EXISTENCE THEORY FOR NONLINEAR STURM–LIOUVILLE PROBLEMS WITH NON–LOCAL BOUNDARY CONDITIONS

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(Communicated by Lingju Kong)

Abstract. In this work we provide conditions for the existence of solutions to nonlinear Sturm-Liouville problems of the form

\[(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(x(t))\]

subject to non-local boundary conditions

\[ax(0) + bx'(0) = \eta_1(x) \quad \text{and} \quad cx(1) + dx'(1) = \eta_2(x).\]

Our approach will be topological, utilizing Schaefer’s fixed point theorem and the Lyapunov-Schmidt procedure.

1. Introduction

In this paper we provide criteria for the solvability of nonlinear Sturm-Liouville problems of the form,

\[(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(x(t)) \quad t \in [0, 1],\] (1)

subject to non-local boundary conditions

\[ax(0) + bx'(0) = \eta_1(x) \quad \text{and} \quad cx(1) + dx'(1) = \eta_2(x).\] (2)

There are several standard ways in which one may define a solution to problem (1)–(2), and so to maintain completeness, we mention that in this paper we will be interested in proving the existence of classical solutions to (1)–(2). Formally, by a solution to (1)–(2) we mean a function \(x : [0, 1] \to \mathbb{R}\) such that \(px'\) is continuously differentiable and satisfies (1)–(2).

Throughout our analysis, we will assume that \(p, q : [0, 1] \to \mathbb{R}\) are continuous, \(p(t) > 0\) for all \(t \in [0, 1]\), \(a^2 + b^2 > 0\) and \(c^2 + d^2 > 0\), \(\lambda\) is an eigenvalue of the associated linear Sturm-Liouville problem, \(f : \mathbb{R} \to \mathbb{R}\) is continuous, and for \(i = 1, 2\), 

\[\eta_i(x) = \int_{[0, 1]} g_i(x) d\mu_i,\]

where \(g_1, g_2 : \mathbb{R} \to \mathbb{R}\) are continuous and \(\mu_1\) and \(\mu_2\) are finite Borel measures on \([0, 1]\).

The focus of this paper is the analysis of nonlinear Sturm-Liouville problems at resonance subject to non-local boundary conditions, where by resonance we mean that

\[\text{Mathematics subject classification (2010): 34B24.}\]

\[\text{Keywords and phrases: Existence theory, Sturm-Liouville problem, boundary conditions.}\]
the linear homogeneous problem (7)–(8) has nontrivial solutions. Since the pioneering work of Landesman-Lazer, [12], much has been written about resonant nonlinear Sturm-Liouville boundary value problems with linear boundary conditions. Pertinent references from the point of view of this paper are [2, 4, 5, 6, 8, 11, 12, 14, 15, 18, 19]. Less has been said in regard to problems with nonlocal boundary conditions, even for the case of nonresonance; readers interested in results in this direction may consult [1, 7, 10, 13, 17, 20, 21, 22, 23].

The novelty of this work is due in large part to the generality of the nonlinear boundary conditions \( \eta_1 \) and \( \eta_2 \). As an important special case we point out that, by taking \( \mu_1 \) and \( \mu_2 \) to be point-supported measures, our integral boundary conditions allow for nonlinear multipoint boundary conditions of the form

\[
\eta_1(x) = \sum_{k=1}^{n} f_k(x(t_k)), \quad \eta_2(x) = \sum_{j=1}^{m} h_j(x(t_j)),
\]

where each \( f_k, h_j \) is a continuous function and each \( t_k, t_j \in [0, 1] \).

Our main result, Theorem 3.1, provides conditions for the existence of solutions to (1)–(2) under a suitable interaction of the eigenspace of the linear Sturm-Liouville problem and the nonlinearities in both the differential equation and the boundary conditions. We would like to remark that the result we obtain in Theorem 3.1 constitutes a significant extension of the work found in [15] by allowing for much more generality in the boundary conditions, (2).

2. Preliminaries

The nonlinear boundary value problem (1)–(2) will be viewed as an operator equation. We let \( C := C[0, 1] \) denote the space of real-valued continuous functions topologized by the supremum norm, \( \| \cdot \|_C \). As usual, \( L^2 := L^2[0, 1] \) will denote the space of real-valued square-integrable functions defined on \([0, 1]\). The topology on \( L^2 \) will be that induced by the standard \( L^2 \)-norm, \( \| \cdot \|_{L^2} \). We use \( H^2 \) to denote the Sobolev space of functions with two weak derivatives in \( L^2 \); that is,

\[
H^2 = \{ x \in L^2 \mid x' \text{ is absolutely continuous and } x'' \in L^2 \}.
\]

Unless otherwise stated, the topology on \( H^2 \) will be the subspace topology inherited from \( L^2 \). However, we will, on several occasions, topologize \( H^2 \) with the Sobolev norm,

\[
\| x \|_{H^2} = \| x \|_{L^2} + \| x' \|_{L^2} + \| x'' \|_{L^2}.
\]

