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NONLOCAL BOUNDARY VALUE PROBLEMS FOR (p,q)–DIFFERENCE EQUATIONS

NATTAPONG KAMSRISUK, CHANON PROMSAKON, SOTIRIS K. NTOUYAS AND JESSADA TARIBOON

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Abstract. In this paper we study existence and uniqueness of solutions for a boundary value problem for (p,q)-difference equations with nonlocal integral boundary conditions, by using classical fixed point theorems. Examples illustrating the main results are also presented.

1. Introduction

Quantum calculus or q-calculus is known as the study of calculus without limits. The study of q-calculus initiated by Euler on studying infinite series. Jackson [15, 16] was the first one, who establised the q-derivative or q-difference operator for a function f on $[0,\infty)$ by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0, \quad D_q f(0) = \lim_{t\to 0} D_q f(t), \quad t = 0, \tag{1}$$

and studied its properties. The q-integral of a function f on $[0,\infty)$ is defined by

$$\int_0^t f(s)d_q s = (1-q)t \sum_{n=0}^{\infty} q^n f(q^n t), \tag{2}$$

provided that infinite series converges. The study of q-difference equations also initiated by Jackson [15, 16]. Details of its basic notions, results and methods can be found in the text [18]. For other papers on the subject see [1, 6, 10, 20]. In recent years, the topic has attracted the attention of several researchers and a variety of new results can be found in [2, 8, 9, 11, 12, 19] and references therein.

Recently Tariboon and Ntouyas [24], generalized the classical quantum calculus by defining quantum calculus on finite intervals, the so called q_k-calculus. The new concepts of q_k-derivative and q_k-integral are discussed in [24] and as applications initial and boundary value problems for impulsive q_k-difference equations and inclusions are studied. Also several classical inequalities were transformed in context of q_k-calculus. For more details we refer to the recent monograph [3].

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Another generalization of quantum calculus is (p,q)-calculus introduced in [7]. For some recent results see [5, 14, 17, 21, 22, 23] and references cited therein. To the best of our knowledge, there is no work on boundary value problems for (p,q)-difference equations in the literature. So, in this paper we initiate the study of boundary value problems for (p,q)-difference equations. To be more precisely, in the present paper we study the existence and uniqueness of solution for (p,q)-difference equation subject to a nonlocal condition of the form

$$D_{p,q}x(t) = f(t,x(pt)), \quad t \in [0,T/p], \tag{3}$$

$$x(0) = \alpha x(T) + \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} x(s) d_{p_i,q_i} s, \tag{4}$$

where $0 < q < p \leqslant 1$, $0 < q_i < p_i \leqslant 1$, $i = 1,2,\ldots,m$ are quantum numbers, $D_{p,q}$ is (p,q)-difference operator, $f \in C([0,T/p] \times \mathbb{R},\mathbb{R})$, $T > 0$, α, β_i, $i = 1,2,\ldots,m$ are given constants, $\eta_i \in [0,p_iT]$, $i = 1,2,\ldots,m$. We prove existence and uniqueness results for the problem (3)–(4) by using the classical fixed point theorems, such as Banach's fixed point theorem, Boyd and Wong fixed point theorem for nonlinear contractions and Leray-Schauder nonlinear alternative.

The paper is organized as follows: In Section 2 we recall some definitions and basic facts from (p,q)-calculus. The main existence and uniqueness results are given in Section 3. Examples illustrating the obtained results are presented in Section 4.

2. Preliminaries

In this section, we recall some basic concepts of (p,q)-calculus. The (p,q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \tag{5}$$

where $0 < q < p \leqslant 1$. For each $k,n \in \mathbb{N}$, $n \geqslant k \geqslant 0$, the (p,q)-factorial and (p,q)-binomial are defined by

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \quad n \geqslant 1, \quad [0]_{p,q}! = 1, \tag{6}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}, \tag{7}$$

respectively.

Let $f : [0,T] \to \mathbb{R}$. The (p,q)-derivative of function f is defined as

$$D_{p,q}f(t) = \frac{f(pt) - f(qt)}{(p-q)t}, \quad t \neq 0, \tag{8}$$

and $D_{p,q}f(0) = \lim_{t\to 0} D_{p,q}f(t)$. *Observe that the function* $g(t) = D_{p,q}f(t)$ *is defined on* $[0,T/p]$. We say that f is (p,q)-differentiable on $[0,T/p]$ provided $D_{p,q}f(t)$ exists for all $t \in [0,T/p]$.

