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ON SIGN–CHANGING SOLUTIONS FOR RESONANT (p,q)–LAPLACE EQUATIONS

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Abstract. We provide two existence results for sign-changing solutions to the Dirichlet problem for the family of equations $-\Delta_p u - \Delta_q u = \alpha|u|^{p-2}u + \beta|u|^{q-2}u$, where $1 < q < p$ and α, β are parameters. First, we show the existence in the resonant case $\alpha \in \sigma(-\Delta_p)$ for sufficiently large β, thereby generalizing previously known results. The obtained solutions have negative energy. Second, we show the existence for any $\beta \geqslant \lambda_1(q)$ and sufficiently large α under an additional nonresonant assumption, where $\lambda_1(q)$ is the first eigenvalue of the q-Laplacian. The obtained solutions have positive energy.

1. Introduction and main results

In this note, we consider the following generalized eigenvalue problem:

$$\begin{cases} -\Delta_p u - \Delta_q u = \alpha|u|^{p-2}u + \beta|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (GEV;\alpha,\beta)$$

where $1 < q < p < \infty$, $\alpha, \beta \in \mathbb{R}$, and $\Delta_r u := \operatorname{div}\left(|\nabla u|^{r-2}\nabla u\right)$ with $r \in \{p,q\}$ is the r-Laplace operator. In view of the symmetry, the assumption $q < p$ is imposed without loss of generality. We assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary $\partial\Omega$, $N \geqslant 1$.

The problem $(GEV;\alpha,\beta)$ is of variational type with the energy functional $E_{\alpha,\beta} \in C^1(W_0^{1,p},\mathbb{R})$ defined by

$$E_{\alpha,\beta}(u) := \frac{1}{p}H_\alpha(u) + \frac{1}{q}G_\beta(u),$$

where

$$H_\alpha(u) := \|\nabla u\|_p^p - \alpha\|u\|_p^p \quad \text{and} \quad G_\beta(u) := \|\nabla u\|_q^q - \beta\|u\|_q^q.$$

Here we denote $W_0^{1,r} := W_0^{1,r}(\Omega)$, and $\|u\|_r := \left(\int_\Omega |u|^r\,dx\right)^{1/r}$ stands for the norm of $L^r(\Omega)$, $r \in (1,\infty)$.

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By definition, critical points of $E_{\alpha,\beta}$ are (weak) solutions to $(GEV;\alpha,\beta)$. Notice that any weak solution belongs to $C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$, see [2, Remark 1.1]. If a solution u can be split as $u = u^+ - u^-$, where $u^\pm := \max\{\pm u, 0\} \not\equiv 0$ in Ω, then u is called *nodal* or *sign-changing* solution.

Various boundary value problems with "mixed" differential operators of the (p,q)-Laplacian type are actively investigated nowadays, see [12] and references below for a survey of the corresponding results, as well as for the physical backgrounds of the (p,q)-Laplacian. The particular problem $(GEV;\alpha,\beta)$ appears to be interesting since it can be considered as a pure combination of the (homogeneous) eigenvalue problems for the p- and q-Laplacians, which forces the associated energy functional to demonstrate the indefinite-type behavior in the sense of signs of its critical points, and hence combined effects of positive and negative terms can be observed. The systematic study of the existence of constant-sign solutions to $(GEV;\alpha,\beta)$ was performed in [1, 3, 11, 14], see also references therein. On the other hand, the existence of sign-changing solutions to $(GEV;\alpha,\beta)$ was the subject of the work [2].

The main aim of the present note is to develop several existence and nonexistence results for nodal solutions to $(GEV;\alpha,\beta)$ obtained in [2]. The note is organized as follows. In the subsequent subsections, we introduce notations and formulate the main results. In Section 2, we give the corresponding proofs.

1.1. Notations

We say that λ is an *eigenvalue* of the r-Laplacian ($r > 1$), if the problem

$$\begin{cases} -\Delta_r u = \lambda |u|^{r-2} u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{$EV;r,\lambda$}$$

has a nontrivial (weak) solution. Analogously to the linear case, the set of all eigenvalues of $-\Delta_r$ (its spectrum) will be denoted as $\sigma(-\Delta_r)$. Along this note, we will work with two sequences of eigenvalues of $-\Delta_r$. The first one, denoted here as $\{\lambda_k(r)\}_{k\in\mathbb{N}}$, was introduced in [8] and can be defined as

$$\lambda_k(r) := \inf\left\{ \max_{z\in S^{k-1}} \|\nabla h(z)\|_r^r : \; h \in C(S^{k-1}, M(r)), \; h \text{ is odd} \right\}, \tag{1}$$

where S^{k-1} is the unit sphere in $\mathbb{R}^k$ and

$$M(r) := \{u \in W_0^{1,r} : \; \|u\|_r = 1\}.$$

The second sequence, denoted here as $\{\mu_k(r)\}_{k\in\mathbb{N}}$, is based on the $\mathbb{Z}_2$-cohomological index i of Fadell and Rabinowitz [10], and can be defined as follows (see, e.g., [13]):

$$\mu_k(r) := \inf\left\{ \sup_{u\in A} \frac{1}{\|u\|_r^r} \; : \; A \subset S_r, \; -A = A, \; i(A) \geqslant k \right\},$$
$$S_r := \{u \in W_0^{1,r} : \; \|\nabla u\|_r = 1\}.$$

Although it is not known whether either of these sequences (or even their union) exhausts $\sigma(-\Delta_r)$ (apart the case $N=1$ or $r=2$), the first and second eigenvalues of the r-Laplacian coincide with $\lambda_1(r)$, $\mu_1(r)$ and $\lambda_2(r)$, $\mu_2(r)$, respectively. Moreover, $\lambda_k(r), \mu_k(r) \to \infty$ as $k \to \infty$, see [8, 13]. Note that $\lambda_1(r)$ (and hence $\mu_1(r)$) is simple and isolated, and the corresponding eigenfunction (which can be taken strictly positive in Ω and has $C^1(\overline{\Omega})$-regularity) will be denoted as φ_r.

