^2_{L^2(\Omega)} + \sum_{k=1}^{\infty} (\varepsilon^2 \mu_k)^s |u_k|^2.$$

We refer readers to [28, 23, 22] and references therein for more details.

When $s = 1$, the problem (1.1) reduces to the Laplace case which is considered in the famous paper [19]. In [19], Lin-Ni-Takagi studied the existence of solutions to the one-parameter semilinear Neumann boundary problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = g(u) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $g(t)$ is a suitable nonnegative nonlinearity on $\mathbb{R}$ vanishing for $t \leqslant 0$, growing superlinearly at infinity. They showed that if ε is small enough, there exists a positive smooth solution u_ε that satisfies $J_\varepsilon(u) \leqslant C\varepsilon^{N/2}$ where C is a positive constant independent of ε with J_ε is the energy functional of problem (1.2), that is,

$$J_\varepsilon(u) = \frac{1}{2} \int_\Omega \left(\varepsilon^2 |\nabla u|^2 + u^2\right) dx - \int_\Omega G(u) dx,$$

where G is an primitive of g.

Recently, Stinga-Volzone [22] extended the results in [19] to the square root of Laplacian case, that is, they considered the following Neumann problem:

$$\begin{cases} (-\varepsilon^2\Delta)^{1/2}u + u = g(u) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where

$$g(t) = \begin{cases} t^p & \text{if } t \geqslant 0, \\ 0 & \text{if } t \leqslant 0, \end{cases} \tag{1.4}$$

for $1 < p < (N+1)/(N-1)$. Notice that $(N+1)/(N-1)$ is the critical Sobolev exponent. By applying the Mountain Pass Lemma of Ambrosetti and Rabinowitz [2], they proved the existence of nonconstant solutions of (1.3) provided ε is small. They also studied the regularity and Harnack inequality in the same paper. Moreover, there are some works related to concentration phenomena for Schrödinger equations involving the integral fractional Laplacian, see for example [13, 14, 17, 7].

In the present paper, we aim to study the fractional case for $s \in (0, s_0)$ with $s_0 \geqslant 1/2$. Our main result is

THEOREM 1. *Suppose* $0 < s \leqslant s_0$ *with some* $1/2 \leqslant s_0 < 1$*, then there exists at least one nonconstant solution* u_ε *to problem* (1.1) *provided* $\varepsilon > 0$ *is small. Moreover,* $u_\varepsilon \in L^\infty(\Omega)$.

REMARK 1. We should remark that we just consider the case $s \in (0, s_0)$ in our article since we study the existence of nonconstant solutions. In fact, we can prove the existence of nontrivial solutions for $s \in (0,1)$.

As we know, by using the language of semigroups (see for example [23, 10]), one can check that $(-\varepsilon\Delta_N)^s$ is indeed a nonlocal operator. In fact,

$$(-\varepsilon^2\Delta_N)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^{+\infty} (e^{t\varepsilon^2\Delta_N} u(x) - u(x)) \frac{dt}{t^{1+s}}, \tag{1.5}$$

where $e^{t\Delta_N} u(x)$ is the heat diffusion semigroup generated by the Neumann Laplacian acting on u. Then, as in [23, 10], one can check that for a smooth function u we have the pointwise integro-differential formula

$$(-\varepsilon^2\Delta_N)^s u(x) = C(N,s,\Omega) P.V. \int_\Omega (u(x) - u(y)) K(x,y) dy, \quad x \in \Omega,$$

where $C(N,s,\Omega)$ is a positive constant and the kernel $K(x,y)$ satisfies the estimate $K(x,y) \sim \varepsilon^{2s}|x-y|^{-(N+2s)}$ for $x,y \in \Omega$.

REMARK 2. (1) We remark that Dipierro-Ros-Oton-Valdinoci in [15] introduced a Neumann type condition for integral fractional elliptic problem, that is,

$$\int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = 0 \quad \text{for } x \in \mathbb{R}^N \setminus \bar{\Omega}.$$

Recently, Chen in [7] studied perturbed problem involving the integral fractional Laplacian with the above Neumann type condition. In our article, we consider the spectral fractional Laplacian which is different from the integral fractional Laplacian, see [27, 20].
(2) We also refer that Dipierro, Soave and Valdinoci consider the stable solutions of fractional Neumann equations in [16].

It is already known that the fractional operator (1.5) can be described as Dirichlet-to-Neumann maps for an extension problem in the spirit of the extension problem for the fractional Laplacian on $\mathbb{R}^N$ of [9], see [23, 28]. Indeed, let us define

$$\mathscr{C} = \Omega \times (0,+\infty), \quad \partial_L \mathscr{C} = \partial\Omega \times [0,+\infty).$$

We write points in the cylinder $\mathscr{C}$ by $(x,y) \in \mathscr{C} = \Omega \times (0,+\infty)$. Consider the space

$$\mathscr{H}^1(\mathscr{C}, y^{1-2s}) = \left\{ w \in H^1(\mathscr{C}, y^{1-2s}) : \frac{1}{|\Omega|} \int_\Omega w(x,y)dx = 0, \forall y \geqslant 0 \right\},$$

where $H^1(\mathscr{C}, y^{1-2s})$ is the weighted Sobolev space with respect to the weight y^{1-2s}. By Lemma 2.2 in [21], the space $\mathscr{H}^1(\mathscr{C}, y^{1-2s})$ can be equipped with the norm

$$\|w\| = \left(\int_{\mathscr{C}} y^{1-2s} |\nabla w|^2 dxdy \right)^{\frac{1}{2}}.$$