Let $f:[0,T]\to\mathbb{R}$. Then the (p,q)-integral of f is defined by

$$\int_0^t f(s)d_{p,q}s=(p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}t\right), \tag{9}$$

provided that the right hand side is convergent. *Note that the function* $\phi(t)=\int_0^t f(s)d_{p,q}s$ *is defined on* $[0,pT]$.

In the following theorems we collect the basic properties of (p,q)-differentiation and (p,q)-integration, respectively. See [23].

THEOREM 1. *Suppose that* $f,g:[0,T]\to\mathbb{R}$ *is* (p,q)*-differentiable on* $[0,T/p]$. *Then:*

(a) $f+g:[0,T]\to\mathbb{R}$ *is* (p,q)*-differentiable on* $[0,T/p]$, *and*

$$D_{p,q}(f(t)+g(t))=D_{p,q}f(t)+D_{p,q}g(t). \tag{10}$$

(b) $\lambda f:[0,T]\to\mathbb{R}$ *is* (p,q)*-differentiable on* $[0,T/p]$ *for any constant* λ, *and*

$$D_{p,q}(\lambda f)(t)=\lambda D_{p,q}f(t). \tag{11}$$

(c) $fg:[0,T]\to\mathbb{R}$ *is* (p,q)*-differentiable on* $[0,T/p]$, *and*

$$D_{p,q}(fg)(t)=f(pt)D_{p,q}g(t)+g(qt)D_{p,q}f(t). \tag{12}$$

(d) *If* $g(t)\neq 0$, *then* $\dfrac{f}{g}$ *is* (p,q)*-differentiable on* $[0,T/p]$ *with*

$$D_{p,q}\left(\frac{f}{g}\right)(t)=\frac{g(qt)D_{p,q}f(t)-f(qt)D_{p,q}g(t)}{g(pt)g(qt)}. \tag{13}$$

THEOREM 2. *Let* $f,g:[0,T]\to\mathbb{R}$ *are continuous functions and* $0<q<p\leqslant 1$. *The following formulas hold:*

(a) *The* (p,q)*-integration by parts is given by*

$$\int_a^b f(px)D_{p,q}g(x)d_{p,q}t=f(x)g(x)|_a^b-\int_a^b g(qx)D_{p,q}f(x)d_{p,q}t.$$

(b) $D_{p,q}\int_0^t f(s)d_{p,q}s=f(t)$.

(c) $\int_0^t D_{p,q}f(s)d_{p,q}s=f(t)-f(0)$.

(d) $\int_a^t D_{p,q}f(s)d_{p,q}s=f(t)-f(a)$ *where* $a\in(0,t)$.

THEOREM 3. *Let a function $f:[0,T]\to\mathbb{R}$ and constants $0<q<p\leqslant 1$. Then for $t\in[0,p^2T]$, we have*

$$\int_0^t\int_0^s f(r)d_{p,q}rd_{p,q}s=\frac{1}{p}\int_0^t (t-qs)f\left(\frac{1}{p}s\right)d_{p,q}s. \tag{14}$$

Proof. From the definition of (p,q)-integral in (9), for $t\in[0,p^2T]$, we obtain

$$\begin{aligned}\int_0^t\int_0^s f(r)d_{p,q}rd_{p,q}s &= \int_0^t\left((p-q)s\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}s\right)\right)d_{p,q}s\\ &= (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\int_0^t sf\left(\frac{q^n}{p^{n+1}}s\right)d_{p,q}s\right).\end{aligned}$$

Note that

$$\begin{aligned}\int_0^t sf\left(\frac{q^n}{p^{n+1}}s\right)d_{p,q}s &= (p-q)t\sum_{m=0}^{\infty}\frac{q^m}{p^{m+1}}\left(\frac{q^m}{p^{m+1}}t\right)f\left(\frac{q^n}{p^{n+1}}\frac{q^m}{p^{m+1}}t\right)\\ &= (p-q)t^2\sum_{m=0}^{\infty}\frac{q^{2m}}{p^{2m+2}}f\left(\frac{q^{n+m}}{p^{n+m+2}}t\right).\end{aligned}$$

Hence

$$\int_0^t\int_0^s f(r)d_{p,q}rd_{p,q}s=(p-q)^2t^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n+2m}}{p^{n+2m+3}}f\left(\frac{q^{n+m}}{p^{n+m+2}}t\right).$$