Finally, we will use the following notation for the eigenspace of $-\Delta_r$ at $\lambda \in \mathbb{R}$:

$$ES(r;\lambda) := \{v \in W_0^{1,r} : \ v \text{ is a solution of } (EV;r,\lambda)\}.$$

Evidently, $ES(r;\lambda) \neq \{0\}$ if and only if $\lambda \in \sigma(-\Delta_r)$.

1.2. Main results

Let us start with the *nonexistence* of nodal solutions to $(GEV;\alpha,\beta)$. The following result was proved in [2].

THEOREM 1. ([2, Theorems 1.3 and 1.4]) *Assume that*

$$(\alpha,\beta) \in (-\infty,\lambda_2(p)] \times (-\infty,\lambda_1(q)] \cup (-\infty,\lambda_1(p)] \times (-\infty,\lambda_2(q)].$$

Then $(GEV;\alpha,\beta)$ *has no nodal solution. Moreover, if* $N=1$, *then the nonexistence holds in* $(-\infty,\lambda_2(p)] \times (-\infty,\lambda_2(q)]$.

In order to extend the set of the nonexistence in the higher-dimensional case, we introduce the following family of critical points:

$$\beta_{\mathscr{N}}(\alpha) := \inf\left\{ \frac{\|\nabla u^+\|_q^q}{\|u^+\|_q^q} : \ u \in W_0^{1,p}, \ u^\pm \not\equiv 0, \ \frac{\|\nabla u^-\|_p^p}{\|u^-\|_p^p} \leqslant \alpha \leqslant \frac{\|\nabla u^+\|_p^p}{\|u^+\|_p^p} \right\}, \tag{2}$$

and we put $\beta_{\mathscr{N}}(\alpha) = \infty$ if the admissible set is empty. In Lemma 1 below, we study some basic properties of $\beta_{\mathscr{N}}(\alpha)$. In particular, we show that $\beta_{\mathscr{N}}(\alpha) < \infty$ if and only if $\alpha > \lambda_1(p)$, and $\beta_{\mathscr{N}}(\alpha) > \lambda_1(q)$ provided $\alpha > \lambda_1(p)$, see Fig. 1.

We generalize Theorem 1 in the following way.

PROPOSITION 1. *For any* $\alpha \leqslant \lambda_2(p)$ *and* $\beta \leqslant \min\{\lambda_2(q), \beta_{\mathscr{N}}(\alpha)\}$ *the problem* $(GEV;\alpha,\beta)$ *has no nodal solution.*

Let us now treat the *existence* of nodal solutions to $(GEV;\alpha,\beta)$. We define

$$k_\alpha := \min\{l \in \mathbb{N} : \alpha < \lambda_{l+1}(p)\} \tag{3}$$

and note that $\lambda_{k_\alpha+1}(q) \geqslant \lambda_2(q)$ for all $\alpha \in \mathbb{R}$. The following theorem was shown in [2].

THEOREM 2. ([2, Theorem 1.6]) *Let* $\alpha \in \overline{\mathbb{R} \setminus \sigma(-\Delta_p)}$. *Then for all*

$$\beta > \max\{\beta_{\mathscr{U}}^*(\alpha), \lambda_{k_\alpha+1}(q)\} \tag{4}$$

the problem ($GEV;\alpha,\beta$) *has a nodal solution u satisfying* $E_{\alpha,\beta}(u)<0$, *where*

$$\beta^*_{\mathscr{U}}(\alpha) := \sup\left\{\frac{\|\nabla\varphi\|_q^q}{\|\varphi\|_q^q}:\ \varphi\in ES(p;\alpha)\setminus\{0\}\right\}\quad \textit{for } \alpha\in\sigma(-\Delta_p),$$

and $\beta^*_{\mathscr{U}}(\alpha)=-\infty$ *provided* $\alpha\notin\sigma(-\Delta_p)$.

Actually, Theorem 2 remains valid under a weaker assumption on β, namely, $\beta>\lambda_{k_\alpha+1}(q)$ such that

$$G_\beta(\varphi)=\|\nabla\varphi\|_q^q-\beta\|\varphi\|_q^q\neq 0\quad \text{for all } \varphi\in ES(p;\alpha)\setminus\{0\}, \tag{5}$$

as it follows from the proof of [2, Theorem 1.6] in combination with [2, Remark 3.12]. In fact, the assumption (5) is needed to guarantee the validity of the Palais–Smale condition for $E_{\alpha,\beta}$, and the assumption (4) only represents an explicit lower bound for β satisfying (5). Roughly speaking, this can be interpreted in a way that the "true spectrum" of the (p,q)-Laplacian (in the sense that $E_{\alpha,\beta}$ possibly undergoes the lack of compactness) consists of points (α,β) where $\alpha\in\sigma(-\Delta_p)$ and β is such that $G_\beta(\varphi)=0$ for some $\varphi\in ES(p;\alpha)\setminus\{0\}$.