Hence, we can study problem (1.1) by variational methods for a local problem. More precisely, problem (1.1) can be reduced to the problem

$$\begin{cases} \varepsilon^{2s} \Delta_x w + \dfrac{1-2s}{y} \dfrac{\partial w}{\partial y} + \dfrac{\partial^2 w}{\partial y^2} = 0 & \text{in } \mathscr{C}, \\ \partial_\nu w = 0 & \text{on } \partial_L \mathscr{C}, \\ -\lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y} = g(w(x,0)) - w(x,0) & \text{in } \Omega, \end{cases} \tag{1.6}$$

where $g(t)$ is defined as (1.4). If positive function w satisfies (1.6), then the trace w on $\Omega \times \{0\}$ will be a solution of problem (1.1). We look for a positive nonconstant weak solution w to (1.6) as a positive nonconstant critical point over $\mathscr{H}^\varepsilon(\mathscr{C}, y^{1-2s})$ (see Section 2) of the functional

$$\mathscr{I}_\varepsilon(w) = \frac{1}{2} \int_{\mathscr{C}} y^{1-2s} \left(\varepsilon^{2s} |\nabla_x w|^2 + w_y^2 \right) dxdy + \frac{1}{2} \int_\Omega w^2(x,0)dx - \int_\Omega G(w(x,0))dx, \tag{1.7}$$

where

$$G(t) = \int_0^t g(s)ds = \begin{cases} \dfrac{1}{p+1} t^{p+1}, & \text{if } t \geqslant 0, \\ 0, & \text{if } t \leqslant 0. \end{cases}$$

Using the local formulation established above, then Theorem 1 will follow as a corollary of the following result for the degenerated equation (1.6).

THEOREM 2. *Suppose $0 < s \leqslant s_0$ with some $1/2 \leqslant s_0 < 1$, then there exists at least one positive nonconstant weak solution w_ε to problem (1.6) provided $\varepsilon > 0$ is small. In this case there exists a positive constant C, depending only on p, s and Ω, such that*

$$\mathscr{I}_\varepsilon(w_\varepsilon) \leqslant C\varepsilon^{sN}.$$

We use variational methods (cf. [24, 25]) to find positive solutions of equation (1.6). We will prove Theorem 2 by using the Mountain Pass Lemma of Ambrosetti and Rabinowitz to energy functional $\mathscr{I}_\varepsilon$ of problem (1.6). However, there are some difficulties appear compare our article with [22]. The main difficulties are two points: the operator in (1.6) is degenerated and the nonlinearity is on the boundary. By adapting Moser-Nash iteration (see [18]) to the problem (1.6), we can obtain the uniform bound in Theorem 1. Although the weighted function y^{1-2s} is possibly singular or degenerates at $y = 0$, we still may establish an inverse Hölder inequality for $w(\cdot, 0) = u(\cdot)$, and we may iterate the inequality for u. The Moser-Nash iteration method has been used to study the uniformly bounds for fractional elliptic problem, see for example [1, 3, 4, 5, 6, 8, 11, 21, 26, 30] and references therein.

The paper is organized as follows. In section 2, we give some notations, preliminaries and prove some estimate lemmas. Section 3 is devoted to prove Theorems 1 and 2.

2. Variational setting and preliminaries

We start this section by recalling the following important extension result proved in [28].

THEOREM 3. *Let $u \in \mathscr{H}_\varepsilon^s(\Omega)$ such that $(1/|\Omega|)\int_\Omega u\mathrm{d}x = 0$. Define*

$$w(x,y) := \sum_{k=1}^{\infty} \rho(\varepsilon^{2s}\lambda_k^{\frac{1}{2}} y) u_k \varphi_k(x).$$

where the function $\rho(t)$ solves the problem

$$\begin{cases} \rho''(t) + \frac{1-2s}{t}\rho'(t) = \rho(t) & t > 0, \\ -\lim_{t\to 0^+} t^{1-2s}\rho'(t) = k_s, \\ \rho(0) = 1, \end{cases}$$

where

$$k_s := \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}.$$

Then $w \in \mathscr{H}^1(\mathscr{C}, y^{1-2s})$ is the unique weak solution to the extension problem

$$\begin{cases} \varepsilon^{2s}\Delta_x w + \dfrac{1-2s}{y}\dfrac{\partial w}{\partial y} + \dfrac{\partial^2 w}{\partial y^2} = 0, & \text{in } \mathscr{C} := \Omega \times (0,\infty), \\ \partial_\nu w = 0, & \text{on } \partial_L \mathscr{C} := \partial\Omega \times [0,\infty), \\ w(x,0) = u(x), & \text{on } \Omega, \end{cases}$$

where ν *is the outward normal to the lateral boundary* $\partial_L\mathscr{C}$ *of* $\mathscr{C}$. *More precisely,*

$$\int_{\mathscr{C}} y^{1-2s}\left(\varepsilon^{2s}\nabla_x w\cdot\nabla_x\psi + w_y\psi_y\right)\mathrm{d}x\mathrm{d}y = 0,$$

for all test functions $\psi\in\mathscr{H}^1(\mathscr{C},y^{1-2s})$ *with zero trace over* Ω, *that is* $tr_\Omega\psi = 0$, *and*

$$\lim_{y\to 0^+} w(x,y) = u(x) \ \ in \ \ L^2(\Omega).$$

Moreover, the function w *is the unique minimizer of the energy functional*

$$\mathscr{F}(w) = \frac{1}{2}\int_{\mathscr{C}} y^{1-2s}\left(\varepsilon^{2s}|\nabla_x w|^2 + |w_y|^2\right)dxdy,$$

over the set

$$\mathscr{U} = \{w\in\mathscr{H}^1(\mathscr{C},y^{1-2s}) : tr_\Omega w = u\}.$$

What's more,

$$-\frac{1}{k_s}\lim_{y\to 0^+} y^{1-2s}w_y = (-\varepsilon^2\Delta_N)^s u, \ in \ \mathscr{H}_\varepsilon^{-s}(\Omega).$$

