

## VANISHING MAGNETIC FIELD LIMITS OF SOLUTIONS TO THE PRESSURELESS MAGNETOGASDYNAMICS

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*Abstract.* We are interested in the system of conservation laws modeling the pressureless magnetogasdynamics. Firstly, we solve the Riemann problem and obtain five kinds of structures consisting of combinations of shocks, rarefaction waves and contact discontinuities. Secondly, we study the vanishing magnetic field limits of the Riemann solutions to the pressureless magnetogasdynamics and show that the density and velocity in the Riemann solutions to the pressureless magnetogasdynamics converge to the Riemann solutions to the pressureless gas dynamics. The formation processes of delta-shocks and vacuum states are discussed in details.

### 1. Introduction

Magnetogasdynamics has been the subject of great interest from both mathematical and physical point of view due to its applications in the variety of fields. The partial differential equations which govern the continuous motion of a perfectly conducting gas in one space dimension in the presence of transverse magnetic field with infinite electrical conductivity may be written in the form [26]

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho(u_t + uu_x) + (p + b^2/2)_x = 0, \\ p_t + up_x + \rho c^2 u_x = 0, \\ b_t + (bu)_x = 0, \end{cases} \quad (1.1)$$

where  $\rho \geq 0$  and  $u$  represent the density and velocity respectively,  $p$  the pressure,  $b \geq 0$  the transverse magnetic field, and  $c = \sqrt{\partial p / \partial \rho}$  the speed of sound. Raja Sekhar and Sharma [27] studied the Riemann problem and elementary wave interactions for (1.1) with  $p = k_1 \rho^\gamma$  ( $1 < \gamma \leq 2$ ) and  $b = k_2 \rho$ , where  $k_1$  and  $k_2$  are positive constants, and  $\gamma$  is the adiabatic constant. Liu and Sun [24] studied the Riemann problem and wave interactions for (1.1) only with  $p = k_1 \rho^\gamma$  ( $1 < \gamma \leq 2$ ). Raja Sekhar and Sharma [26] solved approximately the Riemann problem for (1.1) with  $p$  describing the ideal

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nonisentropic polytropic gas and  $b = k_2\rho$ . Also see the papers [17, 18, 19, 13] for investigations with respect to (1.1).

It is well known that the motion of fluid particles is determined by some kinds of effects, such as the effect of inertia and the effect of pressure difference, etc. For some situations, the effect of pressure difference may be so small as to be negligible. The well-known pressureless Euler equations of gas dynamics have been obtained just by neglecting the effect of pressure difference in the Euler equations of gas dynamics. In this paper, we impose the pressure  $p = 0$  on (1.1) and propose the following pressureless magnetogasdynamics flow in the conservative form

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_x + (\rho u^2 + \kappa^2 B^2/2)_x = 0, \\ (\kappa B)_t + (\kappa B u)_x = 0, \end{cases} \quad (1.2)$$

where we have replaced the magnetic field  $b$  by  $\kappa B$  with a scaling parameter  $\kappa > 0$  modeling the strength of transverse magnetic field.

The first task of this paper is to solve the Riemann problem for (1.2). It has three characteristics, in which two are genuinely nonlinear, and the other is linearly degenerate, thus the classical basic waves contain shocks, rarefaction waves and contact discontinuities. By the analysis method in state space, the Riemann solutions are constructed with five different structures: (i) a backward rarefaction wave, a contact discontinuity, and a forward rarefaction wave, (ii) a backward rarefaction wave, a contact discontinuity, and a forward shock, (iii) a backward shock, a contact discontinuity, and a forward rarefaction wave, (iv) a backward shock, a contact discontinuity, and a forward shock, (v) a backward rarefaction wave, a vacuum intermediary state, and a forward rarefaction wave.

As  $\kappa \rightarrow 0^+$ , the system (1.2) formally tends to the pressureless gas dynamics

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_x + (\rho u^2)_x = 0. \end{cases} \quad (1.3)$$

The model (1.3) has a number of origins, such as the flux-splitting numerical schemes [21, 1], adhesion particle dynamics [28, 33], pressureless isentropic gas dynamics [2, 3], and the electron-sheet evolution [25], etc. In earlier papers [30, 23], Sheng, Zhang and Li solved the 1-D and 2-D Riemann problems for (1.3) completely. A distinctive feature is that delta-shocks and vacuum states develop in solutions, and (1.3) has been one of popular systems admitting delta-shocks. A delta-shock is a kind of discontinuity on which at least one of the state variables becomes a singular measure in the form of a weighted Dirac delta function. Physically, the delta-shocks can describe the concentration of mass. With respect to delta-shocks, we also refer the readers to papers [16, 31, 32, 11, 14, 15, 22, 34, 9, 29, 7, 8] for more details. One can see that the magnetic field prevents the appearance of delta-shocks in solutions to (1.2).

The second task of this paper is to study the limits of the Riemann solutions to (1.2) as the magnetic field vanishes (i.e.,  $\kappa \rightarrow 0^+$ ). We rigorously prove that as  $\kappa \rightarrow 0^+$ , the

limits of the density and velocity in the Riemann solution containing two shocks and possibly one contact discontinuity to (1.2) are just the delta-shock Riemann solution to (1.3). The intermediate densities between the two shocks tend to a weighted  $\delta$ -measure that forms the delta-shock. By contrast, we also show that as  $\kappa \rightarrow 0^+$ , the density and velocity in the Riemann solution containing two rarefaction waves and possibly one contact discontinuity to (1.2) tend to the vacuum Riemann solution to (1.3). The nonvacuum intermediate states between the two rarefaction waves tend to the vacuum state.

We remark that Li [20], and Chen and Liu [5, 6] studied the limits of the Riemann solutions of the isentropic and nonisentropic Euler equations as the pressure vanishes. They showed that the vanishing pressure limits of the Riemann solutions of the isentropic and nonisentropic Euler equations are just the Riemann solutions of the pressureless gas dynamics. Especially, the phenomena of concentration and cavitation and the formation of delta-shocks and vacuum states in the limits were identified and analyzed. Besides, the results were extended to the relativistic Euler equations by Yin and Sheng [37]. Also see [35, 36] for the researches in this area.

The rest of the article is organized as follows. In Section 2, we recall the Riemann problem for (1.3). In Section 3, we solve the Riemann problem for (1.2). In Sections 4 and 5, we discuss the limits of the Riemann solutions to (1.2) as  $\kappa \rightarrow 0^+$ .

## 2. Solutions of the Riemann problem for the pressureless gas dynamics

In this section, we briefly review the Riemann problem for (1.3) with initial data

$$(u, \rho)(x, t = 0) = (u_{\pm}, \rho_{\pm}), \quad \pm x > 0 \tag{2.1}$$

with  $\rho_{\pm} > 0$ . The detailed investigations can be found in [30, 23]. The eigenvalue is  $\lambda = u$  with associated eigenvector  $r = (1, 0)^T$  satisfying  $\nabla \lambda \cdot r = 0$ . So (1.3) is extremely nonstrictly hyperbolic.

Since both system (1.3) and initial data (2.1) remain invariant under a uniform expansion of coordinates  $x \rightarrow \alpha x', t \rightarrow \alpha t', \alpha > 0$ , we should look for the self-similar solution  $(u, \rho)(x, t) = (u, \rho)(\xi)$  ( $\xi = x/t$ ), then we obtain

$$\begin{cases} -\xi \rho_{\xi} + (\rho u)_{\xi} = 0, \\ -\xi (\rho u)_{\xi} + (\rho u^2)_{\xi} = 0 \end{cases} \tag{2.2}$$

and

$$(u, \rho)(\pm\infty) = (\rho_{\pm}, u_{\pm}). \tag{2.3}$$

This is a two-point boundary value problem of first-order ordinary differential equations with the boundary values at the infinity.