Since

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n+2m}}{p^{n+2m+3}}f\left(\frac{q^{m+m}}{p^{n+m+2}}t\right)=\sum_{n=0}^{\infty}\frac{q^n}{p^{n+3}}\frac{1}{(p-q)}\left(p-\frac{q^{n+1}}{p^n}\right)f\left(\frac{q^n}{p^{n+2}}t\right),$$

we get that

$$\begin{aligned}\int_0^t\int_0^s f(r)d_{p,q}rd_{p,q}s &= (p-q)t^2\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+2}}-\frac{q^{2n+1}}{p^{2n+3}}\right)f\left(\frac{q^n}{p^{n+2}}t\right)\\ &= (p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{t}{p}-\frac{q^{n+1}}{p^{n+2}}t\right)f\left(\frac{q^n}{p^{n+2}}t\right)\\ &= (p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{t}{p}-\left(\frac{q}{p}\right)\left(\frac{q^n}{p^{n+1}}t\right)\right)f\left(\frac{1}{p}\left(\frac{q^n}{p^{n+1}}\right)\right)\\ &= \int_0^t\left(\frac{t}{p}-\frac{q}{p}s\right)f\left(\frac{1}{p}s\right)d_{p,q}s.\end{aligned}$$

Therefore, the right-hand side of (14) holds. The proof is completed. □

Let Λ be a nonzero constant defined by

$$\Lambda=1-\alpha-\sum_{i=1}^{m}\beta_i\eta_i. \tag{15}$$

The following lemma deals with a linear variant of problem (3)–(4).

LEMMA 1. *Let* $\Lambda \neq 0$, $\eta_i \in [0, p_iT]$, $i = 1, 2, \ldots, m$, *and* $h \in C([0, T/p], \mathbb{R})$. *The function* x *is a solution of the nonlocal* (p,q)*-difference boundary value problem*

$$D_{p,q}x(t) = h(t), \quad t \in [0, T/p], \tag{16}$$

$$x(0) = \alpha x(T) + \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} x(s) d_{p_i,q_i}s, \tag{17}$$

if and only if

$$x(t) = \frac{1}{\Lambda}\left[\alpha \int_0^T h(s) d_{p,q}s + \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} \int_0^s h(r) d_{p,q}r d_{p_i,q_i}s\right] + \int_0^t h(s) d_{p,q}s, \tag{18}$$

for $t \in [0, T]$.

Proof. Applying the (p,q)-integration to (16), we have

$$x(t) = x(0) + \int_0^t h(s) d_{p,q}s, \quad t \in [0, T/p]. \tag{19}$$

In particular, for $t = T$, we have

$$x(T) = x(0) + \int_0^T h(s) d_{p,q}s. \tag{20}$$

Next, by (p,q)-integration with respect to t in (19)

$$\int_0^t x(s) d_{p_i,q_i}s = x(0) \int_0^t d_{p_i,q_i}s + \int_0^t \int_0^s h(r) d_{p,q}r d_{p_i,q_i}s, \quad t \in [0, p_iT].$$

By substituting $t = \eta_i$ we have

$$\int_0^{\eta_i} x(s) d_{p_i,q_i}s = \eta_i x(0) + \int_0^{\eta_i} \int_0^s h(r) d_{p,q}r d_{p_i,q_i}s,$$

from which we get

$$\sum_{i=1}^{m} \beta_i \int_0^{\eta_i} x(s) d_{p_i,q_i}s = x(0) \sum_{i=1}^{m} \beta_i \eta_i + \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} \int_0^s h(r) d_{p,q}r d_{p_i,q_i}s.$$

Using the nonlocal boundary condition (4), we obtain

$$x(0) = \frac{1}{\Lambda}\left[\alpha \int_0^T h(s) d_{p,q}s + \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} \int_0^s h(r) d_{p,q}r d_{p_i,q_i}s\right]. \tag{21}$$

By substituting the value of $x(0)$ in (19) we obtain the solution (18). The converse follows by direct computation. The proof is completed. □

