Differential
Equations
& Applications
Volume 10, Number 2 (2018), 235–238
doi:10.7153/dea-2018-10-16

ON A HINGED PLATE EQUATION OF NONCONSTANT THICKNESS

CRISTIAN-PAUL DANET

(*Communicated by Ulisse Stefanelli*)

Abstract. This note is concerned with the problem of existence and uniqueness of solutions for a fourth order boundary value problem that models the deflection of a hinged plate of nonconstant thickness.

1. Introduction

In this note we consider a planar bounded hinged plate Ω, $\partial\Omega \in C^2$ of thickness $D \geqslant D_0 > 0$, $D \in C^2(\Omega)$ with an external vertical load $f \in C(\mathbb{R})$. The physically relevant Steklov boundary value problem for the hinged plate is (see [4], p. 7)

$$\begin{cases} \Delta(D(x)\Delta u) - (1-\nu)[D,u] + c(x)f(u) = 0 \text{ in } \Omega, \\ u = \Delta u - (1-\nu)k\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

Here $c \in C^0(\overline{\Omega})$, $c_M \geqslant c(x) \geqslant c_m > 0$ in $\Omega, 0 < \nu < 1$ is the elastic constant (Poisson ratio), $[u,v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + v_{xx}u_{yy}$ and k is the curvature of Ω. The paper [8], p. 153–154 explains why the above boundary condition is physically relevant. On polygonal domains the boundary condition leads to $u = \Delta u = 0$ on $\partial\Omega$ with some singularity in the corners.

Note that the author (see [3]) studied using P functions methods the same equation but under the boundary conditions $u = \Delta u = 0$ on $\partial\Omega$ and proved that if

(i_1) $\Delta(1/c) \leqslant 0$,
(i_2) $\Delta D \geqslant (1-\nu)^2/2(1-2\nu)$, $\Delta D - 4D^{-1}|\nabla D|^2 \geqslant 0$,
(i_3) $F(s) = \int_0^s f(t)dt \geqslant 0$, $f'c > \gamma > 0$, $\gamma \geqslant D \geqslant D_{ij}D_{ij}$, $FF'' - (F')^2 \geqslant 0$,
(i_4) $\frac{\partial D}{\partial n} - 2kD < 0$ on $\partial\Omega$,

then the only classical solution $(C^4(\Omega) \cap C^2(\overline{\Omega}))$ is the trivial one.

It is the purpose of this note to use variational methods to show that existence/uniqueness results can be obtained only under the relaxed assumption (i_3), namely if

$$F(s) \geqslant -\beta|s|^{\alpha}, \quad \text{where } \beta > 0, \ 1 \leqslant \alpha < 2 \tag{2}$$

Mathematics subject classification (2010): 35J35, 35J40, 74K20.
Keywords and phrases: Fourth order, plate theory.
This research was carried out while the author was visiting the Mathematisches Institut der Universität zu Köln, Germany. This opportunity was made available by a KAAD grant. It is a pleasure to thank to Prof. dr. Guido Sweers for several valuable discussions which we had during my stay in Cologne.
I also would like to thank to the reviewer for his/her helpful comments and suggestions which led to the completion of this paper.

and/or

$$f' \geqslant 0 \text{ in } \mathbb{R}.$$

We mention that the methods developed in [3] can't be applied to the problem (1). Interestig positivity preserving properties related to the hinged plate model can be found in the works [2], [7], [8] and [9].

2. Main results

Let $X(\Omega) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

A weak solution of (1) is a function $u \in X(\Omega)$ such that

$$\int_\Omega \Big[D\big(\Delta u \Delta\varphi - (1-\nu)(u_{xx}\varphi_{xx} + u_{yy}\varphi_{yy} - 2u_{xy}\varphi_{xy})\big) + c(x) f\varphi \Big] dx = 0 \quad \forall \varphi \in X(\Omega).$$

It is well known (see [6], Chapter 1) that $X(\Omega)$ becomes a Hilbert space endowed with the scalar product

$$(u,v) \to \int_\Omega \Delta u \Delta v dx, \quad u,v \in X(\Omega).$$

This scalar product induces a norm equivalent to $|| \cdot ||_{W^{2,2}(\Omega)}$.

Weak solutions of (1) are critical points of the elastic energy functional $J : X(\Omega) \to \mathbb{R}$

$$J(u) = \int_\Omega \Big[\frac{D}{2} \Big((\Delta u)^2 - (1-\nu)[u,u] \Big) + c(x) F(u) \Big] dx.$$

LEMMA 2.1. *Suppose that F satisfies (2). Then there exists a minimizer $\tilde{u} \in X(\Omega)$ of J. If in addition we admit that $f' \geqslant 0$ then the minimizer is unique.*

Proof. Let us first establish that J is coercive.

By the Cauchy inequality

$$F(u) \geqslant -\varepsilon u^2 - C(\varepsilon,\alpha)\beta^{\frac{2}{2-\alpha}}, \quad \text{where } \varepsilon > 0. \tag{3}$$

We next use the inequality (see [5], p. 222)

$$\det \mathrm{A} \det \mathrm{B} \leqslant \left(\frac{\mathrm{trace AB}}{n} \right)^n, \quad \mathrm{A,B} \geqslant 0$$

to show that (we have taken $\mathrm{A{=}D}^2 u$, $\mathrm{B{=}I}_2$)

$$[u,u]/2 = \det \mathrm{D}^2 u \leqslant (\Delta u)^2/4, \tag{4}$$

where $\mathrm{D}^2 u$ is the Hessian matrix.