Besides the constant state solution

$$(u, \rho)(\xi) = \text{Constant}, \tag{2.4}$$

and the singular solution (the vacuum denoted by  $Vac$ )

$$\xi = u, \rho = 0, \tag{2.5}$$

it is easy to check that the elementary wave of (1.3) is nothing but contact discontinuity

$$\xi = u_l = u_r, \tag{2.6}$$

where the indices  $l$  and  $r$  denote left and right states respectively.

The Riemann problem can be solved by the following two cases. For the case  $u_- \leq u_+$ , the Riemann solution consists of two contact discontinuities and an intermediary vacuum state:

$$(u, \rho)(\xi) = \begin{cases} (u_-, \rho_-) & -\infty < \xi \leq u_-, \\ (\xi, 0) & u_- \leq \xi \leq u_+, \\ (u_+, \rho_+) & u_+ \leq \xi < +\infty. \end{cases} \tag{2.7}$$

However, for the case  $u_- > u_+$ , the singularity cannot be a jump with finite amplitude, that is, there is no solution which is piecewise smooth and bounded. In order to establish the existence in a space of measures from the mathematical point of view, the delta-shock should be introduced.

To define the measure solutions, the weighted  $\delta$ -measure  $w(s)\delta_L$  supported on a smooth curve  $L$  parameterized as  $x = x(s), t = t(s) (c \leq s \leq d)$  is defined by

$$\langle w(s)\delta_L, \psi(x, t) \rangle = \int_c^d w(s)\psi(x(s), t(s)) ds \tag{2.8}$$

for all test functions  $\psi(x, t) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ .

With this definition, for the case  $u_- > u_+$ , the following delta-shock Riemann solutions has been constructed in [30, 23]

$$(u, \rho)(t, x) = \begin{cases} (u_-, \rho_-), & x < \sigma t, \\ (\sigma, w(t)\delta(x - \sigma t)), & x = \sigma t, \\ (u_+, \rho_+), & x > \sigma t, \end{cases} \tag{2.9}$$

where the weight  $w(t)$  and velocity  $\sigma$  satisfy the generalized Rankine-Hugoniot relation

$$\begin{cases} \frac{dw(t)}{dt} = -\sigma[\rho] + [\rho u], \\ \frac{dw(t)\sigma}{dt} = -\sigma[\rho u] + [\rho u^2] \end{cases} \tag{2.10}$$

and the entropy condition

$$u_+ < \sigma < u_- \tag{2.11}$$

with  $[a] = a_- - a_+$  being the jump of  $a$  across the discontinuities. Under (2.11), solving (2.10) with initial data  $w(0) = 0$  gives

$$w(t) = \sqrt{\rho_- \rho_+} (u_- - u_+) t, \quad \sigma = \frac{\sqrt{\rho_-} u_- + \sqrt{\rho_+} u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}. \tag{2.12}$$

**THEOREM 2.1.** *The Riemann problem for the pressureless gas dynamics (1.3) with initial data (2.1) admits a unique entropy solution, which includes a vacuum state when  $u_- \leq u_+$  and a delta-shock when  $u_- > u_+$ .*

### 3. Solutions of the Riemann problem for the pressureless magnetogasdynamics

In this section, we solve the Riemann problem for the pressureless magnetogasdynamics (1.2) with initial data

$$(u, \rho, B)(x, t = 0) = (u_{\pm}, \rho_{\pm}, B_{\pm}), \quad \pm x > 0 \tag{3.1}$$

with  $\rho_{\pm} > 0$ , and examine the dependence of the Riemann solutions on the parameter  $\kappa > 0$ .

#### 3.1. Hyperbolicity and characteristics

For any smooth solution, system (1.2) can be rewritten as

$$\begin{pmatrix} \rho \\ u \\ B \end{pmatrix}_t + \begin{pmatrix} u \rho & 0 \\ 0 & u \frac{\kappa^2 B}{\rho} \\ 0 & B & u \end{pmatrix} \begin{pmatrix} \rho \\ u \\ B \end{pmatrix}_x = 0 \tag{3.2}$$

with the characteristic equation  $(u - \lambda) [(u - \lambda)^2 - (\kappa^2 B^2) / \rho] = 0$ . It defines the eigenvalues

$$\lambda_0^{\kappa} = u, \quad \lambda_{\pm}^{\kappa} = u \pm \frac{\kappa B}{\sqrt{\rho}}$$

with corresponding right eigenvectors

$$r_0^{\kappa} = (1, 0, 0)^T, \quad r_{\pm}^{\kappa} = \left( \pm \rho, \frac{\kappa B}{\sqrt{\rho}}, \pm B \right)^T.$$

Then it follows

$$\nabla \lambda_0^{\kappa} \cdot r_0^{\kappa} \equiv 0, \quad \nabla \lambda_{\pm}^{\kappa} \cdot r_{\pm}^{\kappa} = \frac{3}{2} \cdot \frac{\kappa B}{\sqrt{\rho}}.$$

The three eigenvalues are real and distinct, therefore the system is strictly hyperbolic. It can also be seen that the characteristic field  $\lambda_0^{\kappa}$  is linearly degenerate and the characteristic fields  $\lambda_{\pm}^{\kappa}$  are genuinely nonlinear.

#### 3.2. Elementary waves

As usual, we look for the self-similar solution

$$(u, \rho, B)(x, t) = (u, \rho, B)(\xi), \quad \xi = x/t,$$

then we obtain the two-point boundary value problem:

$$\begin{cases} -\xi \rho_\xi + (\rho u)_\xi = 0, \\ -\xi (\rho u)_\xi + (\rho u^2 + \kappa^2 B^2 / 2)_\xi = 0, \\ -\xi (\kappa B)_\xi + (\kappa B u)_\xi = 0 \end{cases} \tag{3.3}$$

and

$$(u, \rho, B)(\pm\infty) = (u_\pm, \rho_\pm, B_\pm). \tag{3.4}$$

(i) *Smooth solutions*

For any smooth solution, (3.3) becomes

$$\begin{pmatrix} u - \xi & \rho & 0 \\ 0 & \rho(u - \xi) & \kappa^2 B \\ 0 & B & u - \xi \end{pmatrix} \begin{pmatrix} \rho \\ u \\ B \end{pmatrix}_\xi = 0. \tag{3.5}$$

Besides the general solution (constant state)

$$(u, \rho, B)(\xi) = Constant, \tag{3.6}$$

(3.5) provides the vacuum state

$$\begin{cases} \xi = u, \\ \rho = B = 0; \end{cases} \tag{3.7}$$

the backward rarefaction wave, symbolized by  $\overleftarrow{R}$ ,

$$\begin{cases} \xi = \lambda_-^\kappa = u - \frac{\kappa B}{\sqrt{\rho}}, \\ \frac{d\rho}{\rho} = \frac{dB}{B}, \\ du = -\frac{\kappa B}{\rho\sqrt{\rho}} d\rho; \end{cases} \tag{3.8}$$

and the forward rarefaction wave, symbolized by  $\overrightarrow{R}$ ,

$$\begin{cases} \xi = \lambda_+^\kappa = u + \frac{\kappa B}{\sqrt{\rho}}, \\ \frac{d\rho}{\rho} = \frac{dB}{B}, \\ du = \frac{\kappa B}{\rho\sqrt{\rho}} d\rho. \end{cases} \tag{3.9}$$

For the backward rarefaction wave,

$$\frac{d\lambda_-^\kappa}{d\rho} = \frac{\partial \lambda_-^\kappa}{\partial u} \frac{du}{d\rho} + \frac{\partial \lambda_-^\kappa}{\partial \rho} + \frac{\partial \lambda_-^\kappa}{\partial B} \frac{dB}{d\rho} = -\frac{3\kappa B}{2\rho\sqrt{\rho}} < 0.$$