Let us emphasize that the assumption on α in Theorem 2 is restrictive, since it is not known whether $\overline{\mathbb{R}\setminus\sigma(-\Delta_p)}$ coincides with $\mathbb{R}$, apart the cases $N=1$ or $p=2$. One of the main aims of the present note is to dispose of the assumption on α. Thereby, we prove the following existence result.

THEOREM 3. *Let* $\alpha\in\mathbb{R}$. *Then for every* $\beta>\lambda_{k_\alpha+1}(q)$ *such that*

$$G_\beta(\varphi)=\|\nabla\varphi\|_q^q-\beta\|\varphi\|_q^q\neq 0\quad \textit{for all } \varphi\in ES(p;\alpha)\setminus\{0\} \tag{6}$$

the problem ($GEV;\alpha,\beta$) *has a nodal solution u satisfying* $E_{\alpha,\beta}(u)<0$, *where* k_α *is the natural number defined by* (3).

Note that if $\alpha=\lambda_1(p)$, then (6) is violated only for $\beta=\|\nabla\varphi_p\|_q^q/\|\varphi_p\|_q^q$. In this case, combining the existence result [3, Proposition 2.5 (iii)] with the general existence result for nodal solutions [2, Proposition 3.14], we can specify Theorem 3 as follows (cf. [2, Theorem 1.7]).

THEOREM 4. *Assume* $p>2q$ *and suppose that* $\partial\Omega$ *is connected if* $N\geqslant 2$. *Let* $(\alpha,\beta)\in(-\infty,\lambda_2(p))\times(\lambda_2(q),\infty)$. *Then* ($GEV;\alpha,\beta$) *has a nodal solution u satisfying* $E_{\alpha,\beta}(u)<0$.

Similarly to the statement of Theorem 3 one can ask whether it is possible to obtain the existence of nodal solutions to ($GEV;\alpha,\beta$) for a fixed β and some ranges of α. If $\beta\leqslant\lambda_1(q)$, then it was shown in [2, Theorems 1.3 and 1.5] that ($GEV;\alpha,\beta$) does not have any nodal solution for $\alpha\leqslant\lambda_2(p)$ and has at least one nodal solution with positive energy for $\alpha>\lambda_2(p)$. (More generally, there was constructed a curve $\beta_{\mathscr{L}}(\alpha)$ such that below this curve one can always find a nodal solution with positive energy, see Fig. 1.) However, it was not clear what happens for $\beta>\lambda_1(q)$ and sufficiently large $\alpha\geqslant\lambda_1(p)$. We have the following result in this direction.

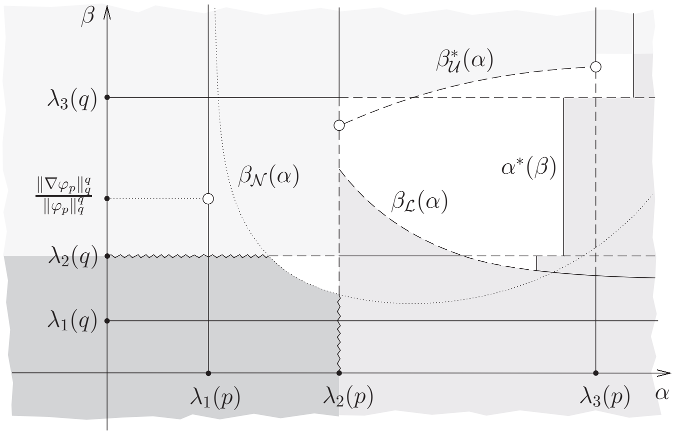

Figure 1: *Schematic plot of the sets of existence and nonexistence of nodal solutions to* ($GEV;\alpha,\beta$)*. Existence – light grays, nonexistence – dark gray.*

THEOREM 5. *Let $\beta \geqslant \lambda_1(q)$. Then there exists $\alpha^*(\beta) \geqslant \lambda_1(p)$ such that for any $\alpha > \alpha^*(\beta)$ satisfying* (6)*, the problem* ($GEV;\alpha,\beta$) *has a nodal solution u with $E_{\alpha,\beta}(u) > 0$.*

2. Proofs

Let us start with the nonexistence result given by Proposition 1. For this end, we first show the properties of $\beta_{\mathcal{N}}(\alpha)$ defined in (2).

LEMMA 1. *The following assertions hold:*

(i) $\beta_{\mathcal{N}}(\alpha) < \infty$ *if and only if* $\alpha > \lambda_1(p)$.

(ii) $\beta_{\mathcal{N}}(\alpha) > \lambda_1(q)$ *for all* $\alpha > \lambda_1(p)$.

(iii) $\beta_{\mathcal{N}}(\alpha) \to \infty$ *as* $\alpha \to \lambda_1(p)$.