On occasion, we may also view \( H^2 \) as a subspace of \( C \). We will use \( | \cdot | \) to denote the Euclidean norm on \( \mathbb{R}^2 \) and \( \langle \cdot, \cdot \rangle_2, \langle \cdot, \cdot \rangle_S, \) and \( \langle \cdot, \cdot \rangle_\mathbb{R} \) will denote the inner products on \( L^2, H^2, \) and \( \mathbb{R}^2 \), respectively. Weak convergence in \( L^2 \) will be denoted by \( \overset{w}{\rightharpoonup} \) and weak convergence in the Sobolev space \( H^2 \) will be denoted by \( \overset{S}{\rightharpoonup} \). We make \( L^2 \times \mathbb{R}^2 \)
an inner product space with inner product
\[
\begin{bmatrix}
  h \\
  w_1 \\
  w_2
\end{bmatrix}, \begin{bmatrix}
  g \\
  v_1 \\
  v_2
\end{bmatrix} := m \left( \langle h, g \rangle_2 + \left[ \begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix}, \begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix} \right] \right),
\]
(3)
where \( m \) is a positive constant which will be chosen later, and we will use \( \| \cdot \|_{L^2 \times \mathbb{R}^2} \) to denote the norm generated by this inner product. Lastly, we give \( C \times \mathbb{R}^2 \) the product topology, and we will use \( \| \cdot \|_{C \times \mathbb{R}^2} \) to denote the standard product norm on this space.

Linear boundary operators \( B_1 \) and \( B_2 \) will be defined as follows:
\[
B_1 : H^2 \to \mathbb{R} \text{ is given by } B_1 x = ax(0) + bx'(0)
\]
and \( B_2 : H^2 \to \mathbb{R} \) is given by
\[
B_2 x = cx(1) + dx'(1).
\]

We define \( \mathcal{L} : H^2 \to L^2 \times \mathbb{R}^2 \)
\[
\mathcal{L} x = \begin{bmatrix}
\mathcal{A} x \\
B_1 x \\
B_2 x
\end{bmatrix},
\]
where \( \mathcal{A} : H^2 \to L^2 \) is defined by
\[
\mathcal{A} x(t) = (p(t)x'(t))' + (q(t) + \lambda)x(t).
\]

Similarly, we define a nonlinear operator \( \mathcal{G} : H^2 \to L^2 \times \mathbb{R}^2 \) by
\[
\mathcal{G} (x) = \begin{bmatrix}
\mathcal{F}(x) \\
\eta_1(x) \\
\eta_2(x)
\end{bmatrix},
\]
where \( \mathcal{F}(x)(t) = f(x(t)) \) and, as before, for \( i = 1, 2 \), \( \eta_i(x) = \int_{[0,1]} g_i(x) d\mu_i \). Solving the nonlinear boundary value problem (1)–(2) is now equivalent to solving
\[
\mathcal{L} x = \mathcal{G}(x).
\]
(4)

The study of the nonlinear boundary value problem (1)–(2) will be intimately related to the linear nonhomogeneous boundary value problem
\[
(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = h(t), \quad t \in [0,1]
\]
(5)
\[
ax(0) + bx'(0) = w_1 \quad \text{and} \quad cx(1) + dx'(1) = w_2,
\]
(6)
where \( h \) is an element of \( L^2 \) and \( w_1 \) and \( w_2 \) are elements of \( \mathbb{R} \). Using our notation from above, we have that solving (5)–(6) is equivalent to solving
\[
\mathcal{L} x = \begin{bmatrix}
h \\
w_1 \\
w_2
\end{bmatrix}.
\]
We begin our study of the nonlinear boundary value problem (1)–(2) by analyzing (5)–(6). To aid in this analysis, we first recall some well-known facts regarding the linear homogeneous Sturm-Liouville problem

\[(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = 0\]

\[ax(0) + bx'(0) = 0 \quad \text{and} \quad cx(1) + dx'(1) = 0.\]  

(7) (8)

For those readers interested in a more detailed introduction to linear Sturm-Liouville problems, we suggest [9].

It is well known that $\lambda$ is a simple eigenvalue; that is, $\text{Ker}(\mathcal{L})$ is one-dimensional. We may therefore choose a vector, $\psi$, which forms a basis for $\text{Ker}(\mathcal{L})$. Without loss of generality, we will assume $\|\psi\|_{L^2} = 1$. Since (7) is a second-order linear homogeneous differential equation, we may choose $\phi$ satisfying (7) so that $\{\psi, \phi\}$ forms a basis for the solution space of this linear homogeneous problem. We will assume $\langle \psi, \phi \rangle_2 = 0$.

For $u, v \in H^2$, let $wr(u, v)$ denote the Wronskian of $u$ and $v$; that is, $wr(u, v) = uv' - vu'$. It follows from standard ode theory that if $u$ and $v$ are linearly independent solutions to (7), then $p \cdot wr(u, v)$ is a nonzero constant. We will assume that $\phi$ has been chosen so that $p \cdot wr(\psi, \phi) = 1$ and define $\omega : [0, 1] \times [0, 1] \to \mathbb{R}$ by

\[\omega(t, s) = \begin{cases} 
\psi(t)\phi(s) & \text{if } 0 \leq t \leq s \leq 1 \\
\psi(s)\phi(t) & \text{if } 0 \leq s \leq t \leq 1.
\end{cases}\]

As a reminder to the reader, $\omega$ is often referred to as a fundamental solution of (7).