For the forward rarefaction wave,

$$\frac{d\lambda_+^\kappa}{d\rho} = \frac{\partial\lambda_+^\kappa}{\partial u} \frac{du}{d\rho} + \frac{\partial\lambda_+^\kappa}{\partial\rho} + \frac{\partial\lambda_+^\kappa}{\partial B} \frac{dB}{d\rho} = \frac{3\kappa B}{2\rho\sqrt{\rho}} > 0.$$

Let  $V_l = (u_l, \rho_l, B_l)$  and  $V_r = (u_r, \rho_r, B_r)$  denote the states connected by a rarefaction wave on the left and right side respectively. Then the condition  $\lambda_-^\kappa(V_r) > \lambda_-^\kappa(V_l)$  and  $\lambda_+^\kappa(V_r) > \lambda_+^\kappa(V_l)$  are required for the backward and forward rarefaction wave, respectively. Then, it follows that the backward rarefaction wave should satisfy

$$\rho_l > \rho_r, \quad u_l < u_r, \quad B_l > B_r \tag{3.10}$$

and the forward rarefaction wave should satisfy

$$\rho_l < \rho_r, \quad u_l < u_r, \quad B_l < B_r. \tag{3.11}$$

For a given left state  $V_l = (u_l, \rho_l, B_l)$ , all possible states which can connect to  $V_l$  on the right by a backward rarefaction wave must be located on the curve

$$\overleftarrow{R}(V_l) : \begin{cases} B = \mu_l \rho, \\ u = u_l - 2\kappa\mu_l(\sqrt{\rho} - \sqrt{\rho_l}), \quad \rho < \rho_l, \end{cases} \tag{3.12}$$

where  $\mu_l = B_l/\rho_l$ . It is easy to see that  $\overleftarrow{R}(V_l)$  intersects with the  $u$ -axis at  $u_0 = u_l + 2\kappa\sqrt{\mu_l B_l}$ . For a given state  $V_r = (u_r, \rho_r, B_r)$ , all possible states which can connect to  $V_r = (u_r, \rho_r, B_r)$  on the left by a forward rarefaction wave must be located on the curve

$$\overrightarrow{R}(V_r) : \begin{cases} B = \mu_r \rho, \\ u = u_r + 2\kappa\mu_r(\sqrt{\rho} - \sqrt{\rho_r}), \quad \rho < \rho_r, \end{cases} \tag{3.13}$$

where  $\mu_r = B_r/\rho_r$ . Obviously,  $\overrightarrow{R}(V_r)$  intersects with the  $u$ -axis at  $u_1 = u_r - 2\kappa\sqrt{\mu_r B_r}$ .

(ii) Discontinuities

Let us turn to the discontinuous solutions. For a bounded discontinuity at  $\xi = \sigma$ , the Rankine-Hugoniot relation reads

$$\begin{cases} -\sigma[\rho] + [\rho u] = 0, \\ -\sigma[\rho u] + [\rho u^2 + \kappa^2 B^2/2] = 0, \\ -\sigma[\kappa B] + [\kappa B u] = 0, \end{cases} \tag{3.14}$$

where  $[a] = a_l - a_r$  is the jump of  $a$  across the discontinuity,  $V_l = (u_l, \rho_l, B_l)$  and  $V_r = (u_r, \rho_r, B_r)$  are the states on the left and right side of the discontinuity. When  $\rho_l = \rho_r$ , it follows  $u_l = u_r$  and  $B_l = B_r$ . In what follows, we assume  $\rho_l \neq \rho_r$ . Eliminating  $\sigma$  in

(3.14), it follows

$$\begin{cases} (u_l - u_r)(\rho_l B_r - \rho_r B_l) = 0, \\ u_r - u_l = \pm \frac{\kappa}{\sqrt{2}} \cdot \sqrt{\frac{1}{\rho_l \rho_r} \cdot \frac{B_l^2 - B_r^2}{\rho_l - \rho_r}} (\rho_r - \rho_l). \end{cases} \quad (3.15)$$

Besides, from the first equation in (3.14), we have

$$\sigma = \frac{\rho_l u_l - \rho_r u_r}{\rho_l - \rho_r} = u_l + \frac{\rho_r (u_l - u_r)}{\rho_l - \rho_r} = u_r + \frac{\rho_l (u_l - u_r)}{\rho_l - \rho_r}. \quad (3.16)$$

Thus we have three kinds of discontinuities. The first is

$$\begin{cases} \sigma = u_l, \\ u_l = u_r, \quad B_l = B_r, \quad \rho_l \neq \rho_r, \end{cases} \quad (3.17)$$

which is a contact discontinuity associating with  $\lambda_0$ , symbolized by  $J$ . Two states  $V_l = (u_l, \rho_l, B_l)$  and  $V_r = (u_r, \rho_r, B_r)$  can be connected by a contact discontinuity if they satisfy  $u_l = u_r$ ,  $B_l = B_r$ ,  $\rho_l \neq \rho_r$ . The rest two are

$$\begin{cases} \sigma = u_l - \frac{\kappa}{\sqrt{2}} \cdot \rho_r \sqrt{\frac{1}{\rho_l \rho_r} \cdot \frac{B_l^2 - B_r^2}{\rho_l - \rho_r}} \\ \quad = u_r - \frac{\kappa}{\sqrt{2}} \cdot \rho_l \sqrt{\frac{1}{\rho_l \rho_r} \cdot \frac{B_l^2 - B_r^2}{\rho_l - \rho_r}}, \\ \frac{\rho_r}{B_r} = \frac{\rho_l}{B_l}, \\ u_r = u_l - \frac{\kappa}{\sqrt{2}} \sqrt{\frac{1}{\rho_l \rho_r} \cdot \frac{B_l^2 - B_r^2}{\rho_l - \rho_r}} (\rho_r - \rho_l), \end{cases} \quad (3.18)$$

and

$$\begin{cases} \sigma = u_l + \frac{\kappa}{\sqrt{2}} \cdot \rho_r \sqrt{\frac{1}{\rho_l \rho_r} \cdot \frac{B_l^2 - B_r^2}{\rho_l - \rho_r}} \\ \quad = u_r + \frac{\kappa}{\sqrt{2}} \cdot \rho_l \sqrt{\frac{1}{\rho_l \rho_r} \cdot \frac{B_l^2 - B_r^2}{\rho_l - \rho_r}}, \\ \frac{\rho_r}{B_r} = \frac{\rho_l}{B_l}, \\ u_r = u_l + \frac{\kappa}{\sqrt{2}} \cdot \sqrt{\frac{1}{\rho_l \rho_r} \cdot \frac{B_l^2 - B_r^2}{\rho_l - \rho_r}} (\rho_r - \rho_l). \end{cases} \quad (3.19)$$



In order to identify the admissible solution, the discontinuity (3.18) associating with  $\lambda_-^K$  should satisfy

$$\sigma < \lambda_-^K(V_l) < \lambda_+^K(V_l), \quad \lambda_-^K(V_r) < \sigma < \lambda_+^K(V_r), \tag{3.20}$$

while the discontinuity (3.19) associating with  $\lambda_+^K$  should satisfy

$$\lambda_-^K(V_l) < \sigma < \lambda_+^K(V_l), \quad \lambda_-^K(V_r) < \lambda_+^K(V_r) < \sigma. \tag{3.21}$$

Then one can find that the inequalities (3.20) are equivalent to

$$-\rho_r B_r \sqrt{\frac{1}{\rho_r}} < -\rho_l \rho_r \sqrt{\frac{1}{\rho_l \rho_r} \cdot \frac{1}{2} \cdot \frac{B_l^2 - B_r^2}{\rho_l - \rho_r}} < -\rho_l B_l \sqrt{\frac{1}{\rho_l}}, \tag{3.22}$$

while those (3.21) are equivalent to

$$\rho_r B_r \sqrt{\frac{1}{\rho_r}} < \rho_l \rho_r \sqrt{\frac{1}{\rho_l \rho_r} \cdot \frac{1}{2} \cdot \frac{B_l^2 - B_r^2}{\rho_l - \rho_r}} < \rho_l B_l \sqrt{\frac{1}{\rho_l}}. \tag{3.23}$$

Moreover, in view of  $\rho_r/B_r = \rho_l/B_l$ , the inequalities (3.22) and (3.23) are respectively equivalent to

$$\rho_l < \rho_r, \quad u_l > u_r, \quad B_l < B_r \tag{3.24}$$

and

$$\rho_l > \rho_r, \quad u_l > u_r, \quad B_l > B_r. \tag{3.25}$$

The discontinuity (3.18) with (3.24) is called as the backward shock and symbolized by  $\overleftarrow{S}$ , and (3.19) with (3.25) is called as the forward shock and symbolized by  $\overrightarrow{S}$ .