Combining Poincare's inequality and inequality (23)(we take into account that $u = 0$ on $\partial\Omega$), p. 20 in [6] we obtain that for all $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$

$$\int_\Omega u^2 dx \leqslant d^2 \int_\Omega (\Delta u)^2 dx, \tag{5}$$

where d is the width of a strip containing Ω.

Consequently by (3), (4) and (5) we get

$$J(u) \geqslant \frac{D_0(1+\nu)}{4} \mid\mid u \mid\mid^2_{X(\Omega)} - \varepsilon d^2 C_M \mid\mid u \mid\mid^2_{X(\Omega)} - C(\varepsilon, \alpha, \beta, C_M, \Omega).$$

Hence J is coercive if we choose ε sufficiently small.

We now show that $J(u)$ is weakly lower semicontinous on the reflexive space $X(\Omega)$.

Since

$$(\Delta u)^2 - (1-\nu)[u,u] = \nu(u_{xx}+u_{yy})^2 + (1-\nu)(u^2_{xx} + 2u^2_{xy} + u^2_{yy})$$

we get that

$$J_1(u) = \frac{1}{2}\int_\Omega D\Big((\Delta u)^2 - (1-\nu)[u,u]\Big)dx$$

is convex.

Hence $J(u)$ can be represented as the sum $J(u) = J_1(u) + J_2(u)$, where $J_1(u)$ is convex and $J_2(u) = \int_\Omega c(x)F(u)dx$ is sequentially weakly continous.

Therefore, $J(u)$ is weakly lower semicontinous by Criterion (6.1.3) in [1], p. 301.

We obtain thus the existence of the minimizer as required.

If $f' \geqslant 0$ then $J(u)$ is convex and uniqueness of the minimizer follows. $\square$

We are now able to prove the main result.

THEOREM 2.1. *Suppose that F satisfies (2). Then the boundary value problem (1) has at least one weak solution. If $f' \geqslant 0$ in $\mathbb{R}$ then the weak solution is unique. If in addition $F \geqslant 0$ in $\mathbb{R}$ then the unique solution is the trivial one.*

If we suppose that

$$\partial\Omega \in C^4,\ \ f \in C^0(\mathbb{R}),\ \ D \in C^2(\overline{\Omega}),\ \ c \in C^0(\overline{\Omega}), \tag{6}$$

then the boundary value problem (1) admits a unique strong solution $u \in W^{4,2}(\Omega)$.

Moreover if we admit that

$$\partial\Omega \in C^6,\ \ f \in C^2(\mathbb{R}),\ \ f'' \in L^\infty(\mathbb{R}),\ \ D \in C^4(\overline{\Omega}),\ \ c \in C^2(\overline{\Omega}), \tag{7}$$

then the solution is a classical solution.

Proof. The existence part of the theorem follows from Lemma 2.1, since the minimizer is the solution to (1).

If the assumption (6) is satisfied then the result follows by Theorem 2.20, p. 46, [4].

To show the regularity we use a bootstrapping argument and Theorem 2.20, p. 46, [4], which is a version of the classical result of Agmon-Douglis-Nirenberg.

Since $f \in C^2(\mathbb{R})$, $f'' \in L^\infty(\mathbb{R})$, $u \in W^{2,2}(\Omega)$ it follows that $f(u(x)) \in W^{2,2}(\Omega)$.

Now by Theorem 2.20, [4] (take $k=6,\ m=2,\ p=2$) it follows that there exists a solution to (1) in $W^{6,2}(\Omega)$.

Consequently, $u\in C^{4,1}(\Omega)$ by the Sobolev imbedding theorem, i.e. u is a classical solution.

$f'\geqslant 0$ and $F\geqslant 0$ implies $f(0)=0$ and hence $u\equiv 0$ is the unique solution. □

REMARK. If $\alpha=2$ we can see that proof of Theorem 2.1 still holds if

$$C_M\beta d^2<\frac{D_0(1+\nu)}{4}.$$

REFERENCES

[1] M. S. BERGER, *Nonlinearity and Functional Analysis. Lectures on Nonlinear Problems in Mathematical Analysis*, Academic Press, 1977.

[2] D. BUOSO, P. LAMBERTI, *Shape deformation for vibrating hinged plates*, Math. Methods Appl. Sci. **37**, 2 (2014), 237–244.

[3] C.-P. DANET, *Two maximum principles for a nonlinear fourth order equation from thin plate theory*, Electron. J. Qual. Theory Differ. Equ. **31**, (2014), 1–9.

[4] F. GAZOLLA, H.-C. GRUNAU, G. SWEERS, *Polyharmonic Boundary Value Problems*, Springer Verlag, 2010.

[5] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, 2001.

[6] O. A. LADYZHENSKAYA, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach Science Publishers, 1969.

[7] S. A. NAZAROV, A. STYLIANOU, G. SWEERS, *Hinged and supported plates with corners*, Z. Angew. Math. Phys. **63**, 5 (2012), 929–960.

[8] S. A. NAZAROV, G. SWEERS, *A hinged plate equation and iterated Dirichlet Laplace operator on domains with concave corners*, J. Differential Equations **233**, 1 (2007), 151–180.

[9] E. PARINI, A. STYLIANOU, *On the positivity preserving property of hinged plates*, SIAM J. Math. Anal. **41**, 5 (2009), 2031–2037.

(Received September 3, 2017)

Cristian-Paul Danet
Department of Applied Mathematics,
University of Craiova
Al. I. Cuza St., 13, 200585 Craiova, Romania
e-mail: cristiandanet@yahoo.com