If we define $K : L^2 \to H^2$ by

\[Kh(t) = \int_0^1 \omega(t, s)h(s)ds,\]

(10)

then it is easy to verify that $K$ is self-adjoint, compact, and satisfies $\mathcal{A}Kh = h$ for every $h \in L^2$. Differentiating under the integral symbol, one easily establishes that for every $h \in L^2$, $B_1Kh = \langle h, \phi \rangle_2B_1\psi = 0$ and $B_2Kh = \langle h, \psi \rangle_2B_2\phi$. Let

\[v_1 = B_1\phi \quad \text{and} \quad v_2 = B_2\phi.\]

Since $\phi$ satisfies (7) and is linearly independent of $\psi$, we must have $B_1\phi \neq 0$ and $B_2\phi \neq 0$; this is a consequence of the uniqueness of solutions to initial value problems and that fact the linear Sturm-Liouville boundary conditions can be thought of as an orthogonality condition.

With the above ideas in hand, we are now in a position characterize the range of $\mathcal{L}$. We have the following result.

**Proposition 2.1.** Let $h \in L^2$ and $w_1, w_2 \in \mathbb{R}$. Then $\vec{h} := \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix} \in \text{Im}(\mathcal{L})$ if and only if $\langle \vec{h}, \vec{\psi} \rangle = 0$, where $\vec{\psi} := \begin{bmatrix} \psi \\ v_1^{-1} \\ -v_2^{-1} \end{bmatrix}$. That is, in $L^2 \times \mathbb{R}^2$, $\text{Im}(\mathcal{L}) = \{\vec{\psi}\}^\perp$. 

Proof. \( \mathcal{L}x = \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix} \) if and only if \( \mathcal{A}x = h, B_1x = w_1, \) and \( B_2x = w_2. \) However, \( \mathcal{A}x = h \) if and only if \( x = c_1\psi + c_2\phi + Kh, \) for some real numbers \( c_1, c_2. \) Applying the boundary map \( B_1 \) and recalling \( B_1Kh = 0, \) we get \( B_1(c_1\psi + c_2\phi + Kh) = c_2v_1. \) Similarly, using \( B_2Kh = \langle h, \psi \rangle_2B_2\phi, \) we get \( B_2(c_1\psi + c_2\phi + Kh) = (c_2 + \langle h, \psi \rangle_2)v_2. \)

Now,

\[
c_2v_1 = w_1 \quad \text{and} \quad (c_2 + \langle h, \psi \rangle_2)v_2 = w_2
\]

if and only if

\[
c_2 = \frac{w_1}{v_1} \quad \text{and} \quad \langle h, \psi \rangle_2 = \frac{w_2}{v_2} - \frac{w_1}{v_1} = \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} -v_1^{-1} \\ v_2^{-1} \end{bmatrix} \right\rangle_\mathbb{R},
\]

which happens if and only if \( \langle \tilde{h}, \tilde{\psi} \rangle = 0. \) \( \square \)

With this characterization of the \( \text{Im}(\mathcal{L}) \) in hand, we make the following definitions which will play a crucial role in our ability to analyze the nonlinear Sturm-Liouville problem, \((1)\)–\((2), \) using a projection scheme.

**Definition 2.2.** Define \( P : L^2 \rightarrow L^2 \) by \( Px = \langle x, \psi \rangle_2\psi. \)

It is clear that \( P \) is the orthogonal projection onto \( \text{Ker}(\mathcal{L}) \).

Now, choose \( m, \) see \((3), \) to be \( m = \frac{1}{1 + \left| \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right|^2}. \) With this choice of \( m, \) \( \bar{\psi} \) is a unit vector in \( L^2 \times \mathbb{R}^2. \)

**Definition 2.3.** Define \( Q : L^2 \times \mathbb{R}^2 \rightarrow L^2 \times \mathbb{R}^2 \) by

\[
Q \left( \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix} \right) = \left\langle \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}, \bar{\psi} \right\rangle \bar{\psi}.
\]

From Proposition 2.1, we have that \( Q \) is the orthogonal projection of \( L^2 \times \mathbb{R}^2 \) on \( \text{Im}(\mathcal{L})^\perp. \) Thus, \( I - Q, \) is a projection onto the \( \text{Im}(\mathcal{L}). \)

In our analysis of the nonlinear Sturm-Liouville problem we will use a projection scheme often referred to as the Lyapunov-Schmidt procedure. The use of the Lyapunov-Schmidt reduction will allow us to write the operator equation \((4)\) as an equivalent equation in which a fixed point argument may be applied to prove the existence of solutions. Interested readers may consult \([3, 16]\) for a more detailed account of these ideas.