For a given state  $V_l = (u_l, \rho_l, B_l)$ , all possible states which can connect to  $V_l$  on the right by a backward shock must be located on the curve

$$\overleftarrow{S}(V_l) : \begin{cases} B = \mu_l \rho, \\ u = u_l - \frac{\kappa}{\sqrt{2}} \cdot \mu_l \sqrt{\frac{1}{\rho} + \frac{1}{\rho_l}} (\rho - \rho_l), \quad \rho > \rho_l. \end{cases} \tag{3.26}$$

where  $\mu_l = B_l/\rho_l$ . For  $\overleftarrow{S}(V_l)$ ,  $u$  tends to the negative infinity as  $\rho$  (or  $B$ ) tends to positive infinity. For a given state  $V_r = (u_r, \rho_r, B_r)$ , all possible states which can connect to  $V_r = (u_r, \rho_r, B_r)$  on the left by a forward shock must be located on the curve

$$\overrightarrow{S}(V_r) : \begin{cases} B = \mu_r \rho, \\ u = u_r + \frac{\kappa}{\sqrt{2}} \cdot \mu_r \sqrt{\frac{1}{\rho} + \frac{1}{\rho_r}} (\rho - \rho_r), \quad \rho > \rho_r, \end{cases} \tag{3.27}$$

where  $\mu_r = B_r/\rho_r$ . For  $\overrightarrow{S}(V_r)$ ,  $u$  tends to the positive infinity as  $\rho$  (or  $B$ ) tends to positive infinity.

We denote  $\overleftarrow{W}(V_l) = \overleftarrow{R}(V_l) \cup \overleftarrow{S}(V_l)$  and  $\overrightarrow{W}(V_r) = \overrightarrow{R}(V_r) \cup \overrightarrow{S}(V_r)$ .

### 3.3. Construction of Riemann solutions

With the above elementary waves, we are ready to construct the solutions to the Riemann problem for (1.2) with initial data (3.1) by using the analysis method in the phase space [4, 38]. Draw the backward wave curve  $\overleftarrow{W}(V_-)$  passing the left state  $V_- = (u_-, \rho_-, B_-)$  and the forward wave curve  $\overrightarrow{W}(V_+)$  passing the right state  $V_+ = (u_+, \rho_+, B_+)$ .

When  $u_- + 2\kappa\sqrt{\mu_- B_-} < u_+ - 2\kappa\sqrt{\mu_+ B_+}$ , it is easy to see that the projections on  $(B, u)$ -plane of  $\overleftarrow{W}(V_-)$  and  $\overrightarrow{W}(V_+)$  do not interact with each other, therefore  $\overleftarrow{W}(V_-)$  and  $\overrightarrow{W}(V_+)$  will not interact with each other. Notice that  $\overleftarrow{W}(V_-)$  and  $\overrightarrow{W}(V_+)$  interact with the  $u$ -axis. At this time, the Riemann solution consists of a backward rarefaction wave, a vacuum intermediary state and a forward rarefaction wave.

When  $u_- + 2\kappa\sqrt{\mu_- B_-} \geq u_+ - 2\kappa\sqrt{\mu_+ B_+}$ , the projections on  $(B, u)$ -plane of  $\overleftarrow{W}(V_-)$  and  $\overrightarrow{W}(V_+)$  will interact with each other, and the interaction point is unique. Then  $\overleftarrow{W}(V_-)$  and  $\overrightarrow{W}(V_+)$  must have pseudo-intersection points. Here the pseudo-intersection points are the points in which the  $u$  and  $B$  coordinates are same while the  $\rho$  coordinates may be different (see Figure 3.1). The projections on  $(B, u)$ -plane of the pseudo-intersection points of  $\overleftarrow{W}(V_-)$  and  $\overrightarrow{W}(V_+)$  are just the interaction point of the projections on  $(B, u)$ -plane of  $\overleftarrow{W}(V_-)$  and  $\overrightarrow{W}(V_+)$ . The states at the pseudo-intersection points can be connected by a contact discontinuity (3.17) in the Riemann solutions. The Riemann solutions can be constructed according to the different locations on  $\overleftarrow{W}(V_-)$  and  $\overrightarrow{W}(V_+)$  of the pseudo-intersection points. To be precise, the Riemann solution contains a backward rarefaction wave, a contact discontinuity and a forward rarefaction wave when the pseudo-intersection points lie on  $\overleftarrow{R}(V_-)$  and  $\overrightarrow{R}(V_+)$ , a backward rarefaction wave, a contact discontinuity and a forward shock wave when the pseudo-intersection points lie on  $\overleftarrow{R}(V_-)$  and  $\overrightarrow{S}(V_+)$ , a backward shock, a contact discontinuity and a forward rarefaction wave when the pseudo-intersection points lie on  $\overleftarrow{S}(V_-)$  and  $\overrightarrow{R}(V_+)$ , a backward shock, a contact discontinuity and a forward shock wave when the pseudo-intersection points lie on  $\overleftarrow{S}(V_-)$  and  $\overrightarrow{S}(V_+)$ .

The conclusion can be stated in the following theorem.

**THEOREM 3.1.** *There exists a unique piecewise smooth solution, which consists of shocks, rarefaction waves, contact discontinuities and vacuum states, to the Riemann problem for (1.2) with initial data (3.1).*

### 4. Limits of solutions to (1.2) and (3.1) for $u_- > u_+$

In this section, we study the vanishing magnetic field limits of the Riemann solutions to the pressureless the magnetogasdynamics when  $u_- > u_+$ , and show the phenomenon of concentration and the formation of delta-shocks in the limit. It can be checked that when  $u_- > u_+$ , there must exist  $M_0 > 0$  such that the solution to (1.2) and (3.1) is the  $\overleftarrow{S} J \overrightarrow{S}$  solution when  $0 < \kappa < M_0$ .

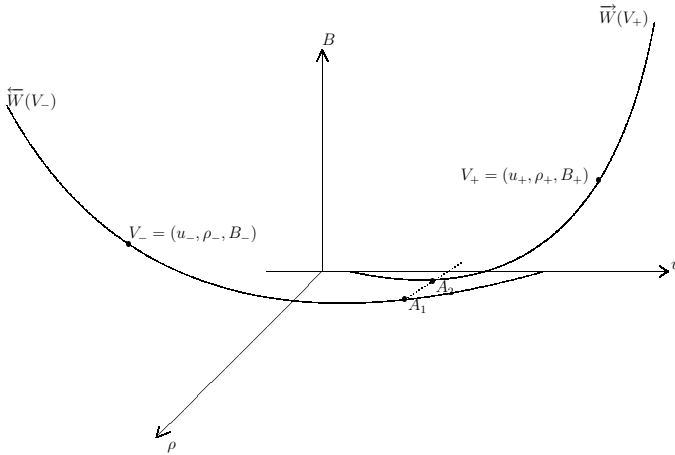


Figure 3.1: The pseudo-intersection points of the wave curves

For fixed  $\kappa \in (0, M_0)$ , the  $\overleftarrow{S} J \overrightarrow{S}$  solution is expressed as

