Differential Equations & Applications
Volume 10, Number 2 (2018), 219–226 doi:10.7153/dea-2018-10-14

LYAPUNOV–TYPE INEQUALITIES FOR THIRD–ORDER LINEAR DIFFERENTIAL EQUATIONS UNDER THE NON–CONJUGATE BOUNDARY CONDITIONS

MUSTAFA FAHRI AKTAŞ AND DEVRIM ÇAKMAK

Dedicated to the memory
of Prof. Dr. Aydın Tiryaki
(June 1956 – May 2016)

(*Communicated by Hiroyuki Usami*)

Abstract. In this paper, we obtain the best constant in the Lyapunov-type inequality for third-order linear differential equations under the non-conjugate boundary conditions by bounding the Green function of the same problem. In this direction, to the best of our knowledge, there is no paper dealing with Lyapunov-type inequalities for the non-conjugate boundary value problems. By using such inequalities, we obtain sharp lower bounds for the eigenvalues of corresponding equations.

1. Introduction

There has been a great deal of research work on the theory of higher order differential equations for different types of boundary value problems. We refer the reader to the papers by Cabada [7], Jackson [19], Klaasen [20], Ma [23], Torres [27], Ward [30], Webb [31], Yang [32, 33], Yang [34, 35], Zhang and Sun [36], the monographs by Coppel [8], Gregus [16], Mawhin [24], and the references cited therein. There are also some useful methods to obtain Lyapunov-type inequalities for various types of boundary value problems [5, 25, 26]. Now, we give one of these methods as follows:

In 1954, Nehari [25] started with the Green's function

$$G(t,s)=\begin{cases}\dfrac{(s-a)(b-t)}{b-a}\ ; & a\leqslant s\leqslant t\\[2mm] \dfrac{(t-a)(b-s)}{b-a}\ ; & t\leqslant s\leqslant b\end{cases} \tag{1.1}$$

of the following problem

$$y''+q(t)y=0 \tag{1.2}$$

Mathematics subject classification (2010): 34C10, 34B05, 34L15.

Keywords and phrases: Lyapunov-type inequalities, Green's functions, non-conjugate boundary conditions.

$$y(a) = y(b) = 0, \tag{1.3}$$

where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and $y(t) \neq 0$ for $t \in (a,b)$. Thus, he wrote the nontrivial solution

$$y(t) = \int_a^b G(t,s)\, q(s)\, y(s)\, ds \tag{1.4}$$

for the problem (1.2)-(1.3). Then, by taking the absolute value of both sides of equation (1.4), choosing $t = t_0$ where $|y(t)|$ is maximized and canceling out $|y(t_0)|$ on both sides of corresponding inequality, he obtained

$$1 \leqslant \max_{a \leqslant t \leqslant b} \int_a^b |G(t,s)|\, |q(s)|\, ds. \tag{1.5}$$

Here, if we find $H(s)$ such that $\max_{a \leqslant t \leqslant b} |G(t,s)| \leqslant H(s)$, then we obtain the following inequality

$$1 \leqslant \int_a^b H(s)\, |q(s)|\, ds. \tag{1.6}$$

Note that we can put $H(t) = \dfrac{(t-a)(b-t)}{b-a}$ in the above problem. Moreover, if we take the absolute maximum of the function $H(t)$ for all $t \in [a,b]$, then we obtain the following inequality

$$\frac{4}{b-a} \leqslant \int_a^b |q(s)|\, ds \tag{1.7}$$

from (1.6), which is known as Lyapunov inequality [22]. This argument and its variants have been used many times to establish such an inequality. We see in the literature that by bounding $G(t,s)$ in various ways, we can obtain the best constant in the Lyapunov-type inequalities for different boundary value problems. Thus, the inequality (1.7) provides a lower bound for the distance between two consecutive zeros of y. The inequality (1.7) is the best possible in the sense that if the constant 4 in the left hand side of (1.7) is replaced by any larger constant, then there exists an example of (1.2) for which (1.7) no longer holds (see [17, p. 345], [21, p. 267]). This result has found many applications in areas like eigenvalue problems, stability, oscillation theory, disconjugate, etc. Since then, there have been several results to generalize the above linear equation in many directions [2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 17, 18, 21, 28, 29]. More recently, $(2n)$th order Green's functions have been typically used to deal with $(2n+1)$th order Lyapunov-type inequalities [1, 14, 15].

