

## ON A HINGED PLATE EQUATION OF NONCONSTANT THICKNESS

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*Abstract.* This note is concerned with the problem of existence and uniqueness of solutions for a fourth order boundary value problem that models the deflection of a hinged plate of nonconstant thickness.

### 1. Introduction

In this note we consider a planar bounded hinged plate  $\Omega$ ,  $\partial\Omega \in C^2$  of thickness  $D \geq D_0 > 0$ ,  $D \in C^2(\Omega)$  with an external vertical load  $f \in C(\mathbb{R})$ . The physically relevant Steklov boundary value problem for the hinged plate is (see [4], p. 7)

$$\begin{cases} \Delta(D(x)\Delta u) - (1 - \nu)[D, u] + c(x)f(u) = 0 \text{ in } \Omega, \\ u = \Delta u - (1 - \nu)k\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \quad (1)$$

Here  $c \in C^0(\overline{\Omega})$ ,  $c_M \geq c(x) \geq c_m > 0$  in  $\Omega$ ,  $0 < \nu < 1$  is the elastic constant (Poisson ratio),  $[u, v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + v_{xx}u_{yy}$  and  $k$  is the curvature of  $\Omega$ . The paper [8], p. 153–154 explains why the above boundary condition is physically relevant. On polygonal domains the boundary condition leads to  $u = \Delta u = 0$  on  $\partial\Omega$  with some singularity in the corners.

Note that the author (see [3]) studied using P functions methods the same equation but under the boundary conditions  $u = \Delta u = 0$  on  $\partial\Omega$  and proved that if

- (i<sub>1</sub>)  $\Delta(1/c) \leq 0$ ,
- (i<sub>2</sub>)  $\Delta D \geq (1 - \nu)^2/2(1 - 2\nu)$ ,  $\Delta D - 4D^{-1}|\nabla D|^2 \geq 0$ ,
- (i<sub>3</sub>)  $F(s) = \int_0^s f(t)dt \geq 0$ ,  $f'c > \gamma > 0$ ,  $\gamma \geq D \geq D_{ij}D_{ij}$ ,  $FF'' - (F')^2 \geq 0$ ,
- (i<sub>4</sub>)  $\frac{\partial D}{\partial n} - 2kD < 0$  on  $\partial\Omega$ ,

then the only classical solution ( $C^4(\Omega) \cap C^2(\overline{\Omega})$ ) is the trivial one.

It is the purpose of this note to use variational methods to show that existence/uniqueness results can be obtained only under the relaxed assumption (i<sub>3</sub>), namely if

$$F(s) \geq -\beta|s|^\alpha, \text{ where } \beta > 0, 1 \leq \alpha < 2 \quad (2)$$

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and/or

$$f' \geq 0 \text{ in } \mathbb{R}.$$

We mention that the methods developed in [3] can't be applied to the problem (1). Interestig positivity preserving properties related to the hinged plate model can be found in the works [2], [7], [8] and [9].

### 2. Main results

Let  $X(\Omega) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ .

A weak solution of (1) is a function  $u \in X(\Omega)$  such that

$$\int_{\Omega} \left[ D(\Delta u \Delta \varphi - (1 - \nu)(u_{xx}\varphi_{xx} + u_{yy}\varphi_{yy} - 2u_{xy}\varphi_{xy})) + c(x)f\varphi \right] dx = 0 \quad \forall \varphi \in X(\Omega).$$

It is well known (see [6], Chapter 1) that  $X(\Omega)$  becomes a Hilbert space endowed with the scalar product

$$(u, v) \rightarrow \int_{\Omega} \Delta u \Delta v dx, \quad u, v \in X(\Omega).$$

This scalar product induces a norm equivalent to  $\| \cdot \|_{W^{2,2}(\Omega)}$ .

Weak solutions of (1) are critical points of the elastic energy functional  $J : X(\Omega) \rightarrow \mathbb{R}$

$$J(u) = \int_{\Omega} \left[ \frac{D}{2} \left( (\Delta u)^2 - (1 - \nu)[u, u] \right) + c(x)F(u) \right] dx.$$

LEMMA 2.1. *Suppose that  $F$  satisfies (2). Then there exists a minimizer  $\tilde{u} \in X(\Omega)$  of  $J$ . If in addition we admit that  $f' \geq 0$  then the minimizer is unique.*

*Proof.* Let us first establish that  $J$  is coercive.