$$U^\kappa(\xi) = (u^\kappa, \rho^\kappa, B^\kappa)(\xi) = \begin{cases} (u_-, \rho_-, B_-) & -\infty < \xi < \sigma_-^\kappa, \\ (u_*^\kappa, \rho_{*1}^\kappa, B_*^\kappa) & \sigma_-^\kappa < \xi < \sigma_0^\kappa, \\ (u_*^\kappa, \rho_{*2}^\kappa, B_*^\kappa) & \sigma_0^\kappa < \xi < \sigma_+^\kappa, \\ (u_+, \rho_+, B_+) & \sigma_+^\kappa < \xi < +\infty, \end{cases} \quad (4.1)$$

where  $(u_-, \rho_-, B_-)$  and  $(u_*^\kappa, \rho_{*1}^\kappa, B_*^\kappa)$  are connected by backward shock  $\overleftarrow{S}$  with speed  $\sigma_-^\kappa$ ,  $(u_*^\kappa, \rho_{*2}^\kappa, B_*^\kappa)$  and  $(u_+, \rho_+, B_+)$  are connected by forward shock  $\overrightarrow{S}$  with speed  $\sigma_+^\kappa$ , and  $(u_*^\kappa, \rho_{*1}^\kappa, B_*^\kappa)$  and  $(u_*^\kappa, \rho_{*2}^\kappa, B_*^\kappa)$  are connected by contact discontinuity with speed  $\sigma_0^\kappa = u_*^\kappa$ . Then it follows

$$\overleftarrow{S} : \begin{cases} \sigma_-^\kappa = u_- - \frac{\kappa}{\sqrt{2}} \cdot \mu_- \rho_{*1}^\kappa \sqrt{\frac{1}{\rho_-} + \frac{1}{\rho_{*1}^\kappa}}, \\ B_*^\kappa = \mu_- \rho_{*1}^\kappa, \\ u_*^\kappa = u_- - \frac{\kappa}{\sqrt{2}} \cdot \mu_- \sqrt{\frac{1}{\rho_-} + \frac{1}{\rho_{*1}^\kappa}} (\rho_{*1}^\kappa - \rho_-), \quad \rho_{*1}^\kappa > \rho_-, \end{cases} \quad (4.2)$$

and

$$\overrightarrow{S} : \begin{cases} \sigma_+^\kappa = u_+ + \frac{\kappa}{\sqrt{2}} \cdot \mu_+ \rho_{*2}^\kappa \sqrt{\frac{1}{\rho_+} + \frac{1}{\rho_{*2}^\kappa}}, \\ B_*^\kappa = \mu_+ \rho_{*2}^\kappa, \\ u_+ = u_*^\kappa + \frac{\kappa}{\sqrt{2}} \cdot \mu_+ \sqrt{\frac{1}{\rho_+} + \frac{1}{\rho_{*2}^\kappa}} (\rho_+ - \rho_{*2}^\kappa), \quad \rho_{*2}^\kappa > \rho_+, \end{cases} \quad (4.3)$$

where  $\mu_{\pm} = B_{\pm}/\rho_{\pm} > 0$ . From (4.2) and (4.3), we can get

$$\begin{aligned}
 & u_- - u_+ \\
 &= \frac{\kappa}{\sqrt{2}} \left( \sqrt{\mu_-} \sqrt{\frac{1}{B_*^{\kappa}} + \frac{1}{B_-}} (B_*^{\kappa} - B_-) + \sqrt{\mu_+} \sqrt{\frac{1}{B_*^{\kappa}} + \frac{1}{B_+}} (B_*^{\kappa} - B_+) \right) \\
 &= \frac{\kappa}{\sqrt{2}} \left( \sqrt{\mu_-} \sqrt{\left(\frac{1}{B_-} - \frac{1}{B_*^{\kappa}}\right) \left((B_*^{\kappa})^2 - (B_-)^2\right)} + \sqrt{\mu_+} \sqrt{\left(\frac{1}{B_+} - \frac{1}{B_*^{\kappa}}\right) \left((B_*^{\kappa})^2 - (B_+)^2\right)} \right) \\
 &:= I(\kappa, B_*^{\kappa}).
 \end{aligned} \tag{4.4}$$

The following Lemmas 4.1–4.4 show the limit behaviors of the states between two shocks.

**THEOREM 4.1.**  $B_*^{\kappa}, \rho_{*1}^{\kappa}$  and  $\rho_{*2}^{\kappa}$  are monotonic decreasing with respect to  $\kappa$ .

*Proof.* Let  $\kappa_1 > \kappa_2$ . Assume  $B_*^{\kappa_1} \geq B_*^{\kappa_2}$ , then one can deduce  $I(\kappa_1, B_*^{\kappa_1}) > I(\kappa_2, B_*^{\kappa_2})$ , which contradicts with  $I(\kappa_1, B_*^{\kappa_1}) = I(\kappa_2, B_*^{\kappa_2}) = u_- - u_+$ . Therefore, we have  $B_*^{\kappa_1} < B_*^{\kappa_2}$  and then  $B_*^{\kappa}$  is monotonic decreasing with respect to  $\kappa$ . Due to  $B_*^{\kappa} = \mu_- \rho_{*1}^{\kappa} = \mu_+ \rho_{*2}^{\kappa}$ , we have  $\rho_{*1}^{\kappa}$  and  $\rho_{*2}^{\kappa}$  are monotonic decreasing with respect to  $\kappa$ .  $\square$

**THEOREM 4.2.**

$$\lim_{\kappa \rightarrow 0^+} B_*^{\kappa} = \lim_{\kappa \rightarrow 0^+} \rho_{*1}^{\kappa} = \lim_{\kappa \rightarrow 0^+} \rho_{*2}^{\kappa} = +\infty.$$

*Proof.* In virtue of the monotonicity of  $B_*^{\kappa}$ , we have either  $\lim_{\kappa \rightarrow 0^+} B_*^{\kappa} = +\infty$  or  $\lim_{\kappa \rightarrow 0^+} B_*^{\kappa} = M \neq +\infty$ . If  $\lim_{\kappa \rightarrow 0^+} B_*^{\kappa} = M \neq +\infty$ , then from (4.4), we can get  $u_- = u_+$ , which contradicts with  $u_- > u_+$ . Therefore we have  $\lim_{\kappa \rightarrow 0^+} B_*^{\kappa} = +\infty$ . From  $B_*^{\kappa} = \mu_- \rho_{*1}^{\kappa} = \mu_+ \rho_{*2}^{\kappa}$ , we obtain  $\lim_{\kappa \rightarrow 0^+} \rho_{*1}^{\kappa} = \lim_{\kappa \rightarrow 0^+} \rho_{*2}^{\kappa} = +\infty$ .  $\square$

**THEOREM 4.3.**  $\kappa B_*^{\kappa}$  is monotonic increasing with respect to  $\kappa$ , and

$$\lim_{\kappa \rightarrow 0^+} \frac{\kappa}{\sqrt{2}} B_*^{\kappa} = \frac{\sqrt{\rho_- \rho_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}} (u_- - u_+). \tag{4.5}$$

*Proof.* Let  $\kappa_1 > \kappa_2$ . Assume  $\kappa_1 B_*^{\kappa_1} \leq \kappa_2 B_*^{\kappa_2}$ , then taking into account that  $B_*^{\kappa}$  is monotonic decreasing, one can deduce  $I(\kappa_1, B_*^{\kappa_1}) < I(\kappa_2, B_*^{\kappa_2})$  from (4.4), which contradicts with  $I(\kappa_1, B_*^{\kappa_1}) = I(\kappa_2, B_*^{\kappa_2}) = u_- - u_+$ . Therefore, we have  $\kappa_1 B_*^{\kappa_1} > \kappa_2 B_*^{\kappa_2}$ . Taking the limit  $\kappa \rightarrow 0^+$  on both sides of (4.4) and taking Lemma 4.2 into account give (4.5).  $\square$

THEOREM 4.4.

$$\lim_{\kappa \rightarrow 0^+} u_*^\kappa = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} := \sigma.$$

*Proof.* Form the last equation in (4.2) (or (4.3)) and Lemmas 4.2–4.3, one can easily obtain the conclusion, and we omit the details.  $\square$

The following Lemma 4.5 shows the limit behaviors of the speeds of shocks and contact discontinuity.