**Proposition 2.4.** Solving \( \mathcal{L}x = \mathcal{G}(x) \) is equivalent to solving the system

\[
(I - P)x - M(I - Q)\mathcal{G}(x) = 0
\]

and

\[
\left( \langle \mathcal{F}(x), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} -v_1^{-1} \\ v_2^{-1} \end{bmatrix} \right\rangle_\mathbb{R} \right) \psi = 0,
\]
where \( M \) denotes \( \left( L_{H^2 \cap \text{Ker}(\mathcal{D})^\perp} \right)^{-1} \).

Proof.

\[
\begin{align*}
\mathcal{L}x = \mathcal{G}(x) &\iff \\
&\iff \\
&\iff \\
&\iff \\
&\iff \\
&\iff \\
&\iff \\
&\iff \\
\end{align*}
\]

3. Main results

We now come to our main result. In what follows, we will assume that the non-linear integral boundary operators \( \eta_1 \) and \( \eta_2 \) are induced by bounded continuous functions \( g_1 \) and \( g_2 \).

To simplify the statement of the theorem, we introduce the following notation. For \( i = 1, 2 \), we let

\[
g_{i,+}(+\infty) := \limsup_{x \to +\infty} g_i(x),
\]
\[ g_i,(-\infty) := \liminf_{x \to -\infty} g_i(x), \]
\[ g_i,(+\infty) := \limsup_{x \to +\infty} g_i(x), \]
and
\[ g_i,(-\infty) := \liminf_{x \to -\infty} \sigma_i(x). \]

We define \( \Theta := \{ t \mid \psi(t) = 0 \} \), \( \Theta^+ := \{ t \mid \psi(t) > 0 \} \), and \( \Theta^- := \{ t \mid \psi(t) < 0 \} \). From Standard Sturm-Liouville theory, we have that \( \Theta_0 \) is a finite set consisting of simple zeros. In what follows, this fact will be used several times, possibly without explicit mention. Finally, for \( i = 1, 2 \), we let
\[ J_i,\pm = g_i,\pm(\pm\infty)\mu_i(\Theta^+) + g_i,\pm(-\infty)\mu_i(\Theta^-). \]

**Theorem 3.1.** Suppose that the following conditions hold:

**C1.** The function \( f \) is “sublinear”; that is, there exists real numbers \( M_1, M_2 \) and \( \beta \), with \( 0 \leq \beta < 1 \), such that for every \( x \in \mathbb{R} \), \( |f(x)| \leq M_1|x|^\beta + M_2 \);

**C2.** There exist positive real numbers \( \hat{z} \) and \( J \) such that for all \( z > \hat{z} \),
\[ f(-z) \leq -J < 0 < J \leq f(z); \]

**C3.** For \( i = 1, 2 \), \( \mu_i(\Theta_0) = 0 \), where again \( u_i \) is the Borel measure in the definition of the boundary operator \( \eta_i \);

**C4.** \(-J \int_0^1 |\psi| dt < \left( \begin{bmatrix} J_{1,\text{sgn}(-v_1)} & v_1^{-1} \\ J_{2,\text{sgn}(v_2)} & -v_2^{-1} \end{bmatrix} \right) \), where for a real number, \( v \), \( \text{sgn}(v) = + \) if \( v > 0 \) and \( \text{sgn}(v) = - \) if \( v < 0 \);

then, there exists a solution to (1)–(2).

**Proof.** We start by defining \( T : L^2 \to H^2 \) by
\[ T(x) = Px - (\langle \mathcal{F}(x), \psi \rangle_2 + \left( \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix} \right)_2 + M(I - Q)\mathcal{F}(x). \]

From Proposition 2.4, we have that the solutions to (1)–(2) are the fixed points of \( T \). Since \( M \) is an integral mapping from \( L^2 \) into \( H^2 \), it is compact, and thus so is \( T \). We will show that
\[ FP := \{ x \in H^2 \mid x = \gamma T(x) \text{ for some } \gamma \in (0, 1) \} \]
is a priori bounded in \( L^2 \). A fixed point will then follow from an application of Schaefer’s fixed point theorem.

To this end, suppose that there exist sequences \( \{x_n\}_{n \in \mathbb{N}} \) and \( \{\gamma_n\}_{n \in \mathbb{N}} \) in \( H^2 \) and \( (0, 1) \), respectively, with \( \|x_n\|_{L^2} \to \infty \) and \( x_n = \gamma_n T(x_n) \). Let \( y_n = \frac{x_n}{\|x_n\|_{H^2}} \). Since
the closed unit ball in the Sobelov space \( H^2 \) is weakly compact, by going to a subsequence if necessary, we may assume that \( y_n \xrightarrow{S} y \), for some \( y \in H^2 \). Again, going to a subsequence if necessary, we may assume that \( y_n \) converges to some \( y \in [0, 1] \).