Proof. (i) Let us show that the admissible set for $\beta_{\mathcal{N}}(\alpha)$ is nonempty if and only if $\alpha > \lambda_1(p)$. If $\alpha \leqslant \lambda_1(p)$, then the emptiness of the admissible set directly follows from the definition of $\lambda_1(p)$ and the fact that the corresponding first eigenfunction φ_p has a constant sign in Ω. Assume that $\alpha > \lambda_1(p)$. Since $\varphi_p \in W_0^{1,p}$, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that $\psi_n \to \varphi_p$ strongly in $W_0^{1,p}$ and $L^p(\Omega)$. Therefore, we can fix sufficiently large $n \in \mathbb{N}$ such that

$$\lambda_1(p) = \frac{\|\nabla\varphi_p\|_p^p}{\|\varphi_p\|_p^p} < \frac{\|\nabla\psi_n\|_p^p}{\|\psi_n\|_p^p} \leqslant \alpha.$$

On the other hand, let us take any $\xi \in C_0^\infty(B)$, where B is a sufficiently small ball such that $B \subset \Omega \setminus \operatorname{supp} \psi_n$ and

$$\alpha \leqslant \frac{\|\nabla \xi\|_p^p}{\|\xi\|_p^p}.$$

Thus, we conclude that $|\xi| - |\psi_n|$ is an admissible function for $\beta_{\mathcal{N}}(\alpha)$.

(ii) Suppose, by contradiction, that there exists $\alpha > \lambda_1(p)$ such that $\beta_{\mathcal{N}}(\alpha) = \lambda_1(q)$. By definition, this means that there exists $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}$ such that $u_n^\pm \not\equiv 0$,

$$\frac{\|\nabla u_n^-\|_p^p}{\|u_n^-\|_p^p} \leqslant \alpha \leqslant \frac{\|\nabla u_n^+\|_p^p}{\|u_n^+\|_p^p} \quad \text{and} \quad \frac{\|\nabla u_n^+\|_q^q}{\|u_n^+\|_q^q} \to \lambda_1(q) \tag{7}$$

as $n \to \infty$. (In fact, the second inequality in (7) is not needed for the proof.) Assume, without loss of generality, that $\|\nabla u_n^+\|_q^q = 1$ for all $n \in \mathbb{N}$. Then the simplicity of $\lambda_1(q)$ implies that $u_n^+ \to \varphi_q$ strongly in $W_0^{1,q}$ and $L^q(\Omega)$, and a.e. in Ω, up to a subsequence, where φ_q is the first eigenfunction of the q-Laplacian. Recall that $\varphi_q \in C^1(\overline{\Omega})$, $\varphi_q > 0$ in Ω, and hence $|\operatorname{supp} \varphi_q| \equiv |\Omega|$, where $|\cdot|$ denotes the Lebesgue measure of a set. Therefore, we get $|\operatorname{supp} u_n^+| \to |\Omega|$ as $n \to \infty$. Indeed, according to Egorov's theorem, for every $\varepsilon > 0$ there exists $E \subset \Omega$ with $|E| < \varepsilon$ such that $u_n^+ \to \varphi_q$ uniformly on $\Omega \setminus E$. Thus, for any $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that $u_n^+ > 0$ on $(\Omega \setminus E) \cap \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ for all $n \geqslant N$, which implies that $|\operatorname{supp} u_n^+| \geqslant |(\Omega \setminus E) \cap \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}| \to |\Omega|$ as $\varepsilon \to 0$.

Consequently, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $|\operatorname{supp} u_n^-| < \varepsilon$. However, this fact contradicts the first inequality in (7). Indeed, let us take some $\gamma > p$ if $p \geqslant N$, and $\gamma = p^*$ if $p < N$. Then, using the Sobolev embedding theorem, the first inequality in (7), and the Hölder inequality, we get

$$C\|u_n^-\|_\gamma^p \leqslant \|\nabla u_n^-\|_p^p \leqslant \alpha \|u_n^-\|_p^p \leqslant \alpha \|u_n^-\|_\gamma^p |\operatorname{supp} u_n^-|^{\frac{\gamma-p}{\gamma}}, \tag{8}$$

where $C > 0$ does not depend on $n \in \mathbb{N}$, which implies that

$$0 < \frac{C}{\alpha} \leqslant |\operatorname{supp} u_n^-|^{\frac{\gamma-p}{\gamma}}. \tag{9}$$

Taking $\varepsilon > 0$ small enough, we get a contradiction for sufficiently large $n \in \mathbb{N}$.

(iii) Suppose, by contradiction, that there exist a constant $C > 0$ and a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\alpha_n > \lambda_1(p)$ and $\alpha_n \to \lambda_1(p)$ as $n \to \infty$, but $\beta_{\mathcal{N}}(\alpha_n) < C$. Then for each $n \in \mathbb{N}$ we can find a point u_n from the admissible set for $\beta_{\mathcal{N}}(\alpha_n)$ such that

$$\frac{\|\nabla u_n^-\|_p^p}{\|u_n^-\|_p^p} \leqslant \alpha_n \leqslant \frac{\|\nabla u_n^+\|_p^p}{\|u_n^+\|_p^p} \quad \text{and} \quad \beta_{\mathcal{N}}(\alpha_n) \leqslant \frac{\|\nabla u_n^+\|_q^q}{\|u_n^+\|_q^q} \leqslant C. \tag{10}$$