THEOREM 4.5.

$$\lim_{\kappa \rightarrow 0^+} \sigma_-^\kappa = \lim_{\kappa \rightarrow 0^+} \sigma_+^\kappa = \lim_{\kappa \rightarrow 0^+} \sigma_0^\kappa = \sigma.$$

*Proof.* Form (4.2), (4.3) and Lemmas 4.2–4.4, one can directly obtain the conclusions.  $\square$

THEOREM 4.6.

$$\lim_{\kappa \rightarrow 0^+} \rho_{*1}^\kappa (\sigma_0^\kappa - \sigma_-^\kappa) = \frac{\rho_- \sqrt{\rho_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}} (u_- - u_+),$$

$$\lim_{\kappa \rightarrow 0^+} \rho_{*2}^\kappa (\sigma_+^\kappa - \sigma_0^\kappa) = \frac{\rho_+ \sqrt{\rho_-}}{\sqrt{\rho_-} + \sqrt{\rho_+}} (u_- - u_+).$$

*Proof.* Note  $\sigma_0^\kappa = u_*^\kappa$  and

$$\sigma_-^\kappa = u_*^\kappa - \frac{\kappa}{\sqrt{2}} \cdot \mu_- \rho_- \sqrt{\frac{1}{\rho_-} + \frac{1}{\rho_{*1}^\kappa}},$$

$$\sigma_+^\kappa = u_*^\kappa + \frac{\kappa}{\sqrt{2}} \cdot \mu_+ \rho_+ \sqrt{\frac{1}{\rho_+} + \frac{1}{\rho_{*1}^\kappa}}.$$

We have

$$\rho_{*1}^\kappa (\sigma_0^\kappa - \sigma_-^\kappa) = \frac{\kappa}{\sqrt{2}} B_*^\kappa \cdot \rho_- \sqrt{\frac{1}{\rho_-} + \frac{1}{\rho_{*1}^\kappa}},$$

and

$$\rho_{*2}^\kappa (\sigma_+^\kappa - \sigma_0^\kappa) = \frac{\kappa}{\sqrt{2}} B_*^\kappa \cdot \rho_+ \sqrt{\frac{1}{\rho_+} + \frac{1}{\rho_{*2}^\kappa}}.$$

Then the conclusions can be obtained with the Lemmas 4.2–4.3.  $\square$

Let us take a sloping test function  $\phi(\xi) \in C_0^\infty(-\infty, +\infty)$  [10, 12] such that  $\phi(\xi) \equiv \phi(\sigma)$  for  $\xi$  in a neighborhood  $\Omega$  of  $\xi = \sigma$ . Then there exists  $M_1 \in (0, M_0)$  such that

when  $\kappa \in (0, M_1)$ , it holds  $\sigma_-^\kappa \in \Omega$  and  $\sigma_+^\kappa \in \Omega$ . It is well known that the solution (4.1) satisfies weak formulations

$$-\int_{-\infty}^{+\infty} \rho^\kappa (u^\kappa - \xi) \phi' d\xi + \int_{-\infty}^{+\infty} \rho^\kappa \phi d\xi = 0, \tag{4.6}$$

$$-\int_{-\infty}^{+\infty} \rho^\kappa u^\kappa (u^\kappa - \xi) \phi' d\xi + \int_{-\infty}^{+\infty} \rho^\kappa u^\kappa \phi d\xi = \frac{\kappa^2}{2} \int_{-\infty}^{+\infty} (B^\kappa)^2 \phi' d\xi. \tag{4.7}$$

Since

$$\int_{-\infty}^{+\infty} \rho^\kappa (u^\kappa - \xi) \phi' d\xi = \left( \int_{-\infty}^{\sigma_-^\kappa} + \int_{\sigma_+^\kappa}^{+\infty} \right) \rho^\kappa (u^\kappa - \xi) \phi' d\xi,$$

we have

$$\begin{aligned} \lim_{\kappa \rightarrow 0^+} \int_{-\infty}^{+\infty} \rho^\kappa (u^\kappa - \xi) \phi' d\xi &= \lim_{\kappa \rightarrow 0^+} \left\{ \int_{-\infty}^{\sigma_-^\kappa} \rho_-(u_- - \xi) \phi' d\xi + \int_{\sigma_+^\kappa}^{+\infty} \rho_+(u_+ - \xi) \phi' d\xi \right\} \\ &= \left( -\sigma[\rho] + [\rho u] \right) \phi(\sigma) + \int_{-\infty}^{+\infty} H(\xi - \sigma) \phi d\xi, \end{aligned}$$

where

$$H(x) = \begin{cases} \rho_-, & x < 0, \\ \rho_+, & x > 0. \end{cases}$$

Returning to (4.6), we get

$$\lim_{\kappa \rightarrow 0^+} \int_{-\infty}^{+\infty} \left( \rho^\kappa - H(\xi - \sigma) \right) \phi d\xi = \left( -\sigma[\rho] + [\rho u] \right) \phi(\sigma). \tag{4.8}$$

Due to

$$\lim_{\kappa \rightarrow 0^+} \frac{\kappa^2}{2} \int_{-\infty}^{+\infty} (B^\kappa)^2 \phi' d\xi = \lim_{\kappa \rightarrow 0^+} \frac{\kappa^2}{2} \left( \int_{-\infty}^{\sigma_-^\kappa} B_-^2 \phi' d\xi + \int_{\sigma_+^\kappa}^{+\infty} B_+^2 \phi' d\xi \right) = 0,$$

we have from (4.7)

$$\lim_{\kappa \rightarrow 0^+} \int_{-\infty}^{+\infty} \left( \rho^\kappa u^\kappa - \tilde{H}(\xi - \sigma) \right) \phi d\xi = \left( -\sigma[\rho u] + [\rho u^2] \right) \phi(\sigma), \tag{4.9}$$

where

$$\tilde{H}(x) = \begin{cases} \rho_- u_-, & x < 0, \\ \rho_+ u_+, & x > 0. \end{cases}$$

For an arbitrary test function  $\varphi(\xi) \in C_0^\infty(-\infty, +\infty)$ , we take a sloping test function  $\phi$  such that  $\phi(\sigma) = \varphi(\sigma)$  and

$$\max_{\xi \in (-\infty, +\infty)} |\phi - \varphi| < \mu.$$

We have

$$\begin{aligned} & \lim_{\kappa \rightarrow 0^+} \int_{-\infty}^{+\infty} (\rho^\kappa - H(\xi - \sigma)) \varphi d\xi \\ &= \lim_{\kappa \rightarrow 0^+} \left\{ \int_{-\infty}^{+\infty} (\rho^\kappa - H(\xi - \sigma)) \phi d\xi + \int_{-\infty}^{+\infty} (\rho^\kappa - H(\xi - \sigma)) (\varphi - \phi) d\xi \right\}. \end{aligned}$$

The first term on the right side

$$\begin{aligned} \lim_{\kappa \rightarrow 0^+} \int_{-\infty}^{+\infty} (\rho^\kappa - H(\xi - \sigma)) \phi d\xi &= (-\sigma[\rho] + [\rho u]) \phi(\sigma) \\ &= (-\sigma[\rho] + [\rho u]) \varphi(\sigma). \end{aligned}$$