In this paper, we consider new Lyapunov-type inequalities for third-order linear differential equations of the form

$$y''' + q(t)\, y = 0, \tag{1.8}$$

where $q \in C([0,\infty), \mathbb{R})$ and $y(t)$ is a real solution of the equation (1.8) satisfying the non-conjugate boundary conditions

$$y(a) = y'(a) = y'(b) = 0, \tag{BC$_1$}$$

or

$$y'(a) = y(b) = y'(b) = 0 \tag{BC$_2$}$$

$a, b \in \mathbb{R}$ with $a < b$ are two points and $y(t) \not\equiv 0$ for $t \in (a,b)$.

To the best of our knowledge, there is no paper dealing with Lyapunov-type inequalities for the equation (1.8) under one of the non-conjugate boundary conditions (BC_i) for $i = 1,2$, by means of the properties of Green's functions. In this paper, we investigate the third-order linear differential equation (1.8) under one of the non-conjugate boundary conditions (BC_i) for $i = 1,2$. Firstly, we construct Green's functions, constant sign, for the equation (1.8) under one of the conditions (BC_i) for $i = 1,2$. And then, by using the Green's functions, we obtain the best constant in Lyapunov-type inequalities for the same problems. Finally, by using such inequalities, we obtain sharp lower bounds for the eigenvalues of corresponding equations.

2. Main results

We state an important lemma which we will use in the proofs of our main results. In this lemma, we construct the Green's functions for third-order nonhomogeneous differential equations

$$y''' = g(t) \tag{2.1}$$

under one of the non-conjugate boundary conditions (BC_i) for $i = 1,2$ as follows.

LEMMA 1. *If $y(t)$ is a solution of (2.1) satisfying one of the non-conjugate boundary conditions (BC_i) for $i = 1,2$, then*

$$y(t) = \int_a^b G_i(t,s)\, g(s)\, ds \tag{2.2}$$

holds, respectively, where

$$G_1(t,s) = \begin{cases} g_1(t,s) := -\dfrac{(s-a)^2 (b-t)}{2(b-a)} - \dfrac{(t-s)(s-a)(2b-a-t)}{2(b-a)} & ; \ a \leqslant s \leqslant t \\ g_2(t,s) := -\dfrac{(t-a)^2 (b-s)}{2(b-a)} & ; \ t \leqslant s \leqslant b \end{cases} \tag{2.3}$$

and

$$G_2(t,s) = \begin{cases} g_3(t,s) := \dfrac{(s-a)(b-t)^2}{2(b-a)} & ; \ a \leqslant s \leqslant t \\ g_4(t,s) := \dfrac{(t-a)(b-s)^2}{2(b-a)} + \dfrac{(s-t)(b-s)(b+t-2a)}{2(b-a)} & ; \ t \leqslant s \leqslant b. \end{cases} \tag{2.4}$$

Proof. Integrating equation (2.1) from a to t to find the solution $y(t)$, we get

$$y''(t) = d_2 + \int_a^t g(s)\, ds, \tag{2.5}$$

$$y'(t) = d_1 + d_2(t-a) + \int_a^t (t-s)\,g(s)\,ds, \tag{2.6}$$

$$y(t) = d_0 + d_1(t-a) + d_2\frac{(t-a)^2}{2} + \int_a^t \frac{(t-s)^2}{2} g(s)\,ds. \tag{2.7}$$

Thus, the general solution of (2.1) is (2.7). By using the boundary conditions (BC_1), we find the constants d_0, d_1, and d_2. Thus, $y(a) = y'(a) = 0$ imply $d_0 = d_1 = 0$, and $y'(b) = 0$ implies

$$d_2 = -\int_a^b \frac{b-s}{b-a} g(s)\,ds. \tag{2.8}$$

Substituting the constants d_0, d_1, and d_2 in the general solution (2.7), we get (2.2) for $i = 1$. Similarly, by using the boundary conditions (BC_2), we obtain (2.2) for $i = 2$, and hence the proof is omitted. This completes the proof. □

REMARK 1. It is easy to see that the functions $G_1(t,s)$ and $G_2(t,s)$ are constant sign, namely, $G_1(t,s) \leqslant 0$ and $G_2(t,s) \geqslant 0$ for all $t,s \in [a,b]$.