Moreover, we can assume that u_n satisfies $\|\nabla u_n^-\|_p = 1$. Therefore, we see from the first inequality in (10) that $u_n^- \to \varphi_p$ strongly in $W_0^{1,p}$, $L^p(\Omega)$, and a.e. in Ω, up to a subsequence. Arguing as in the proof of assertion (ii), we get $|\operatorname{supp} u_n^-| \to |\Omega|$, and hence $|\operatorname{supp} u_n^+| \to 0$ as $n \to \infty$. However, the last inequality in (10) yields a lower bound of $|\operatorname{supp} u_n^+|$ by the same arguments as in (8) and (9) above. A contradiction. $\square$

REMARK 1. We do not know whether $\beta_{\mathcal{N}}(\alpha) \geqslant \lambda_2(q)$ for all $\alpha \in (\lambda_1(p), \lambda_2(p)]$.

Proof of Proposition 1. We will show the nonexistence of nodal solutions to ($GEV;\alpha,\beta$) for $\alpha \in (\lambda_1(p), \lambda_2(p)]$ and $\beta \leqslant \min\{\lambda_2(q), \beta_{\mathcal{N}}(\alpha)\}$, since the case $\alpha \leqslant \lambda_1(p)$ follows from Theorem 1. If we suppose, by contradiction, that there exists a nodal solution u in the considered region for (α,β), then u must satisfy, without loss of generality,

$$0 \geqslant H_\alpha(u^-) = -G_\beta(u^-), \quad 0 \leqslant H_\alpha(u^+) = -G_\beta(u^+), \tag{11}$$

see [2, Lemma 2.5]. This implies that u is an admissible function for the definition of $\beta_{\mathcal{N}}(\alpha)$, and hence $G_\beta(u^+) \geqslant 0$ since $\beta \leqslant \beta_{\mathcal{N}}(\alpha)$. If $G_\beta(u^+) > 0$, then we get a contradiction to (11). (In particular, this happens for $\beta < \beta_{\mathcal{N}}(\alpha)$.) Consequently, $G_\beta(u^+) = 0$. Therefore, we have $H_\alpha(u^+) = 0$ and $H_\alpha(u^-) \leqslant 0$, which reads as

$$\frac{\|\nabla u^+\|_p^p}{\|u_n^+\|_p^p} = \alpha, \quad \frac{\|\nabla u^-\|_p^p}{\|u_n^-\|_p^p} \leqslant \alpha.$$

Let us consider the map

$$h(s,t) = |s|^{\frac{2}{p}-1} s \frac{u^+}{\|u^+\|_p} + |t|^{\frac{2}{p}-1} t \frac{u^-}{\|u^-\|_p}.$$

It can be easily seen that $h \in C(S^1, M(p))$ and h is odd, and hence h is an admissible point for $\lambda_2(p)$, see (1). Thus,

$$\lambda_2(p) \leqslant \max_{(s,t)\in S^1} \|\nabla h(s,t)\|_p^p \leqslant \alpha,$$

which yields $\alpha = \lambda_2(p)$. Moreover, the deformation lemma (see, e.g., [9, Theorem 2.1 and Remark 2.3]) implies that there exists $(s_0,t_0) \in S^1$ such that $h(s_0,t_0) \in ES(p;\lambda_2(p)) \setminus \{0\}$, i.e. $h(s_0,t_0)$ is a second eigenfunction of the p-Laplacian, and hence u^+ and u^- are both first eigenfunctions of the p-Laplacian on their supports. (Note that any second eigenfunction has exactly two nodal domains, see [6, 9].) Recalling that u is a solution of ($GEV;\alpha,\beta$), we see that u^+ and u^- have to be first eigenfunctions of the q-Laplacian on their supports, as well. However, it contradicts the fact that first eigenfunctions of the p- and q-Laplacians are linearly independent provided $p \neq q$, see [3, Proposition A.1]. □

Let us now turn to the existence results. First we prove the following auxiliary fact about the Palais–Smale condition.

LEMMA 2. *Let $\alpha,\beta \in \mathbb{R}$. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}$ satisfies*

$$\sup_{n\in\mathbb{N}} |E_{\alpha,\beta}(u_n)| < \infty \quad \text{and} \quad \|E'_{\alpha,\beta}(u_n)\|_{(W_0^{1,p})^*} \to 0 \tag{12}$$

as $n \to \infty$. If $G_\beta(\varphi) \neq 0$ for any $\varphi \in ES(p,\alpha) \setminus \{0\}$ (in particular, if $\alpha \notin \sigma(-\Delta_p)$), then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}$ and has a subsequence strongly convergent in $W_0^{1,p}$ to a critical point of $E_{\alpha,\beta}$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}$ satisfy (12). Suppose, contrary to our claim, that $\|\nabla u_n\|_p \to \infty$ as $n \to \infty$. Then, applying [2, Lemma 3.3], we see that $v_n := u_n/\|\nabla u_n\|_p$ has a subsequence strongly convergent in $W_0^{1,p}$ to some $v_0 \in ES(p;\alpha)\setminus\{0\}$. Hence, we get a contradiction provided $\alpha \notin \sigma(-\Delta_p)$. If we assume that $\alpha \in \sigma(-\Delta_p)$, then

$$\left(\frac{p}{q}-1\right) G_\beta(v_n) = \frac{1}{\|\nabla u_n\|_p^q}\left(pE_{\alpha,\beta}(u_n) - \langle E'_{\alpha,\beta}(u_n), u_n\rangle\right) \to 0$$

as $n \to \infty$, which implies that $G_\beta(v_0) = 0$. However, this contradicts the assumptions of Lemma 2.