Now,
\[
y_n = \frac{x_n}{\|x_n\|_{H^2}} = \gamma_n \frac{T(x_n)}{\|x_n\|_{H^2}}
\]

\[
P_{x_n} - \left( \langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \right) \psi + M(I - Q)\mathcal{G}(x_n)
\]

\[
= \gamma_n \frac{P_{x_n} - \left( \langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \right) \psi + M(I - Q)\mathcal{G}(x_n)}{\|x_n\|_{H^2}}.
\]

Since \( f \) is sublinear (See C1) and \( g_1 \) and \( g_2 \) are bounded, it follows that
\[
\|\mathcal{G}(x)\|_{L^2 \times \mathbb{R}^2} \leq K_1 \|x\|_{L^2} + K_2,
\]
and
\[
\|\mathcal{G}(x)\|_{C \times \mathbb{R}^2} \leq K_1 \|x\|_{C} + K_2,
\]
for some positive real numbers \( K_1 \) and \( K_2 \) and every \( x \in H^2 \). Thus, from (11),
\[
\gamma_n \frac{\langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \psi + M(I - Q)\mathcal{G}(x_n)}{\|x_n\|_{H^2}} \xrightarrow{2} 0,
\]
so that
\[
P_{x_n} - \left( \langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \right) \psi + M(I - Q)\mathcal{G}(x_n)
\]

\[
= \gamma_n \frac{P_{x_n} - \left( \langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \right) \psi + M(I - Q)\mathcal{G}(x_n)}{\|x_n\|_{H^2}} \xrightarrow{2} \gamma Py.
\]

Since \( y_n \xrightarrow{S} y \), \( y_n \xrightarrow{2} y \), so that we conclude \( y = \gamma Py \). Applying \( P \) gives
\[
P_{y} = \gamma P^2 y = \gamma Py,
\]
from which we deduce that \( \gamma = 1 \) or \( Py = 0 \). Since \( \|y\|_{H^2} = 1 \), it follows that \( \gamma = 1 \).

Thus, \( Py = y \) and we deduce that \( y = \pm \frac{1}{\|\psi\|_{H^2}} \psi \). We will assume that \( y = \frac{1}{\|\psi\|_{H^2}} \psi \), as the other case is similar.

Now, by the compact embedding of \( H^2 \) in \( C \), we have, since \( y_n \xrightarrow{S} y \), that \( y_n \rightarrow y \) in \( C \). Using the fact that \( y_n \xrightarrow{2} \frac{1}{\|\psi\|_{H^2}} \psi \), we have that
\[
\langle y_n, \psi \rangle_2 \rightarrow \frac{1}{\|\psi\|_{H^2}} \langle \psi, \psi \rangle_2 = \frac{1}{\|\psi\|_{H^2}}.
\]
However, \( \langle x_n, \psi \rangle_2 \neq \|x_n\|_{H^2} \langle y_n, \psi \rangle_2 \), so that \( \langle x_n, \psi \rangle_2 \rightarrow \infty \), since \( \|x_n\|_{H^2} \rightarrow \infty \). Without loss of generality, we will assume from now on that \( \langle x_n, \psi \rangle_2 > 0 \) for each \( n \).

From \( x_n = \gamma_n T(x_n) \), it follows that for each \( n \)

\[
(I - P)x_n = \gamma_n M(I - Q)\mathcal{G}(x_n)
\]

and

\[
Px_n = \gamma_n Px_n - \gamma_n \left( \langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \eta_1(x_n), \eta_2(x_n) \right\rangle R \psi \right).
\]

This is equivalent to

\[
(I - P)x_n = \gamma_n M(I - Q)\mathcal{G}(x_n) \tag{14}
\]

and

\[
(1 - \gamma_n)\langle x_n, \psi \rangle_2 + \gamma_n \left( \langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \eta_1(x_n), \eta_2(x_n) \right\rangle R \psi \right) = 0. \tag{15}
\]

Let \( v_n \) denote \( (I - P)x_n \). From (12) and (14), we have that

\[
\|v_n\|_C \leq \gamma_n \|M(I - Q)\| (K_1 \|x_n\|_C + K_2) \leq D_1 \|x_n\|_C + D_2,
\]

where \( \|M(I - Q)\| \) denotes the operator norm of \( M(I - Q) \) and for \( i = 1, 2 \), \( D_i = \|M(I - Q)\| K_i \). Applying the compact embedding theorem again, we may assume, by scaling each \( D_i \), that

\[
\|v_n\|_C \leq D_1 \|x_n\|_{H^2} + D_2.
\]

However, from (13) we have that \( \frac{\langle x_n, \psi \rangle_2}{\|x_n\|_{H^2}} \rightarrow \frac{1}{\|\psi\|_{H^2}} \), so that by rescaling one more time, we may assume

\[
\|v_n\|_C \leq D_1 \langle x_n, \psi \rangle_2 + D_2. \tag{16}
\]

For the moment, fix \( t \in \mathcal{O}_+ \cup \mathcal{O}_- \). Since

\[
|x_n(t)| \geq \langle x_n, \psi \rangle_2 |\psi(t)| - |v_n(t)| \\
\geq \langle x_n, \psi \rangle_2 |\psi(t)| - \|v_n\|_C,
\]

we have, using (16), that

\[
\lim_{n \to \infty} x_n(t) = \pm \infty, \text{ whenever } t \in \mathcal{O}_\pm. \tag{17}
\]