REMARK 1. If $p_i = p$, $q_i = q$ for $i = 1,2,\dots,m,$ then

$$x(t) = \frac{1}{\Lambda}\left[\alpha \int_0^T h(s)d_{p,q}s + \frac{1}{p}\sum_{i=1}^{m}\beta_i \int_0^{\eta_i}(\eta_i - qs)h\left(\frac{s}{p}\right)d_{p,q}s\right] + \int_0^t h(s)d_{p,q}s,$$

is a unique solution of (16) with nonlocal condition

$$x(0) = \alpha x(T) + \sum_{i=1}^{m}\beta_i \int_0^{\eta_i} x(s)d_{p,q}s.$$

3. Main results

Let $\mathscr{C} = C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{[0,T]}|x(t)|$. In view of Lemma 1, we define an operator $\mathscr{A} : \mathscr{C} \to \mathscr{C}$ by

$$\begin{aligned}\mathscr{A}x(t) = {} & \frac{1}{\Lambda}\left[\alpha \int_0^T f(s,x(ps))d_{p,q}s + \sum_{i=1}^{m}\beta_i \int_0^{\eta_i}\int_0^s f(r,x(pr))d_{p,q}r d_{p_i,q_i}s\right] \\ & + \int_0^t f(s,x(ps))d_{p,q}s, \end{aligned} \tag{22}$$

with $\Lambda \neq 0$, defined by (15). It should be noticed that problem (3)–(4) has solutions if and only if the operator $\mathscr{A}$ has fixed points. In addition, if $p_i = p, q_i = q$ for $i = 1,2,\dots,m,$ then the operator $\mathscr{A}$ can be modified by applying Theorem 3 and Remark 1.

For the sake of convenience, we put

$$\Phi = \frac{|\alpha|}{|\Lambda|}T + \frac{1}{|\Lambda|}\sum_{i=1}^{m}\left(\frac{\eta_i^2|\beta_i|}{p_i + q_i}\right) + T. \tag{23}$$

THEOREM 4. *Let $f : [0,T/p]\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_1) *there exists a constant $L > 0$ such that $|f(t,x) - f(t,y)| \leqslant L|x-y|$ for each $t \in [0,T/p]$ and $x,y \in \mathbb{R}$.*

If

$$L\Phi < 1, \tag{24}$$

where Φ is given by (23), then the boundary value problem (3)–(4) has a unique solution on $[0,T]$.

Proof. We transform problem (3)–(4) into a fixed point problem $x = \mathscr{A}x,$ where the operator $\mathscr{A}$ is defined by (22). Applying Banach's contraction mapping principle, we shall show that $\mathscr{A}$ has a unique fixed point.

Define a ball $B_r = \{x \in \mathscr{C} : \|x\| \leqslant r\}$ with the value r satisfying

$$r > \frac{M\Phi}{1 - L\Phi},$$

where $M = \sup_{t\in[0,T/p]} |f(t,0)|$. Now, we will show that $\mathscr{A}B_r \subset B_r$. For any $x \in B_r$, we have

$$\begin{aligned}
|\mathscr{A}x(t)| &\leqslant \frac{1}{|\Lambda|}\left[|\alpha|\int_0^T |f(s,x(ps))|d_{p,q}s + \sum_{i=1}^{m}|\beta_i|\int_0^{\eta_i}\int_0^s |f(r,x(pr))|d_{p,q}r d_{p_i,q_i}s\right] \\
&\quad + \int_0^t |f(s,x(ps))|d_{p,q}s \\
&\leqslant \frac{|\alpha|}{|\Lambda|}\int_0^T (|f(s,x(ps)) - f(s,0)| + |f(s,0)|)d_{p,q}s + \sum_{i=1}^{m}\frac{|\beta_i|}{|\Lambda|}\int_0^{\eta_i}\int_0^s (|f(r,x(pr)) \\
&\quad - f(s,0)| + |f(s,0)|)d_{p,q}r d_{p_i,q_i}s + \int_0^T (|f(s,x(ps)) - f(s,0)| + |f(s,0)|)d_{p,q}s \\
&\leqslant (L\|x\| + M)\left\{\frac{|\alpha|}{|\Lambda|}\int_0^T d_{p,q}s + \sum_{i=1}^{m}\frac{|\beta_i|}{|\Lambda|}\int_0^{\eta_i}\int_0^s d_{p,q}r d_{p_i,q_i}s + \int_0^T d_{p,q}s\right\} \\
&\leqslant (Lr + M)\left\{\frac{|\alpha|}{|\Lambda|}T + \frac{1}{|\Lambda|}\sum_{i=1}^{m}\left(\frac{\eta_i^2|\beta_i|}{p_i + q_i}\right) + T\right\} \\
&= (Lr + M)\Phi < r,
\end{aligned}$$

which means that $\|\mathscr{A}x\| < r$. Therefore we have $\mathscr{A}B_r \subset B_r$.