Thus, $\{u_n\}_{n\in\mathbb{N}}$ is a bounded Palais–Smale sequence for $E_{\alpha,\beta}$. In view of the (S_+)-property for the operator $-\Delta_p - \Delta_q$ (see [2, Remark 3.5]), we conclude that $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence strongly convergent in $W_0^{1,p}$ to a critical point of $E_{\alpha,\beta}$. $\square$

Proof of Theorem 3. The proof will be performed along the same lines as the proof of [2, Theorem 1.6]. As the main step, under the assumptions of Theorem 3, we prove the existence of an abstract solution to $(GEV;\alpha,\beta)$ via the linking arguments, cf. [2, Theorem 3.11]. The assumption $\alpha \in \overline{\mathbb{R}\setminus\sigma(-\Delta_p)}$ in [2, Theorem 3.11] was imposed in order to guarantee the existence of an appropriate bounded Palais–Smale sequence for $E_{\alpha,\beta}$. Here we are able to overcome this assumption and obtain the necessary Palais–Smale sequence for any $\alpha \in \mathbb{R}$.

Since the proof for the case $\alpha \in \overline{\mathbb{R}\setminus\sigma(-\Delta_p)}$ is given in [2, Theorem 3.11], from now on we will assume that

$$\alpha \in \sigma(-\Delta_p), \quad \alpha > \lambda_1(p) \quad \text{and} \quad \lambda_k(p) \leqslant \alpha < \lambda_{k+1}(p),$$

where $k = k_\alpha := \min\{l \in \mathbb{N} : \alpha < \lambda_{l+1}(p)\}$.

Since $\beta > \lambda_{k+1}(q)$ by the assumption, we apply [2, Lemma 3.9] to obtain an odd map $h_0 \in C(S^k, W_0^{1,p})$ and $t_0 > 0$ such that

$$\rho_0 := \max_{z\in S^k} E_{\alpha,\beta}(t_0 h_0(z)) < 0. \tag{13}$$

Consider now the set

$$Y(\lambda_{k+1}(p)) := \{u \in W_0^{1,p} : \ \|\nabla u\|_p^p \geqslant \lambda_{k+1}(p)\|u\|_p^p\}.$$

We easily deduce from $\alpha < \lambda_{k+1}(p)$ that

$$\delta_0 := \inf\{E_{\alpha,\beta}(u) : \ u \in Y(\lambda_{k+1}(p))\} > -\infty,$$

where the last inequality is given by [2, Lemma 3.2]. Moreover, $\delta_0 \leqslant \rho_0$ holds because $t_0 h_0|_{S^k_+}$ links $Y(\lambda_{k+1}(p))$, see [2, Lemma 3.1]. Here S^k_+ denotes a closed hemisphere of S^k.

We claim that for any $\varepsilon \in (0, |\rho_0|/2)$ there exists $u_\varepsilon \in W_0^{1,p}$ such that

$$\delta_0 - 2 \leqslant E_{\alpha,\beta}(u_\varepsilon) \leqslant \rho_0 + 2\varepsilon \ (<0) \quad \text{and} \quad \|E'_{\alpha,\beta}(u_\varepsilon)\|_{(W_0^{1,p})^*} < 2\varepsilon.$$

In fact, if we suppose that the above claim is false, then there exists some $\varepsilon \in (0, |\rho_0|/2)$ such that

$$\|E'_{\alpha,\beta}(u)\|_{(W_0^{1,p})^*} \geqslant 2\varepsilon \quad \text{for all } u \in E^{-1}_{\alpha,\beta}([\delta_0 - 2, \rho_0 + 2\varepsilon]).$$

Noting that $\alpha > \lambda_1(p)$, we get $\inf_{W_0^{1,p}} E_{\alpha,\beta} = -\infty$. Thus, arguing as in the first deformation lemma (see, e.g., [5]), we can find an odd map $\xi \in C(W_0^{1,p}, W_0^{1,p})$ satisfying $E_{\alpha,\beta}(\xi(t_0 h_0(z))) \leqslant \delta_0 - 1$ for all $z \in S^k$. However, since $\xi \circ (t_0 h_0|_{S_+^k})$ links $Y(\lambda_{k+1}(p))$ by [2, Lemma 3.1], we obtain a contradiction:

$$\delta_0 \leqslant \max_{z \in S^k} E_{\alpha,\beta}(\xi(t_0 h_0(z))) \leqslant \delta_0 - 1.$$

Consequently, our claim is shown.

Therefore, for any $n \in \mathbb{N}$, we can choose u_n satisfying

$$\delta_0 - 2 \leqslant E_{\alpha,\beta}(u_n) \leqslant \rho_0 + \frac{1}{n} \quad \text{and} \quad \|E'_{\alpha,\beta}(u_n)\|_{(W_0^{1,p})^*} < \frac{1}{n}.$$

Thanks to Lemma 2, $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W_0^{1,p}$ and has a subsequence strongly convergent in $W_0^{1,p}$ to a critical point u_0 of $E_{\alpha,\beta}$. Moreover, we get from (13) that

$$E_{\alpha,\beta}(u_0) \leqslant \rho_0 < 0,$$

whence $u_0 \neq 0$.