Define

\[
E_n = \{ t \mid \|\psi(t)\| \geq \varepsilon_n \},
\]
where $\varepsilon_n = \frac{\hat{z} + \|v_n\|_C}{\langle x_n, \psi \rangle_2}$. If $t \in E_n$, then

$$|x_n(t)| \geq \langle x_n, \psi \rangle_2 |\psi(t)| - |v_n(t)| \geq \langle x_n, \psi \rangle_2 |\psi(t)| - \|v_n\|_C,$$

$$\geq \langle x_n, \psi \rangle_2 \left( \frac{\hat{z} + \|v_n\|_C}{\langle x_n, \psi \rangle_2} \right) - \|v_n\|_C = \hat{z}.$$

This gives, using C2, that

$$\int_0^1 f(x_n)\psi dt = \int_{E_n} f(x_n)\psi dt + \int_{E_n^c} f(x_n)\psi dt \geq J \int_{E_n} |\psi| dt + \int_{E_n^c} f(x_n)\psi dt \geq J \int_{E_n} |\psi| dt - \int_{E_n^c} |f(x_n)\psi| dt.$$

We claim that $\int_{E_n^c} |f(x_n)\psi| dt \to 0$, so that by Lebesgue’s Dominated Convergence Theorem,

$$\liminf_{n \to \infty} \int_0^1 f(x_n)\psi dt \geq \liminf_{n \to \infty} J \int_{E_n} |\psi| dt = J \int_0^1 |\psi| dt. \quad (18)$$

To see that $\int_{E_n^c} |f(x_n)\psi| dt \to 0$, first note that for any $t \in E_n^c$

$$|x_n(t)| \leq \langle x_n, \psi \rangle_2 e_n + \|v_n\|_C \leq \langle x_n, \psi \rangle_2 \left( \frac{\hat{z} + \|v_n\|_C}{\langle x_n, \psi \rangle_2} \right) + \|v_n\|_C = \hat{z} + 2\|v_n\|_C \leq \hat{z} + 2(D_1\langle x_n, \psi \rangle_2^\beta + D_2) \quad (\text{using } (16)).$$

It then follows, from C1, that

$$|f(x_n)(t)| \leq M_1|x_n(t)|^\beta + M_2 \leq M_1(\hat{z} + 2(D_1\langle x_n, \psi \rangle_2^\beta + D_2))^\beta + M_2,$$

which gives that

$$\int_{E_n^c} |f(x_n)\psi| dt \leq (M_1(\hat{z} + 2(D_1\langle x_n, \psi \rangle_2^\beta + D_2))^\beta + M_2)E_n\mu_L(E_n^c),$$
where $\mu_L$ denotes Lebesgue measure on $[0, 1]$.

Since $\frac{\|v_n\|_C}{\langle x_n, \psi \rangle_2} \to 0$, we have that $E_n^c \to \emptyset_0$. Further, since $\emptyset_0$ consists of finitely many simple zeros, it follows from the Mean Value Theorem that there exists a positive constant, say $L$, with

$$\mu_L(E_n^c) \leq L \varepsilon_n.$$  

We then have that

$$\int_{E_n^c} |f(x_n)\psi| dt \leq (M_1(\hat{z} + 2(D_1\langle x_n, \psi \rangle_2^β + D_2))^β + M_2)L\varepsilon_n^2$$

$$= (M_1(\hat{z} + 2(D_1\langle x_n, \psi \rangle_2^β + D_2))^β + M_2)L\left(\frac{\hat{z} + \|v_n\|_C}{\langle x_n, \psi \rangle_2}\right)^2$$

$$\leq (M_1(\hat{z} + 2(D_1\langle x_n, \psi \rangle_2^β + D_2))^β + M_2)L\left(\frac{\hat{z} + D_1\langle x_n, \psi \rangle_2^β + D_2}{\langle x_n, \psi \rangle_2}\right)^2,$$

so that

$$\int_{E_n^c} |f(x_n)\psi| dt \leq R\frac{\langle x_n, \psi \rangle_2^{2β^2}}{\langle x_n, \psi \rangle_2^2},$$

for some positive constant $R$. Letting $n \to \infty$, and using the fact that $β < 1$, we conclude that

$$\int_{E_n^c} |f(x_n)\psi| dt \to 0.$$  

We now look to analyze $\liminf_{n \to \infty} \int_0^1 g_i(x_n) d\mu_i$ and $\limsup_{n \to \infty} \int_0^1 g_i(x_n) d\mu_i$, for $i = 1, 2$. From (17), if $t \in \emptyset_+$, then

$$g_{i,-}(+∞) \leq \liminf_{n \to \infty} g_i(x_n)(t) \quad \text{and} \quad \limsup_{n \to \infty} g_i(x_n)(t) \leq g_{i,+}(+∞).$$

Similarly, for each $t \in \emptyset_-$ and each $i, i = 1, 2,$

$$g_{i,-}(−∞) \leq \liminf_{n \to \infty} g_i(x_n)(t) \quad \text{and} \quad \limsup_{n \to \infty} g_i(x_n)(t) \leq g_{i,+}(−∞).$$