For $x, y \in \mathscr{C}$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
&|\mathscr{A}x(t) - \mathscr{A}y(t)| \\
&\leqslant \frac{1}{|\Lambda|}\Bigg[|\alpha|\int_0^T |f(s,x(ps)) - f(s,y(ps))|d_{p,q}s + \sum_{i=1}^{m}|\beta_i|\int_0^{\eta_i}\int_0^s |f(r,x(pr)) \\
&\quad - f(r,y(pr))|d_{p,q}r d_{p_i,q_i}s\Bigg] + \int_0^t |f(s,x(ps)) - f(s,y(ps))|d_{p,q}s \\
&\leqslant L\|x - y\|\left\{\frac{|\alpha|}{|\Lambda|}T + \frac{1}{|\Lambda|}\sum_{i=1}^{m}\left(\frac{\eta_i^2|\beta_i|}{p_i + q_i}\right) + T\right\} \\
&= L\Phi\|x - y\|.
\end{aligned}$$

Consequently $\|\mathscr{A}x - \mathscr{A}y\| \leqslant L\Phi\|x - y\|$. As $L\Phi < 1$, $\mathscr{A}$ is a contraction. By the Banach's contraction mapping principle, we deduce that $\mathscr{A}$ has a fixed point which is the unique solution of the problem (3)–(4) on $[0,T]$. This completes the proof. □

Next, we give the second existence and uniqueness result by using nonlinear contractions.

DEFINITION 1. Let E be a Banach space and let $A: E \to E$ be a mapping. A is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\theta) < \theta$ for all $\theta > 0$ with the property:

$$\|Ax - Ay\| \leqslant \Psi(\|x-y\|), \quad \forall x, y \in E. \tag{25}$$

LEMMA 2. (Boyd and Wong) ([4]) *Let E be a Banach space and let $A: E \to E$ be a nonlinear contraction. Then A has a unique fixed point in E.*

THEOREM 5. *Let $f: [0, T/p] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_2) $|f(t,x) - f(t,y)| \leqslant g(t)\dfrac{|x-y|}{G^* + |x-y|}$, $t \in [0, T/p]$, $x, y \in \mathbb{R}$, where $g: [0, T/p] \to \mathbb{R}^+$ is continuous and the positive constant G^* is defined by

$$G^* = \left(\frac{|\alpha| + |\Lambda|}{|\Lambda|}\right)\int_0^T g(s)d_{p,q}s + \frac{1}{|\Lambda|}\sum_{i=1}^m |\beta_i| \int_0^{\eta_i}\int_0^s g(s)d_{p,q}rd_{p_i,q_i}s.$$

Then the boundary value problem (3)–(4) has a unique solution on $[0,T]$.

Proof. Consider the operator $\mathscr{A}: \mathscr{C} \to \mathscr{C}$ defined by (22). Let the continuous nondecreasing function $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Psi(\theta) = \frac{G^*\theta}{G^* + \theta}, \quad \forall \theta \geqslant 0.$$

Note that the function Ψ satisfies $\Psi(0) = 0$ and $\Psi(\theta) < \theta$ for all $\theta > 0$.

For any $x, y \in C$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
&|\mathscr{A}x(t) - \mathscr{A}y(t)| \\
\leqslant\; & \frac{1}{|\Lambda|}\Bigg[|\alpha|\int_0^T |f(s,x(ps)) - f(s,y(ps))|d_{p,q}s + \sum_{i=1}^m |\beta_i|\int_0^{\eta_i}\int_0^s |f(r,x(pr)) \\
& - f(r,y(pr))|d_{p,q}rd_{p_i,q_i}s\Bigg] + \int_0^t |f(s,x(ps)) - f(s,y(ps))|d_{p,q}s \\
\leqslant\; & \frac{|\alpha|}{|\Lambda|}\int_0^T \left(g(s)\frac{|x(ps)-y(ps)|}{G^* + |x(ps)-y(ps)|}\right)d_{p,q}s \\
& + \frac{1}{|\Lambda|}\sum_{i=1}^m |\beta_i|\int_0^{\eta_i}\int_0^s \left(g(r)\frac{|x(pr)-y(pr)|}{G^* + |x(pr)-y(pr)|}\right)d_{p,q}rd_{p_i,q_i}s \\
& + \int_0^t \left(g(s)\frac{|x(ps)-y(ps)|}{G^* + |x(ps)-y(ps)|}\right)d_{p,q}s \\
\leqslant\; & \frac{\Psi(\|x-y\|)}{G^*}\left(\frac{|\alpha|}{|\Lambda|}\int_0^T g(s)d_{p,q}s + \frac{1}{|\Lambda|}\sum_{i=1}^m |\beta_i|\int_0^{\eta_i}\int_0^s g(s)d_{p,q}rd_{p_i,q_i}s + \int_0^T g(s)d_{p,q}s\right) \\
=\; & \Psi(\|x-y\|).
\end{aligned}$$

