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UNIFORM EXPONENTIAL STABILITY IN THE SENSE OF HYERS AND ULAM FOR PERIODIC TIME VARYING LINEAR SYSTEMS

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Abstract. We prove that the uniform exponential stability of time depended p-periodic system

$$\dot{\Psi}(t) = \Pi(t)\Psi(t), \quad t \in \mathbb{R}_+, \;\; \Psi(t) \in \mathbb{C}^n$$

is equivalent to its Hyers–Ulam stability. As a tool, we consider the exact solution of the Cauchy problem

$$\begin{cases} \dot{\Theta}(t) = \Pi(t)\Theta(t) + e^{i\alpha t}\zeta(t), & t \in \mathbb{R}_+ \\ \Theta(0) = \Theta_0 \end{cases}$$

as the approximate solution of $\dot{\Psi}(t) = \Pi(t)\Psi(t)$, $t \in \mathbb{R}_+$, $\Psi(t) \in \mathbb{C}^n$, where α is any real number, $\zeta(t)$ with $\zeta(0) = 0$, is a p-periodic bounded function on the Banach space $\mathscr{S}(\mathbb{R}_+, \mathbb{C}^n)$. More precisely we prove that the system $\dot{\Psi}(t) = \Pi(t)\Psi(t)$, $t \in \mathbb{R}_+$, $\Psi(t) \in \mathbb{C}^n$ is Hyers–Ulam stable if and only if it is exponentially stable. We argue that Hyers-Ulam stability concept is quite significant in realistic problems in numerical analysis and economics.

1. Introduction

Theory of stability is of great interest, the recent advances of stability theory interact with spectral theory, harmonic analysis, modern topics of complex functions theory and also with control theory. The main interest is the asymptotic behavior of solutions and different types of stabilities in the study of such systems. Results related to stability of different system can be found in [3, 6, 5, 7, 9, 16, 15, 4, 22].

In 1940, some open problems were posed by S. M. Ulam, see [19] and [20]. The pursuit of solutions to these problems, to its generalizations and modifications for different classes of difference, functional, differential and integral equations, is a growing region of research and has led to the development of what is now frequently called Ulams type stability theory or the Hyers–Ulam stability theory. One of these problems refers to the stability of a certain functional equation. To this problem, the first answer was given by D. H. Hyers in 1941, [12]. Later on, it was called the Hyers–Ulam problem and its study became a widely studied subject for many mathematicians. M. Obłoza for the first time investigated the stability of differential equations, [17]. Just after, C. Alsina and R. Ger, [1], proved Hyers-Ulam stability of first order linear differential equations, which was then generalized for the Banach space valued first order

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linear differential equation, by S. E. Takahasi, H. Takagi, T. Miura and S. Miyajima in [18]. Different researchers presented their works with different approaches to study Hyers-Ulam stability, e.g., see [2, 11, 8, 10, 13, 14]. Very recently, in [21], Zada et al. generalized the concept of Hyers–Ulam stability of the non-autonomous p-periodic linear differential matrix system to its dichotomy.

In this paper we consider the first order linear non-autonomous system $\dot{\Psi}(t) = \Pi(t)\Psi(t)$, $t \in \mathbb{R}_+$, $\Psi(t) \in \mathbb{C}^n$. We show that the p-periodic system $\dot{\Psi}(t) = \Pi(t)\Psi(t)$ is Hyers–Ulam stable if and only if it is exponentially stable.

2. Notation and preliminaries

Let $\mathscr{C}$ be a complex Banach space and $\mathscr{S}(\mathscr{C})$ the Banach algebra of all bounded linear operators acting on $\mathscr{C}$. We denote by $\|\cdot\|$, the norms in $\mathscr{C}$ and in $\mathscr{S}(\mathscr{C})$. $\mathbb{R}_+$ denotes the set of all positive real numbers and the spectral radius of $\mathscr{W}$ is denoted by $r(\mathscr{W})$.