Let us define the space $\mathscr{H}^\varepsilon(\mathscr{C},y^{1-2s})$ as the completion of $\mathscr{H}^1(\mathscr{C},y^{1-2s})$ under the scalar product

$$\langle v,w\rangle_\varepsilon = \int_{\mathscr{C}} y^{1-2s}\left(\varepsilon^{2s}\nabla_x v\nabla_x w + v_y w_y\right)dxdy + \int_\Omega v(x,0)w(x,0)dx.$$

We denote by $\|\cdot\|_\varepsilon$ the associated norm:

$$\|w\|_\varepsilon^2 = \int_{\mathscr{C}} y^{1-2s}\left(\varepsilon^{2s}|\nabla_x w|^2 + w_y^2\right)dxdy + \int_\Omega w^2(x,0)dx.$$

For $\varepsilon > 0$, we notice that

$$\mathscr{H}^1(\mathscr{C},y^{1-2s}) \subset \mathscr{H}^\varepsilon(\mathscr{C},y^{1-2s}),$$

as Hilbert spaces, where the inclusion is strict, since constant functions belong to $\mathscr{H}^\varepsilon(\mathscr{C},y^{1-2s})$ but not to $\mathscr{H}^1(\mathscr{C},y^{1-2s})$.

By a similar argument as Lemma 2.4 and Corollary 2.7 in [22], then we have the following proposition.

PROPOSTION 1. *Fix* $\varepsilon > 0$. *Then the embedding*

$$\mathscr{H}^\varepsilon(\mathscr{C},y^{1-2s}) \subset L^q(\Omega)$$

for all $1\leqslant q\leqslant 2N/(N-2s)$, *is continuous. Moreover, the embedding is compact provided* $1\leqslant q < 2N/(N-2s)$.

We will use the Mountain Pass Lemma to obtain the existence result in Theorem 2. Thus, by multiplying equation (1.6) by a test function and integrating by parts, we can give the following suitable definition of weak solution.

DEFINITION 1. Let $\phi \in \mathscr{H}^{\varepsilon}(\mathscr{C}, y^{1-2s})$. We say that a function $w \in \mathscr{H}^{\varepsilon}(\mathscr{C}, y^{1-2s})$ is a weak solution to (1.6) if

$$\int_{\mathscr{C}} y^{1-2s}\left(\varepsilon^{2s}\nabla_x w \nabla_x \phi + w_y \phi_y\right) dxdy + \int_{\Omega} w(x,0)\phi(x,0)dx = \int_{\Omega} g(w(x,0))\phi(x,0)dx.$$

In order to use the Mountain Pass Lemma, we need the following lemmas related to energy functional $\mathscr{I}_{\varepsilon}$ which is defined as (1.7).

LEMMA 1. *$\mathscr{I}_{\varepsilon}$ satisfies Palais-Smale condition.*

Proof. Let $\{w_n\}$ be a Palais-Smale sequence such that

$$|\mathscr{I}_{\varepsilon}(w_n)| \leqslant c \quad \text{for all } n \in \mathbb{N}$$

and

$$\mathscr{I}_{\varepsilon}'(w_n) \to 0 \quad \text{as } n \to \infty.$$

Hence, for any $\delta > 0$, there exists $N = N(\delta)$ such that $n \geqslant N$,

$$\delta \|w_n\|_{\varepsilon} \geqslant |\langle \mathscr{I}_{\varepsilon}'(w_n), w_n \rangle_{\varepsilon}| = \left| \|w_n\|_{\varepsilon}^2 - \int_{\Omega} g(w_n(x,0))w_n(x,0)dx \right|.$$

Choosing $\delta = 1$, we have that

$$\left| \int_{\Omega} g(w_n(x,0))w_n(x,0)dx \right| \leqslant \|w_n\|_{\varepsilon}^2 + \|w_n\|_{\varepsilon}.$$

On the other hand, since $|\mathscr{I}_{\varepsilon}(w_n)| \leqslant c$ and $g(t) = t^p$ for $t > 0$ and $p > 1$, we have

$$\begin{aligned} c &\geqslant \left| \frac{1}{2}\|w_n\|_{\varepsilon}^2 - \int_{\Omega} G(w_n(x,0))dx \right| \\ &\geqslant \frac{1}{2}\|w_n\|_{\varepsilon}^2 - \frac{1}{p+1}\left(\|w_n\|_{\varepsilon}^2 + \|w_n\|_{\varepsilon} \right) \\ &= \left(\frac{1}{2} - \frac{1}{p+1} \right)\|w_n\|_{\varepsilon}^2 - \frac{1}{p+1}\|w_n\|_{\varepsilon}. \end{aligned}$$

Therefore, $\{w_n\}$ is bounded in $\mathscr{H}^{\varepsilon}(\mathscr{C}, y^{1-2s})$.