Then $\|\mathscr{A}x-\mathscr{A}y\| \leqslant \Psi(\|x-y\|)$ and $\mathscr{A}$ is a nonlinear contraction and it has a unique fixed point in $\mathscr{C}$ by Lemma 2. This completes the proof. $\square$

Our next existence result is based on Leray-Schauder's Nonlinear Alternative.

LEMMA 3. (Nonlinear alternative for single-value maps) ([13]) *Let E be a Banach space, C be a closed, covex subset of E, U be an open subset of C and $0 \in U$. Suppose that $A : \bar{U} \to C$ is a continuous, compact (that is, $A(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) A has a fixed point in $\bar{U}$, or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda A(u)$.

THEOREM 6. *Assume that $f : [0,T/p] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In addition we suppose that:*

(H_3) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T/p],\mathbb{R}^+)$ such that

$$|f(t,x)| \leqslant p(t)\psi(\|x\|) \quad \textit{for each} \quad (t,x) \in [0,T/p] \times \mathbb{R};$$

(H_4) there exists a constant $N > 0$ such that $\frac{N}{\|p\|\psi(N)\Phi} > 1$, where Φ is defined by (23).

Then the boundary value problem (3)–(4) has at least one solution on $[0,T]$.

Proof. Firstly, we will show that the operator $\mathscr{A}$, difined by (22), maps bounded sets (balls) into bounded sets in $\mathscr{C}$. For a positive number R, let $B_R = \{x \in \mathscr{C} : \|x\| \leqslant R\}$ be a bounded ball in $\mathscr{C}$. Then for $t \in [0,T]$ we have

$$\begin{aligned}
|\mathscr{A}x(t)| &\leqslant \frac{|\alpha|}{|\Lambda|}\int_0^T |f(s,x(ps))|d_{p,q}s + \sum_{i=1}^m \frac{|\beta_i|}{|\Lambda|}\int_0^{\eta_i}\int_0^s |f(r,x(pr)d_{p,q}rd_{p_i,q_i}s \\
&\quad + \int_0^T |f(s,x(ps))d_{p,q}s \\
&\leqslant \|p\|\psi(R)\left\{\frac{|\alpha|}{|\Lambda|}T + \frac{1}{|\Lambda|}\sum_{i=1}^m \left(\frac{\eta_i^2|\beta_i|}{p_i+q_i}\right) + T\right\} \\
&= \|p\|\psi(R)\Phi.
\end{aligned}$$

Therefore, we conclude that $\|\mathscr{A}x\| \leqslant \|p\|\psi(R)\Phi$.

Secondly, we show that $\mathscr{A}$ maps bounded sets into equicontinuous sets of $\mathscr{C}$. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $x \in B_R$. Then we have

$$\begin{aligned}
|(\mathscr{A}x)(t_2) - (\mathscr{A}x)(t_1)| &= \left|\int_0^{t_2} f(s,x(ps))d_{p,q}s - \int_0^{t_1} f(s,x(ps))d_{p,q}s\right| \\
&\leqslant \|p\|\psi(R)|t_2 - t_1|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independent of $x \in B_R$ as $t_1 \to t_2$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathscr{A} : \mathscr{C} \to \mathscr{C}$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Lemma 3) once we have proved the boundedness of the set of all solutions to equations $x = \theta \mathscr{A} x$ for $\theta \in [0,1]$.