Since $g_1$ and $g_2$ are bounded, we have, by Fatou’s lemma, that for each $i$,

$$J_{i,-} = g_{i,-}(+∞)\mu_i(\emptyset_+) + g_{i,-}(−∞)\mu_i(\emptyset_-)$$

$$= \int_{\emptyset_+} g_{i,-}(+∞)d\mu_i + \int_{\emptyset_-} g_{i,-}(−∞)d\mu_i$$

$$\leq \liminf_{n \to \infty} g_i(x_n)d\mu_i$$

$$= \int_{[0,1]} \liminf_{n \to \infty} g_i(x_n)d\mu_i \quad \text{(using C3)}$$

$$\leq \liminf_{n \to \infty} \int_{[0,1]} g_i(x_n)d\mu_i$$

$$\leq \limsup_{n \to \infty} \int_{[0,1]} g_i(x_n)d\mu_i.$$
must be a priori bounded, and the proof is complete.

Suppose for the moment that \( v_1 > 0 \) and \( -v_2 > 0 \) and let \( s \) and \( r \) be positive real numbers. Using the definitions of limit inferior and limit superior, see (18) and (19), there exists an \( n_s \) and an \( n_r \) such that if \( n \geq n_s \), then

\[
\int_0^1 |\psi| dt - s < \langle f(x_n), \psi \rangle_2 < \langle \mathcal{F}(x_n), \psi \rangle_2,
\]

and if \( n \geq n_r \), then

\[
J_{i,-} - r < \int_{[0,1]} g_i(x_n) d\mu_i < J_{i,+} + r.
\]

Since \( v_1 > 0 \) and \( -v_2 > 0 \), it follows that

\[
\left\langle \begin{bmatrix} J_{1,-} - r & v_1^{-1} \\ J_{2,-} - r & -v_2^{-1} \end{bmatrix} \right\rangle \leq \left\langle \int_{[0,1]} g_1(x_n) d\mu_1 \right\rangle \leq \left\langle \int_{[0,1]} g_2(x_n) d\mu_2 \right\rangle \leq \left\langle \begin{bmatrix} J_{1,+} + r & v_1^{-1} \\ J_{2,+} + r & -v_2^{-1} \end{bmatrix} \right\rangle.
\]

However,

\[
\left\langle \begin{bmatrix} J_{1,-} & v_1^{-1} \\ J_{2,-} & -v_2^{-1} \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} J_{1,\text{sgn}(v_1)} & v_1^{-1} \\ J_{2,\text{sgn}(v_2)} & -v_2^{-1} \end{bmatrix} \right\rangle > -J \int_0^1 |\psi| dt.
\]

Thus, it follows, from (20), (21), (22), and (23), that we may choose \( r \) and \( s \) small enough so that

\[
\langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle > 0,
\]

for large enough \( n \). The other cases for the sign of \( v_1 \) and \( -v_2 \) are similar. In each case, the conclusion in (24) holds. Recalling that \( \langle x_n, \psi \rangle_2 \to +\infty \), we have that for large enough \( n \),

\[
(1 - \gamma_n) \langle x_n, \psi \rangle_2 + \gamma_n \left( \langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \right) > 0.
\]

However, this contradicts the fact that by (15),

\[
(1 - \gamma_n) \langle x_n, \psi \rangle_2 + \gamma_n \left( \langle \mathcal{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle \right) = 0.
\]

Thus,

\[
FP := \{ x \in H^2 \mid x = \gamma T(x) \text{ for some } \gamma \in (0,1) \}
\]

must be a priori bounded, and the proof is complete. \( \square \)
Remark 3.2. If $\eta_1 = \eta_2 = 0$, then by choosing for each $i$, $i = 1, 2$, $g_i = 0$ and $\mu_i$ to be Lebesgue measure on $[0, 1]$, we have that $J_{i, \pm} = 0$. Thus, condition C4 of Theorem 3.1 is trivially satisfied. This shows that Theorem 3.1 is a generalization of the result found in [15], where they analyze linear homogeneous boundary conditions.

The following corollary isolates the special case in which the boundary operators $\eta_1$ and $\eta_2$ are generated by bounded continuous function $g_1$ and $g_2$ for which we assume that for $i = 1, 2$, $g_i(\pm \infty) := \lim_{x \to \pm \infty} g_i(x)$ exists.

Corollary 3.3. Suppose that the following conditions hold:

$C1^*$. The function $f$ is “sublinear”; that is, there exists real numbers $M_1, M_2$ and $\beta$, with $0 \leq \beta < 1$, such that for every $x \in \mathbb{R}$, $|f(x)| \leq M_1 |x|^\beta + M_2$;

$C2^*$. There exist positive real numbers $\hat{z}$ and $J$ such that for all $z > \hat{z}$,

$$f(-z) \leq -J < 0 < J \leq f(z);$$

$C3^*$. For $i = 1, 2$, $\mu_i(\emptyset_0) = 0$, where again $u_i$ is the Borel measure in the definition of the boundary operator $\eta_i$;

$C4^*$. For $i = 1, 2$, $g_i(\pm \infty) := \lim_{x \to \pm \infty} g_i(x)$ exists;

$C5^*$. $-J \int_0^1 |\psi| dt < \left< \begin{bmatrix} J_{1, +} \\ J_{2, +} \end{bmatrix}, \begin{bmatrix} v_{1, -}^{-1} \\ -v_{2, -}^{-1} \end{bmatrix} \right>_{\mathbb{R}}$;

then, there exists a solution to (1)–(2).