A family $\mathscr{U} = \{U(u,v) : u \geqslant v \geqslant 0\} \subset \mathscr{S}(\mathscr{C})$ is known to be p-periodic evolution family if

1. $U(u,w)U(w,v) = U(u,v)$ for all $u \geqslant v \geqslant w \geqslant 0$,
2. $U(u,u) = I$ for all $u \geqslant 0$,
3. for all $x \in \mathscr{C}$, the map $(u,v) \mapsto U(u,v)x : \{(u,v) \in \mathbb{R}^2 : u \geqslant v \geqslant 0\} \to \mathscr{C}$, is continuous,
4. $U(u+p, v+p) = U(u,v)$ for all $u \geqslant v \geqslant 0$.

A p-periodic evolution family also satisfies:

- $U(wp+u, wp+v) = U(u,v)$ for all $w \in \mathbb{N}$, for all $u \geqslant v \in \mathbb{R}_+$;
- $U(up, vp) = U((u-v)p, 0) = U(p,0)^{u-v}$ for all $u, v \in \mathbb{N}$, $u \geqslant v$.

The evolution family $\mathscr{U}$ is said to be exponentially bounded if there exist $\tau \in \mathbb{R}$ and $K_\tau \geqslant 0$ such that

$$\|U(u,v)\| \leqslant K_\tau e^{\tau(u-v)}, \ \forall \ u \geqslant v \geqslant 0.$$

The evolution family is uniformly exponentially stable if there exist $K > 0$ and $\tau > 0$ such that

$$\|U(u,v)\| \leqslant K e^{-\tau(u-v)}, \ \forall \ u \geqslant v \geqslant 0.$$

PROPOSITION 1. [3] *Consider a strongly continuous and p-periodic evolution family $\mathscr{U} = \{U(u,v) : u \geqslant v \geqslant 0\}$ acting on the Banach space $\mathscr{C}$. Then the following statement are equivalent:*

(A) *$\mathscr{U}$ is uniformly exponentially stable,*

(B) *there exist $M, \tau > 0$ such that $\|U(s,0)\| \leqslant M e^{-\tau s}$, for all $s \geqslant 0$,*

(C) *let* $\mathscr{W} = U(p,0)$, *then* $r(\mathscr{W}) < 1$,

(D) *let* $\mathscr{W} = U(p,0)$, *for each* $\alpha \in \mathbb{R}$, *one has*

$$\sup_{m\geqslant 1}\left\|\sum_{k=1}^{m} e^{-i\alpha k}\mathscr{W}^k\right\| := L(\alpha) < \infty.$$

By $\mathscr{S}(\mathbb{R}_+,\mathbb{C}^n)$ we denotes the space of all $\mathbb{C}^n$–valued bounded functions with "sup" norm and by $\mathfrak{X}_0^p(\mathbb{R}_+,\mathbb{C}^n)$ denotes the set of all continuous and p-periodic functions ζ with $\zeta(0) = 0$, where $\mathbb{C}^n$ the n-dimensional space of all n-tuples complex numbers.

3. Main result

Consider the time dependent p-periodic system

$$\dot{\Psi}(t) = \Pi(t)\Psi(t), \quad t \in \mathbb{R}_+, \;\; \Psi(t) \in \mathbb{C}^n, \tag{1}$$

where $\Pi(t+p) = \Pi(t)$ for all $t \in \mathbb{R}_+$.