Let x be a solution. Then, for $t \in [0,T]$, and following the similar computations as in the first step, we have

$$\begin{aligned} \|x\| &\leqslant \|p\|\psi(\|x\|)\left\{\frac{|\alpha|}{|\Lambda|}T + \frac{1}{|\Lambda|}\sum_{i=1}^{m}\left(\frac{\eta_i^2|\beta_i|}{p_i+q_i}\right) + T\right\} \\ &= \|p\|\psi(\|x\|)\Phi. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\Phi} \leqslant 1.$$

In view of (H_4), there exists N such that $\|x\| \neq N$. Let us set

$$U = \{x \in \mathscr{C} : \|x\| < N\}. \tag{26}$$

Note that the operator $\mathscr{A} : \bar{U} \to \mathscr{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \theta \mathscr{A} x$ for some $\theta \in (0,1)$. Consequently, by nonlinear alternative of Leray-Schauder type we deduce that $\mathscr{A}$ has a fixed point in $\bar{U}$, which is a solution of the boundary value problem (3)–(4). This completes the proof. $\square$

4. Examples

EXAMPLE 1. Consider the following nonlocal boundary value problem for (p,q)-difference equation of the form

$$\begin{cases} D_{\frac{1}{2},\frac{1}{4}}x(t) = \dfrac{|\cos t|}{10}\left(\dfrac{x^2(t/2)+2|x(t/2)|}{1+|x(t/2)|}\right) + e^t, \quad t \in [0,4], \\ x(0) = \dfrac{1}{5}x(2) + \dfrac{1}{3}\displaystyle\int_0^{1/3} x(s)d_{\frac{1}{5},\frac{1}{7}}s + \dfrac{2}{3}\displaystyle\int_0^{1/4} x(s)d_{\frac{3}{4},\frac{3}{5}}s + \dfrac{4}{3}\displaystyle\int_0^{2/3} x(s)d_{\frac{4}{5},\frac{1}{2}}s. \end{cases} \tag{27}$$

Here $p = 1/2$, $q = 1/4$, $T = 2$, $\alpha = 1/5$, $m = 3$, $\beta_1 = 1/3$, $\beta_2 = 2/3$, $\beta_3 = 4/3$, $\eta_1 = 1/3$, $\eta_2 = 1/4$, $\eta_3 = 2/3$, $p_1 = 1/5$, $p_2 = 3/4$, $p_3 = 4/5$, $q_1 = 1/7$, $q_2 = 3/5$, $q_3 = 1/2$ and $f(t,x) = |\cos t|(x^2+2|x|)/(10(1+|x|)) + e^t$. Since

$$|f(t,x) - f(t,y)| \leqslant (1/5)|x-y|,$$

then (H_1) is satisfied with $L = 1/5$. We can show that $\Lambda = 1 - \alpha - \sum_{i=1}^{m}\beta_i\eta_i \approx -0.3666666666$ and

$$\Phi = \frac{|\alpha|}{|\Lambda|}T + \frac{1}{|\Lambda|}\sum_{i=1}^{m}\left(\frac{\eta_i^2|\beta_i|}{p_i+q_i}\right) + T = 4.7128982.$$

Then we have

$$L\Phi = \frac{1}{5}(4.7128982) = 0.94257964 < 1.$$

Hence, by Theorem 3.1, the nonlocal boundary value problem (27) has a unique solution on $[0,2]$.

EXAMPLE 2. Consider the following nonlocal boundary value problem for (p,q)-difference equation of the form

$$\begin{cases} D_{\frac{3}{4},\frac{3}{5}}x(t) = \left(\frac{t}{32}+\frac{1}{4}\right)\frac{|x(3t/4)|}{1+|x(3t/4)|}+\frac{3}{8}, \quad t \in [0,4/5], \\ x(0) = \frac{5}{6}x\left(\frac{3}{5}\right)+\frac{1}{3}\int_0^{1/5}x(s)d_{\frac{1}{2},\frac{1}{4}}s+\frac{2}{5}\int_0^{1/3}x(s)d_{\frac{2}{3},\frac{2}{5}}s+\frac{4}{7}\int_0^{1/2}x(s)d_{\frac{4}{3},\frac{3}{7}}s. \end{cases} \tag{28}$$

Here $p=3/4$, $q=3/5$, $T=3/5$, $\alpha=5/6$, $m=3$, $\beta_1=1/3$, $\beta_2=2/5$, $\beta_3=4/7$, $\eta_1=1/5$, $\eta_2=1/3$, $\eta_3=1/2$, $p_1=1/2$, $p_2=2/3$, $p_3=4/3$, $q_1=1/4$, $q_2=2/5$, $q_3=3/7$ and $f(t,x)=((t/32)+(1/4))|x|/(1+|x|)+3/8$. We choose $g(t)=(t/32)+(1/4)$. Then we find that

$$\Lambda = 1-\alpha-\sum_{i=1}^{m}\beta_i\eta_i = -0.319047619$$

and

$$G^* = \left(\frac{|\alpha|+|\Lambda|}{|\Lambda|}\right)\int_0^T g(s)d_{p,q}s+\frac{1}{|\Lambda|}\sum_{i=1}^{m}|\beta_i|\int_0^{\eta_i}\int_0^s g(s)d_{p,q}rd_{p_i,q_i}s = 0.5735834475.$$