The second term on the right side

$$\begin{aligned} & \int_{-\infty}^{+\infty} (\rho^\kappa - H(\xi - \sigma)) (\varphi - \phi) d\xi \\ &= \int_{\sigma_-^\kappa}^{\sigma_+^\kappa} (\rho^\kappa - H(\xi - \sigma)) (\varphi - \phi) d\xi \\ &= \int_{\sigma_-^\kappa}^{\sigma_+^\kappa} \rho^\kappa (\varphi - \phi) d\xi - \int_{\sigma_-^\kappa}^{\sigma_+^\kappa} H(\xi - \sigma) (\varphi - \phi) d\xi \\ &= \left( \int_{\sigma_-^\kappa}^{\sigma_0^\kappa} + \int_{\sigma_0^\kappa}^{\sigma_+^\kappa} \right) \rho^\kappa (\varphi - \phi) d\xi - \left( \int_{\sigma_-^\kappa}^{\sigma_0^\kappa} + \int_{\sigma_0^\kappa}^{\sigma_+^\kappa} \right) H(\xi - \sigma) (\varphi - \phi) d\xi, \end{aligned}$$

which converges to 0 as  $\kappa \rightarrow 0^+$  by sending  $\mu \rightarrow 0$  and recalling Lemma 4.6. Thus we have that

$$\lim_{\kappa \rightarrow 0^+} \int_{-\infty}^{+\infty} (\rho^\kappa - H(\xi - \sigma)) \varphi d\xi = (-\sigma[\rho] + [\rho u]) \varphi(\sigma) \tag{4.10}$$

for all test functions  $\varphi \in C_0^\infty(-\infty, +\infty)$ . Similarly, we have

$$\lim_{\kappa \rightarrow 0^+} \int_{-\infty}^{+\infty} (\rho^\kappa u^\kappa - \tilde{H}(\xi - \sigma)) \varphi d\xi = (-\sigma[\rho u] + [\rho u^2]) \varphi(\sigma) \tag{4.11}$$

for all test functions  $\varphi \in C_0^\infty(-\infty, +\infty)$ .

Let  $\psi(x, t) \in C_0^\infty((-\infty, +\infty) \times [0, +\infty))$  be an arbitrary test function, and let  $\tilde{\psi}(\xi, t) := \psi(\xi t, t)$ . Then it follows

$$\begin{aligned} \lim_{\kappa \rightarrow 0^+} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho^\kappa(x/t) \psi(x, t) dx dt &= \lim_{\kappa \rightarrow 0^+} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho^\kappa(\xi) \psi(\xi t, t) d(\xi t) dt \\ &= \lim_{\kappa \rightarrow 0^+} \int_0^{+\infty} t \left( \int_{-\infty}^{+\infty} \rho^\kappa(\xi) \tilde{\psi}(\xi, t) d\xi \right) dt, \end{aligned}$$

and with (4.10)

$$\begin{aligned} & \lim_{\kappa \rightarrow 0^+} \int_{-\infty}^{+\infty} \rho^\kappa(\xi) \tilde{\psi}(\xi, t) d\xi \\ &= \int_{-\infty}^{+\infty} H(\xi - \sigma) \tilde{\psi}(\xi, t) d\xi + (-\sigma[\rho] + [\rho u]) \tilde{\psi}(\sigma, t) \\ &= t^{-1} \int_{-\infty}^{+\infty} H(x - \sigma t) \psi(x, t) dx + (-\sigma[\rho] + [\rho u]) \psi(\sigma t, t). \end{aligned}$$

Combining the two relations above yields

$$\begin{aligned} & \lim_{\kappa \rightarrow 0^+} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho^\kappa(x/t) \psi(x, t) dx dt \\ &= \int_0^{+\infty} \int_{-\infty}^{+\infty} H(x - \sigma t) \psi(x, t) dx dt + \int_0^{+\infty} (-\sigma[\rho] + [\rho u]) t \psi(\sigma t, t) dt. \end{aligned} \tag{4.12}$$

By the definition, the last term

$$\int_0^{+\infty} (-\sigma[\rho] + [\rho u]) t \psi(\sigma t, t) dt = \left\langle w(t) \delta_{x=\sigma t}, \psi(x, t) \right\rangle$$

with

$$w(t) = (-\sigma[\rho] + [\rho u])t.$$

Similarly, we can show from (4.11) that

$$\begin{aligned} & \lim_{\kappa \rightarrow 0^+} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho^\kappa(x/t) u^\kappa(x/t) \psi(x, t) dx dt \\ &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \tilde{H}(x - \sigma t) \psi(x, t) dx dt + \left\langle \tilde{w}(t) \delta_{x=\sigma t}, \psi(x, t) \right\rangle \end{aligned} \tag{4.13}$$

with

$$\tilde{w}(t) = (-\sigma[\rho u] + [\rho u^2])t.$$

Thus we obtain the following conclusion.

**THEOREM 4.7.** *Let  $u_- > u_+$ . For fixed  $\kappa > 0$ , assume that  $(u^\kappa, \rho^\kappa, B^\kappa)(x, t)$  is the two-shock solution to (1.2) and (3.1). Then*

$$\lim_{\kappa \rightarrow 0^+} u^\kappa(t, x) = \begin{cases} u_-, & x < \sigma t \\ \sigma, & x = \sigma t, \\ u_+, & x > \sigma t, \end{cases}$$

$\rho^\kappa$  and  $\rho^\kappa u^\kappa$  converge in the sense of distributions, and the limit functions are all the sum of a step function and a Dirac delta function supported on  $x = \sigma t$  with weights

$$(-\sigma[\rho] + [\rho u])t \quad \text{and} \quad (-\sigma[\rho u] + [\rho u^2])t,$$

respectively, where  $\sigma = (\sqrt{\rho_-} u_- + \sqrt{\rho_+} u_+) / (\sqrt{\rho_-} + \sqrt{\rho_+})$ .



It can be seen that the limits of  $(u^\kappa, \rho^\kappa)(x, t)$  are the delta-shock solution of the Riemann problem for the pressureless gas dynamics obtained in Section 2.

**5. Limits of solutions to (1.2) and (3.1) for  $u_- < u_+$**

In this section, we study the vanishing magnetic field limits of the Riemann solutions to the pressureless magnetogasdynamics when  $u_- < u_+$ , and show the phenomenon of cavitation and the formation of vacuum states in the limit. It can be checked that when  $u_- < u_+$ , there must exist  $N_0 > 0$  such that the Riemann solution to (1.2) and (3.1) containing two rarefaction waves and a contact discontinuity when  $\kappa < N_0$ .

For fixed  $\kappa < N_0$  with  $u_- + 2\kappa\sqrt{\mu_- B_-} > u_+ - 2\kappa\sqrt{\mu_+ B_+}$ , the Riemann solution is the  $\overleftarrow{R}J\overrightarrow{R}$  solution expressed as

$$(u^\kappa, \rho^\kappa, B^\kappa)(\xi) \tag{5.1}$$

$$= \begin{cases} (u_-, \rho_-, B_-), & -\infty < \xi \leq u_- - \kappa\sqrt{\mu_- B_-}, \\ \overleftarrow{R}, & u_- - \kappa\sqrt{\mu_- B_-} \leq \xi \leq u_- + 2\kappa\sqrt{\mu_- B_-} - 3\kappa\sqrt{\mu_- B_*^\kappa}, \\ (u_*^\kappa, \rho_{*1}^\kappa, B_*^\kappa), & u_- + 2\kappa\sqrt{\mu_- B_-} - 3\kappa\sqrt{\mu_- B_*^\kappa} \leq \xi < u_- + 2\kappa\sqrt{\mu_- B_-} - 2\kappa\sqrt{\mu_- B_*^\kappa}, \\ (u_*^\kappa, \rho_{*2}^\kappa, B_*^\kappa), & u_+ - 2\kappa\sqrt{\mu_+ B_+} + 2\kappa\sqrt{\mu_+ B_*^\kappa} < \xi < u_+ - 2\kappa\sqrt{\mu_+ B_+} + 3\kappa\sqrt{\mu_+ B_*^\kappa}, \\ \overrightarrow{R}, & u_+ - 2\kappa\sqrt{\mu_+ B_+} + 3\kappa\sqrt{\mu_+ B_*^\kappa} \leq \xi \leq u_+ + \kappa\sqrt{\mu_+ B_+}, \\ (u_+, \rho_+, B_+), & u_+ + \kappa\sqrt{\mu_+ B_+} \leq \xi < +\infty, \end{cases}$$