Up to a subsequence, we assume that

$$w_n \rightharpoonup w \quad \text{in } \mathscr{H}^{\varepsilon}(\mathscr{C}, y^{1-2s}). \tag{2.1}$$

By Sobolev embedding (see Proposition 1), $w_n \to w$ in $L^{p+1}(\Omega)$. Notice that $p+1 < 2N/(N-2s)$, then we can find that $g(w_n) \to g(w)$ in $L^{(p+1)/p}(\Omega)$. Observe that

$$\|w_n - w\|_{\varepsilon}^2 = \langle \mathscr{I}_{\varepsilon}'(w_n) - \mathscr{I}_{\varepsilon}'(w), w_n - w \rangle_{\varepsilon} + \int_{\Omega} (g(w_n) - g(w))(w_n - w)dx.$$

By (2.1), we have

$$\langle \mathscr{I}_\varepsilon'(w_n) - \mathscr{I}_\varepsilon'(w), w_n - w\rangle_\varepsilon \to 0, \quad n \to \infty.$$

It follows from the Hölder inequality that

$$\left| \int_\Omega \big(g(w_n) - g(w)\big)(w_n - w)dx \right| \leqslant \big\|g(w_n) - g(w)\big\|_{L^{(p+1)/p}(\Omega)} \|w_n - w\|_{L^{p+1}(\Omega)} \to 0$$

as $n \to \infty$. Thus we have proved that

$$\|w_n - w\|_\varepsilon \to 0, \quad n \to \infty.$$

□

LEMMA 2. *There exists a* $\rho > 0$ *such that* $\mathscr{I}_\varepsilon(w) > 0$ *if* $0 < \|w\|_\varepsilon < \rho$ *and* $\mathscr{I}_\varepsilon(w) \geqslant \beta > 0$ *for some* $\beta > 0$ *if* $\|w\|_\varepsilon = \rho$.

Proof. By Sobolev embedding (see Proposition 1),

$$\int_\Omega G(w(x,0))dx \leqslant C\|w\|_\varepsilon^{p+1}.$$

Since $p > 1$, then we can get the conclusions by the definition of functional $\mathscr{I}_\varepsilon$. □

LEMMA 3. *For a sufficiently small* $\varepsilon > 0$, *there exists a nonnegative function* $\Phi \in \mathscr{H}^\varepsilon(\mathscr{C}, y^{1-2s})$ *and positive constants* t_0, C_0 *such that*

$$\mathscr{I}_\varepsilon(t_0\Phi) = 0,$$

and

$$\mathscr{I}_\varepsilon(t\Phi) \leqslant C_0\varepsilon^{sN}, \quad \textit{for all } t \in [0, t_0].$$

Proof. Choosing Φ as

$$\Phi(x,y) = e^{-y/2}\varphi(x),$$

where φ is

$$\varphi(x) = \begin{cases} \varepsilon^{-sN}(1 - \varepsilon^{-s}|x|), & \text{if } |x| < \varepsilon^s, \\ 0, & \text{if } |x| \geqslant \varepsilon^s. \end{cases}$$

We can also suppose that $0 \in \Omega$ and that ε is sufficiently small so that $\Phi \in \mathscr{H}^\varepsilon(\mathscr{C}, y^{1-2s})$. Indeed, we have that

$$\begin{aligned} \int_\mathscr{C} y^{1-2s}|\nabla_x\Phi|^2 dxdy &= \int_\mathscr{C} y^{1-2s}e^{-y}|\nabla\varphi|^2 dxdy \\ &= \int_0^{+\infty} y^{1-2s}e^{-y}dy \int_\Omega |\nabla\varphi|^2 dx \end{aligned}$$

$$= \Gamma(2-2s)\int_\Omega |\nabla\varphi|^2 dx,$$

$$\begin{aligned}\int_{\mathscr{C}} y^{1-2s}|\Phi_y|^2 dxdy &= \frac{1}{4}\int_{\mathscr{C}} y^{1-2s}e^{-y}\varphi^2 dxdy\\ &= \frac{1}{4}\int_0^{+\infty} y^{1-2s}e^{-y}dy\int_\Omega \varphi^2 dx\\ &= \frac{1}{4}\Gamma(2-2s)\int_\Omega \varphi^2 dx,\end{aligned}$$

and

$$\int_\Omega |\Phi|^2 dx = \int_\Omega \varphi^2 dx.$$

Since $0 < s \leqslant s_0$ with some $1/2 \leqslant s_0 < 1$, then $\Gamma(2-2s)$ is finite. Moreover, by straightforward computation, we have that, for $p > 0$,

$$\int_\Omega |\nabla\varphi|^2 dx = C\varepsilon^{-sN-2s},$$
$$\int_\Omega \varphi^p dx = C_p\varepsilon^{(1-p)sN},$$

for positive constants C and C_p. Therefore, by following the same arguments as Lemma 2.4 in [19] (see also [22, 7]) if we set

$$\mathscr{G}(t) = \mathscr{I}_\varepsilon(t\Phi), \quad \text{for } t \geqslant 0, \tag{2.2}$$

it is seen that there exists t_1, t_2 satisfy $0 < t_1 < t_2$ such that $\mathscr{G}'(t) < 0$ if $t > t_1$ and $\mathscr{G}(t) < 0$ if $t > t_2$. We remark that $t_1 = C\varepsilon^{sN}$ for constant $C > 0$.

By Lemma 2, we know that $\mathscr{G}(t) > 0$ for t small enough, then there exists a t_0 such that $\mathscr{G}(t_0) = 0$, that is,

$$\mathscr{I}_\varepsilon(t_0\Phi) = 0 \quad \text{for small } \varepsilon > 0.$$

By using Lemma 2 again, we can get $\|t_0\Phi\|_\varepsilon > \rho$, where ρ is defined in Lemma 2 . Moreover, as in [19], we have

$$\begin{aligned}\max_{t\geqslant 0}\mathscr{G}(t) &= \max_{0\leqslant t\leqslant t_1}\mathscr{G}(t)\\ &= \max_{0\leqslant t\leqslant t_1}\left\{\frac{1}{2}t^2\varepsilon^{-sN}(C+C_2) - \int_\Omega G(w(x,0))dx\right\}\\ &\leqslant \max_{0\leqslant t\leqslant t_1}\frac{1}{2}t^2\varepsilon^{-sN}(C+C_2)\\ &= \frac{1}{2}t_1^2\varepsilon^{-sN}(C+C_2)\\ &\leqslant C_0\varepsilon^{sN},\end{aligned} \tag{2.3}$$

for some constant $C_0 > 0$. For the last inequality, we have used the fact $t_1 = C\varepsilon^{sN}$. □

3. Proofs

This section is devoted to prove of our main results, Theorems 1 and 2.