Proof. If for $i = 1, 2$, $g_i(\pm \infty) := \lim_{x \to \pm \infty} g_i(x)$ exist, then for each of these $i$, $J_{i, -} = J_{i, +}$.

4. Example

In this section we give a concrete example of the application of our main result, Theorem 3.1. We will use an interval of $[0, \pi]$ to simplify calculations.

Consider

$$x'' + m^2 x = f(x(t))$$

subject to

$$x(0) = \int_{[0, \pi]} g_1(x) du_1 \text{ and } x(\pi) = \int_{[0, \pi]} g_2(x) du_2$$

where $f$, $g_1$, and $g_2$ are real-valued continuous functions with $g_1$ and $g_2$ bounded.

It is well-known that the $L^2$-normalized eigenfunctions corresponding to the Dirichlet problem

$$x'' + m^2 x = 0$$
subject to boundary conditions

\[ x(0) = 0 \text{ and } x(\pi) = 0, \]

are \( \pm \frac{2}{\pi} \sin(mt) \). We choose to take \( \psi(t) = \frac{2}{\pi} \sin(mt) \). This gives that \( \phi \), see (9), is \( -\frac{\pi}{2} \cos(mt) \). Thus, \( v_1 = \phi(0) = -\frac{\pi}{2} \) and \( v_2 = \phi(\pi) = \frac{\pi}{2} \). We also have that

\[
\mathcal{O}_+ = \begin{cases} \bigcup_{i=0}^{j} \left( \frac{2i\pi}{m}, \frac{(2i+1)\pi}{m} \right) & \text{if } m = 2j + 1 \\ \bigcup_{i=0}^{j} \left( \frac{2i\pi}{m}, \frac{(2i+1)\pi}{m} \right) & \text{if } m = 2j \end{cases}
\]

and

\[
\mathcal{O}_- = \begin{cases} \bigcup_{i=0}^{j} \left( \frac{(2i+1)\pi}{m}, \frac{(2i+2)\pi}{m} \right) & \text{if } m = 2j + 1 \\ \bigcup_{i=0}^{j} \left( \frac{(2i+1)\pi}{m}, \frac{(2i+2)\pi}{m} \right) & \text{if } m = 2j \end{cases}
\]

Suppose for the moment that conditions C1-C3 hold, since these can be trivially satisfied by any number of choices for \( f \) and \( \mu_1, \mu_2 \). Condition C4 of Theorem 3.1 in this specific problem becomes

\[
-\frac{4}{\pi} J < \left\langle \begin{bmatrix} J_{1,+} \\ J_{2,+} \end{bmatrix}, \begin{bmatrix} -\frac{2}{\pi} \\ -\frac{2}{\pi} \end{bmatrix} \right\rangle, \]

which is equivalent to \( (J_{1,+} + J_{2,+}) < 2J \). It is clear that there are several bounded continuous functions \( g_1, g_2 \) and Borel measures \( \mu_1, \mu_2 \) which make the above inequality valid.

As a concrete example, let \( E_m = \{ t \mid \sin(mt) = 0 \} \) and fix \( t_0 \notin E_m \). Take \( \mu := \mu_1 = \mu_2 \) to be the measure point-supported at \( t_0 \); that is, for a subset \( A \) of \([0,1]\),

\[
\mu(A) = \begin{cases} 1 & \text{if } t_0 \in A \\ 0 & \text{if } t_0 \notin A \end{cases}
\]

Since \( t_0 \notin E_m \), we have that \( t_0 \) is in \( \mathcal{O}_+ \) or \( \mathcal{O}_- \). If for each \( i, i = 1,2 \), \( g_i(\pm\infty) := \lim_{x \to \pm\infty} g_i(x) \) exists, then when \( t \in \mathcal{O}_+, J_{i,+} = g_i(\infty) \). Similarly, when \( t \in \mathcal{O}_- \), then \( J_{i,+} = g_i(-\infty) \). Thus, if \( t_0 \in \mathcal{O}_\pm \), then provided \( g_1(\pm\infty) + g_2(\pm\infty) < 2J \), we have, from Corollary 3.3, that the nonlinear boundary value problem (25)–(26) has a solution. It is interesting to note that if \( t_0 \notin \bigcup_mE_m \), and both \( g_1(\infty) + g_2(\infty) < 2J \) and \( g_1(-\infty) + g_2(-\infty) < 2J \), then (25)–(26) has a solution for all eigenvalues \( m \).

REFERENCES


(Received May 24, 2017)