To finish the proof, let us consider the sign of u_0. If u_0 is a sign-changing solution to $(GEV;\alpha,\beta)$, then we are done. If u_0 has a constant sign in Ω, then we apply the general existence result [2, Proposition 3.14] to guarantee the existence of a sign-changing solution to $(GEV;\alpha,\beta)$ with negative energy also in this case. □

Proof of Theorem 5. To obtain the required existence we will adopt the arguments of the proof of [4, Theorem 1.4] which are based on the abstract critical point theorem [4, Theorem 1.6]. Here we will use the sequence of eigenvalues $\{\mu_k(q)\}_{k \in \mathbb{N}}$, see Section 1.1 for notations.

Let us fix an arbitrary $\beta \geqslant \mu_1(q)$. Then there exists $k \geqslant 1$ such that $\mu_k(q) \leqslant \beta < \mu_{k+1}(q)$. It was proved in [7, Theorem 2.3] that there is a symmetric cone

$$C \subset \{u \in W_0^{1,q} : \ \|\nabla u\|_q^q \leqslant \mu_k(q)\|u\|_q^q\} \equiv \{u \in W_0^{1,q} : \ G_{\mu_k(q)}(u) \leqslant 0\}$$

such that $C \cap M(q)$ is compact in $C^1(\overline{\Omega})$, and $i(C \setminus \{0\}) = k$.

Define the projection map π_p to the unit sphere S_p in $W_0^{1,p}$ as

$$\pi_p(u) = \frac{u}{\|\nabla u\|_p} \quad \text{for } u \in W_0^{1,p} \setminus \{0\}$$

and consider two sets

$$A_0 := \pi_p(C \cap M(q)), \quad B_0 := \{u \in S_p : \ G_{\mu_{k+1}(q)}(u) \geqslant 0\}.$$

Notice that A_0 is symmetric and compact in $W_0^{1,p}$, B_0 is symmetric and closed in $W_0^{1,p}$, and

$$i(A_0) = i(S_p \setminus B_0) = k,$$

see [4, p. 1969]. Thus, to apply [4, Theorem 1.6], we will show that, for sufficiently large α, there exist $R > r > 0$ and $v \in S_p \setminus A_0$ such that

$$\sup E_{\alpha,\beta}(A) \leqslant 0 < \inf E_{\alpha,\beta}(B) \quad \text{and} \quad \sup E_{\alpha,\beta}(X) < \infty, \tag{14}$$

where

$$A := \{tu : u \in A_0,\ 0 \leqslant t \leqslant R\} \cup \{R\pi_p((1-t)u + tv) : u \in A_0,\ 0 \leqslant t \leqslant 1\},$$
$$B := \{ru : u \in B_0\},$$
$$X := \{tu : u \in A,\ \|\nabla u\|_p = R,\ 0 \leqslant t \leqslant 1\}.$$

First we introduce the following critical value:

$$\underline{\alpha}(k) := \inf_{v \in S_p \setminus A_0} \max\left\{ \frac{\|(1-t)\nabla u + t\nabla v\|_p^p}{\|(1-t)u + tv\|_p^p} : u \in A_0,\ 0 \leqslant t \leqslant 1 \right\}. \tag{15}$$

Since A_0 is compact, we see that $\underline{\alpha}(k) < \infty$.

Let us assume that $\alpha > \underline{\alpha}(k)$. Then we can find $v \in S_p \setminus A_0$ such that

$$\underline{\alpha}(k) \leqslant \bar{\alpha}(k) := \max\left\{ \frac{\|(1-t)\nabla u + t\nabla v\|_p^p}{\|(1-t)u + tv\|_p^p} : u \in A_0,\ 0 \leqslant t \leqslant 1 \right\} < \alpha. \tag{16}$$

Consider the set

$$X_0 := \{\pi_p((1-t)u + tv) : u \in A_0,\ 0 \leqslant t \leqslant 1\}.$$

Since A_0 is compact, X_0 is compact as well, and therefore

$$\rho := \max_{u \in X_0} G_\beta(u) < \infty, \quad \min_{w \in X_0} \|w\|_p^p = \frac{1}{\bar{\alpha}(k)} > \frac{1}{\alpha},$$

see (16). Consequently, for every $u \in X_0$ it holds

$$H_\alpha(u) = 1 - \alpha\|u\|_p^p \leqslant 1 - \frac{\alpha}{\bar{\alpha}(k)} < 0,$$

and hence for any $u \in X_0$ and $t \geqslant 0$ we have

$$E_{\alpha,\beta}(tu) \leqslant \frac{t^p}{p}\left(1 - \frac{\alpha}{\bar{\alpha}(k)}\right) + \frac{t^q \rho}{q} =: C(t) < \infty.$$

Moreover, we can find $R > 0$ large enough such that $C(R) < 0$ by $q < p$. Furthermore, for any $u \in A_0 \subset X_0$ and $t \geqslant 0$ we get

$$E_{\alpha,\beta}(tu) \leqslant \frac{t^q}{q} G_\beta(u) \leqslant \frac{t^q}{q}(\mu_k(q) - \beta)\|u\|_q^q \leqslant 0.$$