Proof. [*Proof of Theorem* 2] **(Existence.)** We let $E = \mathscr{H}^{\varepsilon}(\mathscr{C}, y^{1-2s})$, $e = t_0\Phi$ and

$$X = \{\gamma \in C([0,1];E) : \gamma(0) = 0, \gamma(1) = e\}.$$

As the proof in [22], the functional $\mathscr{I}_{\varepsilon}$ is in $C^1(\mathscr{H}^{\varepsilon}(\mathscr{C}, y^{1-2s});\mathbb{R})$. Hence, applying Lemmas 1-3 and the Mountain Pass Lemma (see [2, 29]), then the number

$$c = \min_{\gamma \in X} \max_{t \in [0,1]} \mathscr{I}_{\varepsilon}(\gamma(t))$$

is a critical value of $\mathscr{I}_{\varepsilon}$ in $\mathscr{H}^{\varepsilon}(\mathscr{C}, y^{1-2s})$. Thus, there exists w_{ε} in $\mathscr{H}^{\varepsilon}(\mathscr{C}, y^{1-2s})$ such that

$$\mathscr{I}_{\varepsilon}'(w_{\varepsilon}) = 0, \tag{3.1}$$

that is w_{ε} is a weak solution of equation (1.6).

(L^{∞}-estimate.) Here we use the Moser-Nash iteration method. For the convenience, we denote $w := w_{\varepsilon}$ where w_{ε} is the solution obtained above, and $u := u_{\varepsilon}$ with $u_{\varepsilon} = w_{\varepsilon}(x,0)$.

Letting $\bar{t} = |t| + k$ and $\bar{t}^{+} = t^{+} + k$. For $k > 0$ large, we have

$$|g(t) - t| \leqslant C\big(|\bar{t}|^{p} + \bar{t}\big) \quad \text{for } 1 < p < \frac{N+2s}{N-2s}.$$

Denote $w^{+} = \max\{0, w\}$, $w^{-} = -\min\{0, w\}$. We deal only with w^{+}, it can be done in the same way for w^{-}. Define

$$\bar{w}_L^{+} = \begin{cases} \bar{w}^{+} & \text{if} \quad \bar{w}^{+} < L, \\ L & \text{if} \quad \bar{w}^{+} \geqslant L. \end{cases} \tag{3.2}$$

For any $\varphi \in \mathscr{H}^{\varepsilon}(\mathscr{C}, y^{1-2s})$, by Definition 1,

$$\int_{\mathscr{C}} y^{1-2s}\big(\varepsilon^{2s}\nabla_x w \nabla_x \varphi + w_y \varphi_y\big) = \int_{\Omega \times \{0\}} \big(g(w(x,0)) - w(x,0)\big)\varphi dx. \tag{3.3}$$

For $\beta > 1$ to be determined, we choose in (3.3) that

$$\varphi = \bar{w}^{+}(\bar{w}_L^{+})^{2(\beta-1)} - k^{2(\beta-1)+1}.$$

Then, we have that

$$\nabla\varphi = (\bar{w}_L^{+})^{2(\beta-1)}\nabla\bar{w}^{+} + 2(\beta-1)\bar{w}^{+}(\bar{w}_L^{+})^{2(\beta-1)-1}\nabla\bar{w}_L^{+},$$

and

$$\begin{aligned} &\int_{\mathscr{C}} y^{1-2s}\Big\{\varepsilon^{2s}\nabla_x w\big[(\bar{w}_L^{+})^{2(\beta-1)}\nabla_x\bar{w}^{+} + 2(\beta-1)\bar{w}^{+}(\bar{w}_L^{+})^{2(\beta-1)-1}\nabla_x\bar{w}_L^{+}\big] \\ &\qquad + w_y\big[(\bar{w}_L^{+})^{2(\beta-1)}(\bar{w}^{+})_y + 2(\beta-1)\bar{w}^{+}(\bar{w}_L^{+})^{2(\beta-1)-1}(\bar{w}_L^{+})_y\big]\Big\}\,dxdy \\ &= \int_{\mathscr{C}} y^{1-2s}\Big\{(\bar{w}_L^{+})^{2(\beta-1)}\big[\varepsilon^{2s}|\nabla_x\bar{w}^{+}|^2 + (\bar{w}^{+})_y^2\big] \\ &\qquad + 2(\beta-1)\bar{w}^{+}(\bar{w}_L^{+})^{2(\beta-1)-1}\big[\varepsilon^{2s}|\nabla_x\bar{w}_L^{+}|^2 + (\bar{w}_L^{+})_y^2\big]\Big\}\,dxdy. \end{aligned} \tag{3.4}$$