Clearly,

$$\begin{aligned} |f(t,x)-f(t,y)| &= \left(\frac{t}{32}+\frac{1}{4}\right)\left(\frac{|x|-|y|}{1+|x|+|y|+|x||y|}\right) \\ &\leqslant \left(\frac{t}{32}+\frac{1}{4}\right)\left(\frac{|x-y|}{0.5735834475+|x-y|}\right). \end{aligned}$$

Hence, by Theorem 3.4, the nonlocal boundary value problem (28) has a unique solution on $[0,3/5]$.

EXAMPLE 3. Consider the following nonlocal boundary value problem for (p,q)-difference equation of the form

$$\begin{cases} D_{\frac{1}{4},\frac{1}{6}}x(t) = \frac{1}{(2+t)^2}\left(\frac{2x^{10}(t/4)}{1+|x^9(t/4)|}+3\right), \quad t \in [0,2], \\ x(0) = \frac{4}{3}x\left(\frac{1}{2}\right)+\frac{2}{3}\int_0^{1/7}x(s)d_{\frac{1}{3},\frac{1}{4}}s+\frac{5}{6}\int_0^{1/6}x(s)d_{\frac{2}{5},\frac{1}{3}}s+\frac{4}{5}\int_0^{1/2}x(s)d_{\frac{2}{3},\frac{1}{2}}s. \end{cases} \tag{29}$$

Here $p = 1/4$, $q = 1/6$, $T = 1/2$, $\alpha = 4/3$, $m = 3$, $\beta_1 = 2/3$, $\beta_2 = 5/6$, $\beta_3 = 4/5$, $\eta_1 = 1/7$, $\eta_2 = 1/6$, $\eta_3 = 1/2$, $p_1 = 1/3$, $p_2 = 2/5$, $p_3 = 2/3$, $q_1 = 1/4$, $q_2 = 1/3$, $q_3 = 1/2$ and $f(t,x) = (2x^{10}/(1+|x|^9)+3)/(2+t)^2$. Clearly,

$$|f(t,x)| = \left|\frac{1}{(2+t)^2}\left(\frac{2x^{10}}{1+|x|^9}+3\right)\right| \leqslant \frac{1}{(2+t)^2}(2|x|+3).$$

Choosing $p(t) = 1/(2+t)^2$ and $\psi(|x|) = 2|x|+3$, we can show that $\Lambda = -0.9674603175$ and $\Phi = 1.4180088065$. Then we have

$$\frac{N}{(\frac{1}{4})(2N+3)(1.4180088065)} > 1,$$

which implies that there exists a constant $N > 3.654717173$. Hence, by Theorem 3.9, the nonlocal boundary value problem (29) has at least one solution on $[0,1/2]$.

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Nattapong Kamsrisuk
Nonlinear Dynamic Analysis Research Center, Department of Mathematics
Faculty of Applied Science, King Mongkut's University of
Technology North Bangkok
Bangkok, 10800 Thailand
e-mail: lllarte@hotmail.com

Chanon Promsakon
Nonlinear Dynamic Analysis Research Center, Department of Mathematics
Faculty of Applied Science, King Mongkut's University of
Technology North Bangkok
Bangkok, 10800 Thailand
e-mail: chanon.p@sci.kmutnb.ac.th

Sotiris K. Ntouyas
Department of Mathematics
University of Ioannina
451 10 Ioannina, Greece
and
Nonlinear Analysis and Applied Mathematics (NAAM) – Research Group
Department of Mathematics, Faculty of Science
King Abdulaziz University
P. O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: sntouyas@uoi.gr

Jessada Tariboon
Nonlinear Dynamic Analysis Research Center, Department of Mathematics
Faculty of Applied Science, King Mongkut's University of
Technology North Bangkok
Bangkok, 10800 Thailand
and
Centre of Excellence in Mathematics
CHE, Sri Ayutthaya Rd., Bangkok 10400, Thailand
e-mail: jessada.t@sci.kmutnb.ac.th