Let $\mathfrak{X}_0^p(\mathbb{R}_+,\mathbb{C}^n)$ be the space of all p-periodic bounded functions $\zeta(t)$ with $\zeta(0) = 0$. Consider the Cauchy problem

$$\begin{cases} \dot{\Theta}(t) = \Pi(t)\Theta(t) + e^{i\alpha t}\zeta(t), \quad t \in \mathbb{R}_+ \\ \Theta(0) = \Theta_0. \end{cases} \tag{2}$$

The solution of the Cauchy problem (2) is

$$\Theta(t) = U(t,0)\Theta_0 + \int_0^t U(t,s)e^{i\alpha s}\zeta(s)\,ds. \tag{3}$$

DEFINITION 1. Let ε be a positive number. If there exists a constant $L > 0$ such that for every differentiable function Θ satisfying the relation

$$\sup_{t\in\mathbb{R}_+}\|\dot{\Theta}(t) - \Pi(t)\Theta(t)\| \leqslant \varepsilon,$$

there exists an exact solution $\Psi(t)$ of $\dot{\Psi}(t) = \Pi(t)\Psi(t)$ such that

$$\sup_{t\in\mathbb{R}_+}\|\Theta(t) - \Psi(t)\| \leqslant L\varepsilon, \tag{4}$$

then the system $\dot{\Psi}(t) = \Pi(t)\Psi(t)$ is said to be Hyers–Ulam stable.

REMARK 1. If $\Theta(t)$ is an approximate solution of $\dot{\Psi}(t) = \Pi(t)\Psi(t)$ then $\dot{\Theta}(t) \approx \Pi(t)\Theta(t)$. So let $\zeta(t)$ is an error function then $\Theta(t)$ is the exact solution of $\dot{\Theta}(t) = \Lambda(t)\Theta(t) + e^{i\alpha t}\zeta(t)$.

Thus with the help of Remark 1, Definition 1 can be modified as follows.

DEFINITION 2. Let ε be a positive real number. If there exists a constant $L>0$ such that, for every differentiable function $\Theta(t)$ satisfying (2) and $\sup_{t\in\mathbb{R}_+}\|\zeta(t)\|\leqslant\varepsilon$ for any $t\in\mathbb{R}_+$, there exists an exact solution $\Psi(t)$ of $\dot{\Psi}(t)=\Pi(t)\Psi(t)$ with $\Psi(0)=\Theta_0$ such that (4) holds, then the system $\dot{\Psi}(t)=\Pi(t)\Psi(t)$ is said to be Hyers–Ulam stable.

The following Lemma will help us to prove our main results.

LEMMA 1. *Let us consider the functions* $\lambda_1,\lambda_2:[0,p]\to\mathbb{C}$*, defined by;*

$$\lambda_1(s)=\begin{cases} s\ , & s\in[0,p/2)\\ p-s, & s\in[p/2,p]\end{cases}\quad \textit{and}\ \ \lambda_2(s)=s(p-s),\quad s\in[0,p].$$

Let

$$\Upsilon_j(\alpha)=\int_0^p\lambda_j(s)e^{i\alpha s}ds,\quad \textit{where}\quad j\in\{1,2\}.$$

Then it is easy to verify that $\Upsilon_1(\alpha)\neq 0$ *if and only if* $\alpha\in\mathscr{G}_1=\mathbb{C}\setminus\{\frac{4n\pi}{p}:\ n\in\mathbb{Z}\setminus\{0\}\}$ *and* $\Upsilon_2(\alpha)\neq 0$ *for all* $\alpha\in\mathscr{G}_2=\{\frac{4n\pi}{p}:\ n\in\mathbb{Z}\setminus\{0\}\}$.

Now we are in the position to state and prove our main result.

THEOREM 1. *Let for any real number α the equation (3) represent the approximate solution of the system $\dot{\Psi}(t)=\Pi(t)\Psi(t)$, where $e^{i\alpha t}\zeta(t)$ is the error function. Then the following two statements hold true.*

(1) *If the system $\dot{\Psi}(t)=\Pi(t)\Psi(t)$ is uniformly exponentially stable then for any $\zeta\in\mathfrak{X}_0^p(\mathbb{R}_+,\mathbb{C}^n)$ and any real number α the system $\dot{\Psi}(t)=\Pi(t)\Psi(t)$ is Hyers–Ulam stable.*