Let $\mathscr{W}_L = \bar{w}^+(\bar{w}_L^+)^{\beta-1}$ and then

$$\nabla \mathscr{W}_L = (\bar{w}_L^+)^{\beta-1}\nabla \bar{w}^+ + (\beta-1)\bar{w}^+(\bar{w}_L^+)^{\beta-2}\nabla \bar{w}_L^+,$$

we deduce from (3.3) and (3.4) for $\beta > 1$ that

$$\begin{aligned}
&\int_{\mathscr{C}} y^{1-2s}\big(\varepsilon^{2s}|\nabla_x \mathscr{W}_L|^2 + (\mathscr{W}_L)_y^2\big)\,dxdy \\
&\leqslant C\beta \int_{\mathscr{C}} y^{1-2s}\big\{\varepsilon^{2s}\nabla_x w\big[(\bar{w}_L^+)^{2(\beta-1)}\nabla_x \bar{w}^+ + 2(\beta-1)\bar{w}^+(\bar{w}_L^+)^{2(\beta-1)-1}\nabla_x \bar{w}_L^+\big] \\
&\qquad + w_y\big[(\bar{w}_L^+)^{2(\beta-1)}(\bar{w}^+)_y + 2(\beta-1)\bar{w}^+(\bar{w}_L^+)^{2(\beta-1)-1}(\bar{w}_L^+)_y\big]\big\}\,dxdy \qquad (3.5)\\
&= C\beta \int_{\Omega\times\{0\}} \big(\bar{w}^+(\bar{w}_L^+)^{2(\beta-1)} - k^{2(\beta-1)+1}\big)\big(g(w(x,0)) - w(x,0)\big)\,dxdy \\
&\leqslant C\beta \int_{\Omega\times\{0\}} \bar{w}^+(\bar{w}_L^+)^{2(\beta-1)}\big|g(w(x,0)) - w(x,0)\big|\,dxdy.
\end{aligned}$$

By Young inequality, for any $\theta > 0$, there exists $C_\theta > 0$ such that

$$|g(t) - t| \leqslant C\big(|\bar{t}|^p + |\bar{t}|\big) \leqslant \theta|\bar{t}|^{2_s^*-1} + C_\theta|\bar{t}|.$$

This implies

$$\begin{aligned}
&\int_{\Omega\times\{0\}} \bar{w}^+(\bar{w}_L^+)^{2(\beta-1)}|g(w(x,0)) - w(x,0)|\,dx \\
&\leqslant C\int_{\Omega\times\{0\}} ((\bar{w}^+)^{p+1} + (\bar{w}^+)^2)(\bar{w}_L^+)^{2(\beta-1)}\,dx \qquad (3.6)\\
&\leqslant C\int_{\Omega\times\{0\}} \big[\theta(\bar{w}^+)^{2_s^*}(\bar{w}_L^+)^{2(\beta-1)} + C_\theta(\bar{w}_L^+)^{2(\beta-1)}(\bar{w}^+)^2\big]\,dx.
\end{aligned}$$

By (3.5), (3.6) and the Sobolev embedding theorem,

$$\begin{aligned}
&\Big(\int_{\Omega\times\{0\}} |\mathscr{W}_L|^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} \leqslant C\int_{\mathscr{C}} y^{1-2s}\big(\varepsilon^{2s}|\nabla_x \mathscr{W}_L|^2 + (\mathscr{W}_L)_y^2\big)\,dxdy \\
&\leqslant C\beta\int_{\Omega\times\{0\}} \big[\theta(\bar{w}^+)^{2_s^*}(\bar{w}_L^+)^{2(\beta-1)} + C_\theta(\bar{w}_L^+)^{2(\beta-1)}(\bar{w}^+)^2\big]\,dx.
\end{aligned}$$

Since $\mathrm{tr}_\Omega w = u$, then we have

$$\begin{aligned}
&\Big(\int_\Omega (\bar{u}^+(\bar{u}_L^+)^{\beta-1})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} \\
&\leqslant C\beta\int_\Omega \big[\theta(\bar{u}^+)^{2_s^*}(\bar{u}_L^+)^{2(\beta-1)} + C_\theta(\bar{u}_L^+)^{2(\beta-1)}(\bar{u}^+)^2\big]\,dx.
\end{aligned} \qquad (3.7)$$

Next, we claim that $\bar{u} \in L^{(2_s^*)^2/2}(\Omega)$. In fact, choosing $\beta = 2_s^*/2$, we have

$$\begin{aligned}
&\Big(\int_\Omega (\bar{u}^+(\bar{u}_L^+)^{\frac{2_s^*-2}{2}})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} \\
&\leqslant \int_\Omega \big[\theta(\bar{u}^+)^{2_s^*}(\bar{u}_L^+)^{2_s^*-2} + C_\theta(\bar{u}_L^+)^{2_s^*-2}(\bar{u}^+)^2\big]\,dx \qquad (3.8)\\
&\leqslant \theta\Big(\int_\Omega (\bar{u}^+(\bar{u}_L^+)^{\frac{2_s^*-2}{2}})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}}\Big(\int_\Omega (\bar{u}^+)^{2_s^*}\,dx\Big)^{\frac{2_s^*-2}{2_s^*}} + C_\theta\int_\Omega (\bar{u}_L^+)^{2_s^*-2}(\bar{u}^+)^2\,dx.
\end{aligned}$$

Choosing $\theta > 0$ small enough, we obtain

$$\Big(\int_\Omega (\bar{u}^+ (\bar{u}_L^+)^{\frac{2_s^*-2}{2}})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} \leqslant C\int_\Omega (\bar{u}_L^+)^{2_s^*-2}(\bar{u}^+)^2\,dx.$$

Let $L \to +\infty$, it yields

$$\Big(\int_\Omega (\bar{u}^+)^{\frac{(2_s^*)^2}{2}})\,dx\Big)^{\frac{2}{2_s^*}} \leqslant C\int_\Omega (\bar{u}^+)^{2_s^*}\,dx < +\infty. \tag{3.9}$$