where  $(u_-, \rho_-, B_-)$  and  $(u_*^\kappa, \rho_{*1}^\kappa, B_*^\kappa)$  are connected by backward rarefaction wave  $\overleftarrow{R}$  :

$$\overleftarrow{R} : \begin{cases} \xi = u - \kappa\sqrt{\mu B} \\ B = \mu\rho, \\ u = u_- - 2\kappa\mu_-(\sqrt{\rho} - \sqrt{\rho_-}), \quad \rho < \rho_- \end{cases} \tag{5.2}$$

with  $\mu_- = B_-/\rho_-$ ,  $(u_*^\kappa, \rho_{*2}^\kappa, B_*^\kappa)$  and  $(u_+, \rho_+, B_+)$  are connected by forward rarefaction wave  $\overrightarrow{R}$  :

$$\overrightarrow{R} : \begin{cases} \xi = u + \kappa\sqrt{\mu B}, \\ B = \mu\rho, \\ u = u_+ - 2\kappa\mu_+(\sqrt{\rho_+} - \sqrt{\rho}), \quad \rho < \rho_+ \end{cases} \tag{5.3}$$

with  $\mu_+ = B_+/\rho_+$ , and  $(u_*^\kappa, \rho_{*1}^\kappa, B_*^\kappa)$  and  $(u_*^\kappa, \rho_{*2}^\kappa, B_*^\kappa)$  are connected by a contact discontinuity with speed  $\sigma_0^\kappa = u_*^\kappa$ .

Because  $(u_*^\kappa, \rho_{*1}^\kappa, B_*^\kappa)$  lies on  $\overleftarrow{R}$  and  $(u_*^\kappa, \rho_{*2}^\kappa, B_*^\kappa)$  lies on  $\overrightarrow{R}$ , we have

$$u_*^\kappa = u_- + 2\kappa\sqrt{\mu_- B_-} - 2\kappa\sqrt{\mu_- B_*^\kappa} = u_+ - 2\kappa\sqrt{\mu_+ B_+} + 2\kappa\sqrt{\mu_+ B_*^\kappa}, \tag{5.4}$$

which gives

$$u_+ - u_- = 2\kappa\sqrt{\mu_-} \left( \sqrt{B_-} - \sqrt{B_*^\kappa} \right) + 2\kappa\sqrt{\mu_+} \left( \sqrt{B_+} - \sqrt{B_*^\kappa} \right) := J(\kappa, B_*^\kappa), \quad (5.5)$$

or

$$(u_- + 2\kappa\sqrt{\mu_- B_-}) - (u_+ - 2\kappa\sqrt{\mu_+ B_+}) = 2\kappa(\sqrt{\mu_-} + \sqrt{\mu_+})\sqrt{B_*^\kappa}. \quad (5.6)$$

**THEOREM 5.1.**  $B_*^\kappa$ ,  $\rho_{*1}^\kappa$  and  $\rho_{*2}^\kappa$  are monotonic increasing with respect to  $\kappa$ .

*Proof.* Let  $\kappa_1 > \kappa_2$ . Assume  $B_*^{\kappa_1} \leq B_*^{\kappa_2}$ , then one can deduce  $J(\kappa_1, B_*^{\kappa_1}) > J(\kappa_2, B_*^{\kappa_2})$ , which contradicts with  $J(\kappa_1, B_*^{\kappa_1}) = J(\kappa_2, B_*^{\kappa_2}) = u_- - u_+$ . Therefore, we have  $B_*^{\kappa_1} > B_*^{\kappa_2}$  and then  $B_*^\kappa$  is monotonically increasing with respect to  $\kappa$ . Due to  $\rho_{*1}^\kappa = B_*^\kappa/\mu_-$  and  $\rho_{*2}^\kappa = B_*^\kappa/\mu_+$ , we have  $\rho_{*1}^\kappa$  and  $\rho_{*2}^\kappa$  are monotonically increasing with respect to  $\kappa$ .  $\square$

Assume that  $N_1 > 0$  satisfies

$$u_- + 2N_1\sqrt{\mu_- B_-} = u_+ - 2N_1\sqrt{\mu_+ B_+}.$$

Then taking the limit  $\kappa \rightarrow N_1$  on both sides of (5.6) and recalling  $\rho_{*1}^\kappa = B_*^\kappa/\mu_-$  and  $\rho_{*2}^\kappa = B_*^\kappa/\mu_+$ , we have

$$\lim_{\kappa \rightarrow N_1} B_*^\kappa = \lim_{\kappa \rightarrow N_1} \rho_{*1}^\kappa = \lim_{\kappa \rightarrow N_1} \rho_{*2}^\kappa = 0.$$

Therefore, from  $\kappa = N_1$ , the vacuum state appears. Furthermore, with (5.4), one has

$$\lim_{\kappa \rightarrow N_1} u_*^\kappa = u_- + 2N_1\sqrt{\mu_- B_-} = u_+ - 2N_1\sqrt{\mu_+ B_+} := u_1.$$

Moreover, as  $\kappa \rightarrow N_1$ , both the wave front  $\xi_{front} = u_- + 2\kappa\sqrt{\mu_- B_-} - 3\kappa\sqrt{\mu_- B_*^\kappa}$  of  $\overleftarrow{R}$  and the wave back  $\xi_{back} = u_+ - 2\kappa\sqrt{\mu_+ B_+} + 3\kappa\sqrt{\mu_+ B_*^\kappa}$  of  $\overrightarrow{R}$  tend to  $\xi = u_1$ . That is, the wave front of  $\overleftarrow{R}$ , the wave back of  $\overrightarrow{R}$  and the contact discontinuity  $J$  coincide as  $\kappa \rightarrow N_1$ .

When  $\kappa$  decreases so that  $\kappa < N_1$ , the Riemann solution becomes

$$(u^\kappa, \rho^\kappa, B^\kappa)(\xi) = \begin{cases} (u_-, \rho_-, B_-), & -\infty < \xi \leq u_- - \kappa\sqrt{\mu_- B_-}, \\ \overleftarrow{R}, & u_- - \kappa\sqrt{\mu_- B_-} \leq \xi \leq u_- + 2\kappa\sqrt{\mu_- B_-}, \\ (\xi, 0, 0), & u_- + 2\kappa\sqrt{\mu_- B_-} \leq \xi \leq u_+ - 2\kappa\sqrt{\mu_+ B_+}, \\ \overrightarrow{R}, & u_+ - 2\kappa\sqrt{\mu_+ B_+} \leq \xi \leq u_+ + \kappa\sqrt{\mu_+ B_+}, \\ (u_+, \rho_+, B_+), & u_+ + \kappa\sqrt{\mu_+ B_+} \leq \xi < +\infty \end{cases} \quad (5.7)$$

with (5.2) and (5.3). Then when  $\kappa$  continues to decrease, the rarefaction waves become narrower and narrower and the vacuum region in between becomes wider and wider. Finally, when  $\kappa$  drops to zero, the rarefaction waves become two lines with  $\xi = u_-$

and  $\xi = u_+$ , and between which is a vacuum state. In summary, the limit functions of the velocity and density as  $\kappa \rightarrow 0^+$  are

$$\lim_{\kappa \rightarrow 0^+} (u^\kappa, \rho^\kappa)(\xi) = \begin{cases} (u_-, \rho_-) & -\infty < \xi \leq u_-, \\ (\xi, 0) & u_- \leq \xi \leq u_+, \\ (u_+, \rho_+) & u_+ \leq \xi < +\infty, \end{cases}$$

which are just the vacuum solution of the Riemann problem for the pressureless gas dynamics obtained in Section 2.

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