(2) *Let $\mathbb{C}:=\mathscr{G}_1\cup\mathscr{G}_2$. If for each real number α and each p-periodic function $\zeta(t)$ in $\mathscr{G}\subset\mathfrak{X}_0^p(\mathbb{R}_+,\mathbb{C}^n)$, the system $\dot{\Psi}(t)=\Pi(t)\Psi(t)$ is Hyers–Ulam stable. Then the system $\dot{\Psi}(t)=\Pi(t)\Psi(t)$ is uniformly exponentially stable.*

Proof. **(1)** Let $\varepsilon>0$, and $\Psi(t)$ is the exact solution of $\dot{\Psi}(t)=\Pi(t)\Psi(t)$ with $\Psi(0)=\Theta_0$ and $\Theta(t)$ is the approximate solution, which is an exact solution of the Cauchy problem (2) with $\sup_{t\in\mathbb{R}_+}\|\zeta(t)\|\leqslant\varepsilon$. Then

$$\begin{aligned}\sup_{t\in\mathbb{R}_+}\|\Theta(t)-\Psi(t)\| &= \sup_{t\in\mathbb{R}_+}\|U(t,0)\Theta_0+\int_0^t U(t,s)e^{i\alpha s}\zeta(s)\,ds-U(t,0)\Theta_0\|\\ &=\|\int_0^t U(t,s)e^{i\alpha s}\zeta(s)\,ds\|\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_0^t \|U(t,s)e^{i\alpha s}\zeta(s)\|\,ds \\
&\leqslant \int_0^t \|U(t,s)\|\|\zeta(s)\|\,ds \\
&\leqslant \int_0^t Ke^{-\beta(t-s)}\|\zeta(s)\|\,ds, \;\; \text{where } \; K>0,\; \beta>0 \\
&= Ke^{-\beta t}\int_0^t e^{\beta s}\|\zeta(s)\|\; ds \\
&\leqslant Ke^{-\beta t}\int_0^t e^{\beta s}\varepsilon ds, \\
&= \varepsilon\frac{K}{\beta}(1-e^{-\beta t}) \\
&\leqslant \frac{K}{\beta}\varepsilon \\
&= L\varepsilon, \;\; \text{where } \; L=\frac{K}{\beta}.
\end{aligned}
$$

Thus $\sup_{t\in\mathbb{R}_+}\|\Theta(t)-\Psi(t)\| \leqslant L\varepsilon$. Which implies that the system $\dot{\Psi}(t)=\Pi(t)\Psi(t)$ is Hyers–Ulam stable.

(2) Let $\mathscr{W}=U(p,0)$, $x\in\mathbb{C}^n$ and $\zeta_j\in\mathfrak{X}_0^p(\mathbb{R}_+,\mathbb{C}^n)$ such that:

$$\zeta_j(s)=\lambda_j(s)U(s,0)x, \;\; s\in[0,p].$$

where $\lambda_j(s)$ is defined in Lemma 1 for $j=1,2$. Thus for any natural number n we have

$$
\begin{aligned}
\Theta_j(np) &= \int_0^{np} U(np,s)e^{i\alpha s}\zeta_j(s)\,ds \\
&= \sum_{k=0}^{n-1}\int_{kp}^{kp+p} U(np,s)e^{i\alpha s}\zeta_j(s)\,ds \\
&= \sum_{k=0}^{n-1}\int_0^{p} U(np,kp+r)e^{i\alpha(kp+r)}\zeta_j(kp+r)\,dr \\
&= \sum_{k=0}^{n-1}\int_0^{p} e^{i\alpha kp}U\big((n-k)p,r\big)e^{i\alpha r}\lambda_j(r)U(r,0)x\,dr
\end{aligned}
$$

$$= \sum_{k=0}^{n-1} e^{i\alpha kp} U\big((n-k)p,0\big)x \int_0^p e^{i\alpha r}\lambda_j(r)\,dr$$