Now, we give one of main results of this paper.

THEOREM 1. *If $y(t)$ is a nontrivial solution of the third-order linear differential equations (1.8) satisfying the non-conjugate boundary conditions (BC_1), then the inequality*

$$1 \leqslant \int_a^b |G_1(t_0,s)|\,|q(s)|\,ds \tag{2.9}$$

holds, where $G_1(t,s)$ is given in (2.3) and $|y(t_0)| = \max\{|y(t)| : a \leqslant t \leqslant b\}$.

Proof. Let $y(a) = y'(a) = y'(b) = 0$ where $a,b \in \mathbb{R}$ with $a < b$ are two points, and y is not identically zero on (a,b). From (1.8) and (2.2) for $i = 1$, we get

$$|y(t)| \leqslant \int_a^b |G_1(t,s)|\,|q(s)|\,|y(s)|\,ds. \tag{2.10}$$

Pick $t_0 \in (a,b)$ so that $|y(t_0)| = \max\{|y(t)| : a \leqslant t \leqslant b\}$. From (2.10), we have

$$\begin{aligned} |y(t_0)| &\leqslant \int_a^b |G_1(t_0,s)|\,|q(s)|\,|y(s)|\,ds \\ &\leqslant |y(t_0)| \int_a^b |G_1(t_0,s)|\,|q(s)|\,ds. \end{aligned} \tag{2.11}$$

After dividing by $|y(t_0)|$ to the inequality (2.11), we obtain the inequality (2.9). This completes the proof. □

It is easy to see that the following result for (BC_2) can be easily obtained from those for (BC_1) and hence the proof is omitted.

THEOREM 2. *If $y(t)$ is a nontrivial solution of the third-order linear differential equations (1.8) satisfying the non-conjugate boundary conditions (BC_2), then the inequality*

$$1 \leqslant \int_a^b G_2(t_0,s)\,|q(s)|\,ds \tag{2.12}$$

holds, where $G_2(t,s)$ is given in (2.4) and $|y(t_0)| = \max\{|y(t)| : a \leqslant t \leqslant b\}$.

Now, we find the absolute minimum of Green's function $G_1(t,s)$. Consider

$$g_1(t,s) = -\frac{(s-a)^2(b-t)}{2(b-a)} - \frac{(t-s)(s-a)(2b-a-t)}{2(b-a)} \tag{2.13}$$

for $a \leqslant s \leqslant t$. $g_1(t,s)$ takes its minimum value at the point $(t_0,s_0) = \left(b, \dfrac{a+b}{2}\right)$, and its minimum value is $g_1\left(b, \dfrac{a+b}{2}\right) = -\dfrac{(b-a)^2}{8}$. Thus, we have

$$g_1(t,s) \geqslant g_1\left(b, \frac{a+b}{2}\right) = -\frac{(b-a)^2}{8} \quad \text{for } a \leqslant s \leqslant t.$$

Moreover, we have

$$\begin{aligned} g_2(t,s) &= -\frac{(t-a)^2(b-s)}{2(b-a)} \geqslant g_2(s) := -\frac{(s-a)^2(b-s)}{2(b-a)} \\ &\geqslant g_2\left(\frac{a+2b}{3}\right) = -\frac{2(b-a)^2}{27} \quad \text{for } t \leqslant s \leqslant b. \end{aligned}$$

Therefore, we get

$$G_1(t,s) \geqslant \min\left\{-\frac{(b-a)^2}{8}, -\frac{2(b-a)^2}{27}\right\} = -\frac{(b-a)^2}{8} \tag{2.14}$$

for all $t,s \in [a,b]$. Similarly, by finding the absolute maximum of Green's function $G_2(t,s)$, we have

$$G_2(t,s) \leqslant \frac{(b-a)^2}{8} \tag{2.15}$$

for all $t,s \in [a,b]$. Note that if we use the inequality (2.14) in (2.9) and (2.15) in (2.12), then we have the following result. Hence, the proof is omitted.