Therefore, we obtain

$$\sup E_{\alpha,\beta}(A) \leqslant 0 \quad \text{and} \quad \sup E_{\alpha,\beta}(X) < \infty.$$

Let us show now that there exists $r \in (0,R)$ such that $\inf E_{\alpha,\beta}(B) > 0$. From the definition of B_0 we see that

$$E_{\alpha,\beta}(tu) = \frac{t^p}{p} H_\alpha(u) + \frac{t^q}{q} G_\beta(u) \geqslant \frac{t^p}{p} H_\alpha(u) + \frac{t^q}{q}(\mu_{k+1}(q) - \beta)\|u\|_q^q \tag{17}$$

for any $u \in B_0$ and $t > 0$. Clearly, if $H_\alpha(u) \geqslant 0$, then $E_{\alpha,\beta}(tu) > 0$ for any $t > 0$. First, we will prove that

$$\delta(t) := \inf\{E_{\alpha,\beta}(tu):\ u \in B_0,\ H_\alpha(u) \geqslant 0\} > 0$$

for any $t > 0$. Suppose, by contradiction, that we can find $t_0 > 0$ such that $\delta(t_0) = 0$. This yields the existence of $\{u_n\}_{n\in\mathbb{N}} \subset B_0$ such that $H_\alpha(u_n) \geqslant 0$ and $0 < E(t_0 u_n) \leqslant 1/n$ for all $n \in \mathbb{N}$. Recalling that $\beta < \mu_{k+1}(q)$, we see from (17) that this is possible if and only if $H_\alpha(u_n) \to 0$ and $\|u_n\|_q \to 0$. Since each $u_n \in S_p$, u_n converges weakly in $W_0^{1,p}$ and strongly in $L^p(\Omega)$, up to a subsequence, to some $u_0 \in W_0^{1,p}$. Then, the convergence $\|u_n\|_q \to 0$ implies that $u_0 \equiv 0$ a.e. in Ω. However, this leads to a contradiction since from $1 - \alpha\|u_n\|_p^p = H_\alpha(u_n) \to 0$ we get $\|u_n\|_p^p \to 1/\alpha > 0$ as $n \to \infty$, and hence $u_0 \not\equiv 0$ a.e. in Ω. Thus, we conclude that $\delta(t) > 0$ for all $t > 0$.

Assume now that $u \in B_0$ and $H_\alpha(u) < 0$. Then there exists $C > 0$ independent of u such that

$$\mu_1(p)^{1/p}\|u\|_p \leqslant 1 = \|\nabla u\|_p \leqslant C\|u\|_q,$$

see [14, Lemma 9]. Hence, we deduce from (17) that

$$E_{\alpha,\beta}(tu) \geqslant -\frac{t^p\alpha}{p\mu_1(p)} + \frac{t^q(\mu_{k+1}(q) - \beta)}{qC^q} > 0$$

for any $t > 0$ small enough, and such t is independent of u. Thus, we conclude that $\inf E_{\alpha,\beta}(u) > 0$ for all $u \in B = rB_0$ such that $H_\alpha(u) < 0$, where $r > 0$ is sufficiently small. Combining this with $\delta(r) > 0$, we derive the desired estimate $\inf E_{\alpha,\beta}(B) > 0$.

Therefore, (14) is satisfied. Applying now [4, Theorem 1.6] in combination with Lemma 2, we obtain for any $\alpha > \underline{\alpha}(k)$ a critical point u_α of $E_{\alpha,\beta}$ such that

$$0 < \inf E_{\alpha,\beta}(B) \leqslant c_\alpha = E_{\alpha,\beta}(u_\alpha) \leqslant \sup E_{\alpha,\beta}(X) < \infty.$$

Finally, let us show that we can take α larger, if necessary, to guarantee that u_α is sign-changing. For this end, we recall that [1] contains the construction of a curve $\mathscr{C}$ on the (α,β)-plane which separates the sets of the existence and nonexistence of positive solutions to $(GEV;\alpha,\beta)$, see [1, Theorem 2.2 and Proposition 4]. Moreover, in view of [1, Proposition 3], for any $\beta \geqslant \mu_1(q)$ we can find $\alpha_*(\beta) \geqslant \mu_1(p)$ such that $(GEV;\alpha,\beta)$ has no positive solutions for all $\alpha > \alpha_*(\beta)$. Therefore, taking any $\alpha > \alpha^*(\beta) := \max\{\underline{\alpha}(k), \alpha_*(\beta)\}$ such that (6) holds (where $\underline{\alpha}(k)$ is defined in (15)), we see that u_α is a sign-changing solution to $(GEV;\alpha,\beta)$ with positive energy. □

REMARK 2. Notice that, according to [1, Proposition 3], the critical value $\alpha_*(\beta)$ in the proof of Theorem 5 can be estimated from above by $\|\nabla\varphi_q\|_p^p/\|\varphi_q\|_p^p$ for any $\beta \geqslant \mu_1(q)$. This fact implies that the critical value $\alpha^*(\beta)$ in Theorem 5 possesses the following upper bound:

$$\alpha^*(\beta) \leqslant \max\left\{\underline{\alpha}(k), \frac{\|\nabla\varphi_q\|_p^p}{\|\varphi_q\|_p^p}\right\}.$$

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