$$= \sum_{k=0}^{n-1} e^{i\alpha kp} U\big(p,0\big)^{n-k} x\Upsilon_j(\alpha)$$

$$= \Upsilon_j(\alpha)\sum_{k=0}^{n-1} e^{i\alpha kp}\mathscr{W}^{n-k}x.$$

In view of Lemma 1 we may write

$$\Theta_1(np)\frac{1}{\Upsilon_1(\alpha)} = \sum_{k=0}^{n-1} e^{i\alpha kp}\mathscr{W}^{n-k}x, \quad \text{for } \alpha \in \mathscr{G}_1, \tag{5}$$

and

$$\Theta_2(np)\frac{1}{\Upsilon_2(\alpha)} = \sum_{k=0}^{n-1} e^{i\alpha kp}\mathscr{W}^{n-k}x, \quad \text{for } \alpha \in \mathscr{G}_2. \tag{6}$$

From our assumptions it is obvious that the system $\dot{\Psi}(t) = \Pi(t)\Psi(t)$ is Hyers-Ulam stable, so

$$\|\int_0^{np} U(np,s)e^{i\alpha s}\zeta_j(s)\,ds\| \leqslant L\varepsilon,$$

for any natural number n, we conclude that $\Theta_i(np)$ for $i \in \{1,2\}$ are bounded functions, i.e. there exist two constants $\mathscr{L}_1$ and $\mathscr{L}_2$ such that

$$\|\Theta_1(np)\| \leqslant \mathscr{L}_1 \text{ and } \|\Theta_2(np)\| \leqslant \mathscr{L}_2 \text{ for all } n = 1,2,3\ldots.$$

Thus from (5) it follows that if $\alpha \in \mathscr{G}_1$ then

$$\Big\|\sum_{k=0}^{n-1} e^{i\alpha kp}\mathscr{W}^{n-k}x\Big\| \leqslant \frac{\mathscr{L}_1}{|\Upsilon_1(\alpha)|} = \mathrm{E}_1, \tag{7}$$

and from (6) it follows that if $\alpha \in \mathscr{G}_2$ then

$$\Big\|\sum_{k=0}^{n-1} e^{i\alpha kp}\mathscr{W}^{n-k}x\Big\| \leqslant \frac{\mathscr{L}_2}{|\Upsilon_2(\alpha)|} = \mathrm{E}_2. \tag{8}$$

Thus from (7) and (8), for any $\alpha \in \mathscr{G}_1 \cup \mathscr{G}_2 = \mathbb{C}$, we have

$$\Big\|\sum_{k=0}^{n-1} e^{i\alpha kp}\mathscr{W}^{n-k}x\Big\| \leqslant \mathrm{E}_1 + \mathrm{E}_2. \tag{9}$$

Taking $n-k = l$ then

$$\sum_{k=0}^{n-1} e^{i\alpha kp}\mathscr{W}^{n-k}x = e^{i\alpha n}\sum_{l=1}^{n} e^{-i\alpha lp}\mathscr{W}^{l}x.$$

So from (9) we have

$$\left\| \sum_{l=1}^{n} e^{-i\alpha l p} \mathscr{W}^{l} \right\| \leqslant L < \infty.$$

By Proposition 1 we conclude that the system $\dot{\Psi}(t) = \Pi(t)\Psi(t)$ is uniformly exponentially stable. $\square$

COROLLARY 1. *The system* (1) *is uniformly exponentially stable if and only if it is Hyers–Ulam stable.*

4. Conclusion

We showed that the uniform exponential stability of time dependent periodic system is equivalent to its Hyers-Ulam stability. This concept has applicable importance, it means that if one is studying Hyers–Ulam stable system then one does not have to reach the exact solution, which is quite difficult or time consuming. All what is required is to get a function which satisfies Definition 2. Hyers-Ulam stability guarantees that there is a close exact solution.

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