THEOREM 3. *If $y(t)$ is a nontrivial solution of the third-order linear differential equations (1.8) satisfying one of the non-conjugate boundary conditions (BC_i) for $i = 1,2$, then the inequality*

$$\frac{8}{(b-a)^2} \leqslant \int_a^b |q(s)|\,ds \tag{2.16}$$

holds.

REMARK 2. We believe that the Lyapunov-type inequality in Theorem 3 is the best possibility for the equation (1.8) under one of the boundary conditions (BC_i) for $i=1,2$ in the sense that the constant 8 in the left hand side of the inequality (2.16) cannot be replaced by any larger constant.

Here, we note that the following inequalities can be given from the Green's functions (2.3) and (2.4):

$$\begin{aligned} g_1(t,s) &\geqslant -\frac{(s-a)^2(b-s)}{2(b-a)} - \frac{(b-s)(s-a)(2b-a-s)}{2(b-a)} \\ &= -(s-a)(b-s), \end{aligned} \tag{2.17}$$

$$g_2(t,s) \geqslant -\frac{(s-a)^2(b-s)}{2(b-a)}, \tag{2.18}$$

$$g_3(t,s) \leqslant \frac{(s-a)(b-s)^2}{2(b-a)}, \tag{2.19}$$

$$\begin{aligned} g_4(t,s) &\leqslant \frac{(s-a)(b-s)^2}{2(b-a)} + \frac{(s-a)(b-s)(b+s-2a)}{2(b-a)} \\ &= (s-a)(b-s). \end{aligned} \tag{2.20}$$

Hence we get

$$G_1(t,s) \geqslant \min\left\{-(s-a)(b-s), -\frac{(s-a)^2(b-s)}{2(b-a)}\right\} = -(s-a)(b-s) \tag{2.21}$$

$$G_2(t,s) \leqslant \max\left\{\frac{(s-a)(b-s)^2}{2(b-a)}, (s-a)(b-s)\right\} = (s-a)(b-s). \tag{2.22}$$

Therefore, if we use the inequality (2.21) in (2.9) and (2.22) in (2.12), then we have the following result.

THEOREM 4. *If $y(t)$ is a nontrivial solution of the third-order linear differential equations (1.8) satisfying one of the non-conjugate boundary conditions (BC_i) for $i=1,2$, then the inequality*

$$1 \leqslant \int_a^b (s-a)(b-s)|q(s)|\,ds \tag{2.23}$$

holds.

We may adopt a different point of view and use (2.16) to obtain an extension of the following oscillation criterion due originally to Liapounoff (cf. [5]): $y''(t)$ and $y''(t)y^{-1}(t)$ are continuous for $a \leqslant t \leqslant b$, with $y(a)=y(b)=0$, then

$$\frac{4}{b-a} \leqslant \int_a^b \left|y''(s)y^{-1}(s)\right| ds \tag{2.24}$$

from the inequality (1.7). Thus, (2.16) leads to the following extension: If $y'''(t)$ and $y'''(t)y^{-1}(t)$ are continuous for $a \leqslant t \leqslant b$, $y(t)$ has one of the non-conjugate boundary conditions (BC_i) for $i = 1,2$, then

$$\frac{8}{(b-a)^2} \leqslant \int_a^b \left| y'''(s)y^{-1}(s) \right| ds.$$

Now, we give an another application of the obtained Lyapunov-type inequality (2.16) for the following eigenvalue problem: If $y(t)$ is a nontrivial solution of the equation

$$y''' + \lambda h(t) y = 0 \tag{2.25}$$

under one of the non-conjugate boundary conditions (BC_i) for $i = 1,2$, then we have

$$\frac{8}{(b-a)^2 \int_a^b |h(s)| ds} \leqslant |\lambda|. \tag{2.26}$$

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(Received April 28, 2017)

Mustafa Fahri Aktaş
Gazi University, Faculty of Sciences
Department of Mathematics
06500 Teknikokullar, Ankara, Turkey
e-mail: mfahri@gazi.edu.tr

Devrim Çakmak
Gazi University, Faculty of Education
Department of Mathematics Education
06500 Teknikokullar, Ankara, Turkey
e-mail: dcakmak@gazi.edu.tr