Letting $t = (2_s^*)^2/[2(2_s^*-2)]$, then $2t/(t-1) < 2_s^*$. We estimate the right-hand side of (3.7). By the Hölder inequality,

$$\begin{aligned} &\int_\Omega (\bar{u}^+)^{2_s^*}(\bar{u}_L^+)^{2(\beta-1)}\,dx \\ &\leqslant \Big(\int_\Omega (\bar{u}^+)^{(2_s^*-2)t}\,dx\Big)^{\frac{1}{t}}\Big(\int_\Omega (\bar{u}^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{1-\frac{1}{t}} \\ &\leqslant C\Big(\int_\Omega (\bar{u}^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{1-\frac{1}{t}} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\int_\Omega (\bar{u}^+)^{2\beta}\,dx \\ &\leqslant \Big(\int_\Omega (\bar{u}^+)^{(2_s^*-2)t}\,dx\Big)^{\frac{1}{t}}\Big(\int_\Omega (\bar{u}^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{1-\frac{1}{t}} \\ &\leqslant C\Big(\int_\Omega (\bar{u}^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{1-\frac{1}{t}}. \end{aligned} \tag{3.11}$$

We deduce from (3.7), (3.10) and (3.11) that

$$\Big(\int_\Omega (\bar{u}^+ (\bar{u}_L^+)^{\beta-1})^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} \leqslant C\beta^2\Big(\int_\Omega (\bar{u}^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{1-\frac{1}{t}},$$

that is,

$$\Big(\int_\Omega (\bar{u}^+)^{\beta 2_s^*}\,dx\Big)^{\frac{1}{\beta}} \leqslant C^{\frac{1}{\beta}}\beta^{\frac{2_s^*}{\beta}}\Big(\int_\Omega (\bar{u}^+)^{\frac{2\beta t}{t-1}}\,dx\Big)^{\frac{(t-1)2_s^*}{2t\beta}}$$

For $i \geqslant 1$, we define β_{i+1} inductively so that

$$\frac{2t\beta_{i+1}}{t-1} = \beta_i 2_s^*,$$

that is,

$$\beta_{i+1} = \frac{2_s^*(t-1)}{2t}\beta_i,$$

and $\beta_1 = 2_s^*/2$. Therefore, we have

$$\Big(\int_\Omega (\bar{u}^+)^{\beta_{i+1}2_s^*}\,dx\Big)^{\frac{1}{\beta_{i+1}}} \leqslant C^{\frac{1}{\beta_{i+1}}}\beta_{i+1}^{\frac{2_s^*}{\beta_{i+1}}}\Big(\int_\Omega (\bar{u}^+)^{\beta_i 2_s^*}\,dx\Big)^{\frac{1}{\beta_i}}. \tag{3.12}$$

Let

$$A_i = \Big(\int_\Omega (\bar{u}^+)^{\beta_i 2_s^*}\, dx \Big)^{\frac{1}{\beta_i}}.$$

Iterating by (3.12), we obtain

$$A_{i+1} \leqslant \Pi_{m=2}^{i+1} C^{\frac{1}{\beta_m}} \beta_m^{\frac{2_s^*}{\beta_m}} A_1 \leqslant C_0 A_1.$$

This implies that

$$\|u^+\|_{L^\infty(\Omega)} \leqslant C_0 A_1.$$

We complete the proof by using the fact

$$A_1 = \Big(\int_\Omega (\bar{u}^+)^{\frac{(2_s^*)^2}{2}}\, dx \Big)^{\frac{2}{2_s^*}} < +\infty.$$

Similarly, we can prove $\|u^-\|_{L^\infty(\Omega)} < +\infty$. Therefore $w_\varepsilon(x,0) = u_\varepsilon(x) \in L^\infty(\Omega)$.

(Positivity.) Since $u_\varepsilon(x) = w_\varepsilon(x,0) \in L^\infty(\Omega)$, then by bootstrap argument we can prove that w_ε is a classical solution of (1.6) (see for example [22, 30]). In order to prove that $w_\varepsilon > 0$ everywhere in $\bar{\mathscr{C}}$, let us choose w_ε^- in the weak formulation

$$\langle \mathscr{I}_\varepsilon'(w_\varepsilon), w_\varepsilon^- \rangle = 0,$$

which implies that $w_\varepsilon \geqslant 0$ in $\mathscr{C}$. Hence, using the strong maximum principle (see [21]), we have $w_\varepsilon > 0$ in $\mathscr{C}$.

(Nonconstant solutions.) By (2.3),

$$\mathscr{I}_\varepsilon(w_\varepsilon) = c \leqslant \max_{t\in[0,t_0]} \mathscr{I}_\varepsilon(t\Phi) \leqslant C_0 \varepsilon^{sN}.$$

Let us argue by contradiction. Suppose that $w_\varepsilon = c_0$, where c_0 is positive real number. Hence,

$$\mathscr{I}_\varepsilon(w_\varepsilon) = \Big(\frac{1}{2} c_0^2 - \frac{1}{p+1} c_0^{p+1} \Big) |\Omega|.$$

Since (3.1) holds ture, then by using the equation (1.6), we have that $g(c_0) = c_0$ which implies that $c_0 = 1$. Thus,

$$\mathscr{I}_\varepsilon(w_\varepsilon) = \Big(\frac{1}{2} - \frac{1}{p+1} \Big) |\Omega|,$$

which contradicts the inequality $\mathscr{I}_\varepsilon(w_\varepsilon) \leqslant C_0 \varepsilon^{sN}$ for ε small.

So, we conclude that for small ε the function $\mathscr{I}_\varepsilon$ has at least one positive nonconstant critical point.

Now we are in position to prove Theorem 1.

Proof. [*Proof of Theorem* 1] A direct conclusion of Theorem 2 is that there exists at least one positive nonconstant solution $u_\varepsilon(x) = w_\varepsilon(x,0)$ to problem (1.1) provided $\varepsilon > 0$ is small. Moreover, $u_\varepsilon(x) = w_\varepsilon(x,0) \in L^\infty(\Omega)$. □

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REFERENCES

[1] C.O. ALVES AND O.H. MIYAGAKI, *Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^N$ via penalization method*, Calc. Var. Partial Differential Equations **55** (2016), no. 3, Art. 47, 19 pp.

[2] A. AMBROSETTI AND P.H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381.

[3] V. AMBROSIO, *Periodic solutions for a pseudo-relativistic Schrödinger equation*, Nonlinear Anal. **120** (2015), 262–284.

[4] V. AMBROSIO, *Periodic solutions for the non-local operator $(-\Delta+m^2)^s - m^{2s}$ with $m \geqslant 0$*, Topol. Methods Nonlinear Anal. **49** (2017), no. 1, 75–104.

[5] V. AMBROSIO, *Multiplicity of positive solutions for a class of fractional Schrödinger equations via penalization method*, Ann. Mat. Pura Appl. **(4) 196** (2017), no. 6, 2043–2062.

[6] C. BRÄNDLE, E. COLORADO, A. DE PABLO AND U. SÁNCHEZ, *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), no. 1, 39–71.

[7] G. CHEN, *Singularly perturbed Neumann problem for fractional Schrödinger equations*, Sci. China Math., 2017, **60**, doi: 10.1007/s11425-016-0420-2.

[8] X. CABRÉAND J. TAN, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. **224** (2010), no. 5, 2052–2093.

[9] L. CAFFARELLI AND L. SILVESTRE, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007) 1245–1260.

[10] L. CAFFARELLI AND P.R. STINGA, *Fractional elliptic equations, Caccioppoli estimates and regularity*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 3, 767–807.

[11] S. DIPIERRO, A. FIGALLI AND E. VALDINOCI, *Strongly nonlocal dislocation dynamics in crys- tals*, Comm. Partial Differential Equations **39** (2014), no. 12, 2351–2387.

[12] S. DIPIERRO, M. MEDINA, I. PERAL AND E. VALDINOCI, *Bifurcation results for a fractional elliptic equation with critical exponent in $\mathbb{R}^N$*, Manuscripta Math. **153** (2017), no. 1-2, 183–230.

[13] J. DÁVILA, M. DEL PINO AND J. WEI, *Concentrating standing waves for the fractional nonlinear Schrödinger equation*, J. Differential Equations **256** (2014), no. 2, 858–892.

[14] J. DÁVILA, M. DEL PINO, S. DIPIERRO AND E. VALDINOCI, *Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum*, Anal. PDE **8** (2015), no. 5, 1165–1235.

[15] S. DIPIERRO, X. ROS-OTON AND E. VALDINOCI, *Nonlocal problems with Neumann boundary conditions*, Rev. Mat. Iberoam. **33** (2017), 377–416.

[16] S. DIPIERRO, N. SOAVE AND E. VALDINOCI, *On stable solutions of boundary reaction-diffusion equations and applications to nonlocal problems with Neumann data*, to appear in Indiana Univ. Math. J.

[17] M. M. FALL, F. MAHMOUDI AND E. VALDINOCI, *Ground states and concentration phenomena for the fractional Schrödinger equation*, Nonlinearity **28** (2015), no. 6, 1937–1961.

[18] D. GILBARG AND N.S. TRUDINGER, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[19] C.-S. LIN, W.M. NI AND I. TAKAGI, *Large amplitude stationary solutions to a chemotaxis system*, J. Differential Equations **72** (1988), no. 1, 1–27.

[20] R. MUSINA AND A. I. NAZAROV, *On fractional Laplacians*, Comm. Partial Differential Equations **39** (2014), no. 9, 1780–1790.

[21] E. MONTEFUSCO, B. PELLACCI AND G. VERZINI, *Fractional diffusion with Neumann boundary conditions: the logistic equation*, Discrete Contin. Dyn. Syst. Ser. B **18** (2013) 2175–2202.

[22] P.R. STINGA AND B. VOLZONE, *Fractional semilinear Neumann problems arising from a fractional Keller-Segel model*, Calc. Var. Partial Differential Equations **54** (2015), no. 1, 1009–1042.

[23] P.R. STINGA AND J.L. TORREA, *Extension problem and Harnack's inequality for some fractional operators*, Comm. Partial Differential Equations **35** (2010), no. 11, 2092–2122.

[24] R. SERVADEI AND E. VALDINOCI, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (2013), no. 5, 2105–2137.

[25] R. SERVADEI AND E. VALDINOCI, *Mountain pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. **389** (2012), no. 2, 887–898.

[26] R. SERVADEI AND E. VALDINOCI, *Weak and viscosity solutions of the fractional Laplace equation*. Publ. Mat. **58** (2014), no. 1, 133–154.

[27] R. SERVADEI AND E. VALDINOCI, *On the spectrum of two different fractional operators*, Proc. Roy. Soc. Edinburgh Sect. A **144** (2014), no. 4, 831–855.

[28] B. VOLZONE, *Symmetrization for fractional Neumann problems*, Nonlinear Anal. **147** (2016), 1–25.

[29] M. WILLEM, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[30] A. XIA AND J. YANG, *Regularity of nonlinear equations for fractional Laplacian*, Proc. Amer. Math. Soc. **141** (2013), no. 8, 2665–2672.

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