By the Cauchy inequality

$$F(u) \geq -\epsilon u^2 - C(\epsilon, \alpha)\beta^{\frac{2}{2-\alpha}}, \quad \text{where } \epsilon > 0. \tag{3}$$

We next use the inequality (see [5], p. 222)

$$\det A \det B \leq \left( \frac{\text{trace} AB}{n} \right)^n, \quad A, B \geq 0$$

to show that (we have taken  $A=D^2u, B=I_2$ )

$$[u, u]/2 = \det D^2u \leq (\Delta u)^2/4, \tag{4}$$

where  $D^2u$  is the Hessian matrix.

Combining Poincare's inequality and inequality (23)(we take into account that  $u = 0$  on  $\partial\Omega$ ), p. 20 in [6] we obtain that for all  $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$

$$\int_{\Omega} u^2 dx \leq d^2 \int_{\Omega} (\Delta u)^2 dx, \tag{5}$$

where  $d$  is the width of a strip containing  $\Omega$ .

Consequently by (3), (4) and (5) we get

$$J(u) \geq \frac{D_0(1+\nu)}{4} \|u\|_{X(\Omega)}^2 - \varepsilon d^2 C_M \|u\|_{X(\Omega)}^2 - C(\varepsilon, \alpha, \beta, C_M, \Omega).$$

Hence  $J$  is coercive if we choose  $\varepsilon$  sufficiently small.

We now show that  $J(u)$  is weakly lower semicontinuous on the reflexive space  $X(\Omega)$ .

Since

$$(\Delta u)^2 - (1-\nu)[u, u] = \nu(u_{xx} + u_{yy})^2 + (1-\nu)(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)$$

we get that

$$J_1(u) = \frac{1}{2} \int_{\Omega} D \left( (\Delta u)^2 - (1-\nu)[u, u] \right) dx$$

is convex.

Hence  $J(u)$  can be represented as the sum  $J(u) = J_1(u) + J_2(u)$ , where  $J_1(u)$  is convex and  $J_2(u) = \int_{\Omega} c(x)F(u)dx$  is sequentially weakly continuous.

Therefore,  $J(u)$  is weakly lower semicontinuous by Criterion (6.1.3) in [1], p. 301.

We obtain thus the existence of the minimizer as required.

If  $f' \geq 0$  then  $J(u)$  is convex and uniqueness of the minimizer follows.  $\square$

We are now able to prove the main result.

**THEOREM 2.1.** *Suppose that  $F$  satisfies (2). Then the boundary value problem (1) has at least one weak solution. If  $f' \geq 0$  in  $\mathbb{R}$  then the weak solution is unique. If in addition  $F \geq 0$  in  $\mathbb{R}$  then the unique solution is the trivial one.*

*If we suppose that*

$$\partial\Omega \in C^4, \quad f \in C^0(\mathbb{R}), \quad D \in C^2(\overline{\Omega}), \quad c \in C^0(\overline{\Omega}), \tag{6}$$

*then the boundary value problem (1) admits a unique strong solution  $u \in W^{4,2}(\Omega)$ .*

*Moreover if we admit that*

$$\partial\Omega \in C^6, \quad f \in C^2(\mathbb{R}), \quad f'' \in L^\infty(\mathbb{R}), \quad D \in C^4(\overline{\Omega}), \quad c \in C^2(\overline{\Omega}), \tag{7}$$

*then the solution is a classical solution.*

*Proof.* The existence part of the theorem follows from Lemma 2.1, since the minimizer is the solution to (1).

If the assumption (6) is satisfied then the result follows by Theorem 2.20, p. 46, [4].

To show the regularity we use a bootstrapping argument and Theorem 2.20, p. 46, [4], which is a version of the classical result of Agmon-Douglis-Nirenberg.

Since  $f \in C^2(\mathbb{R})$ ,  $f'' \in L^\infty(\mathbb{R})$ ,  $u \in W^{2,2}(\Omega)$  it follows that  $f(u(x)) \in W^{2,2}(\Omega)$ .

Now by Theorem 2.20, [4] (take  $k = 6$ ,  $m = 2$ ,  $p = 2$ ) it follows that there exists a solution to (1) in  $W^{6,2}(\Omega)$ .

Consequently,  $u \in C^{4,1}(\Omega)$  by the Sobolev imbedding theorem, i.e.  $u$  is a classical solution.

$f' \geq 0$  and  $F \geq 0$  implies  $f(0) = 0$  and hence  $u \equiv 0$  is the unique solution.  $\square$

REMARK. If  $\alpha = 2$  we can see that proof of Theorem 2.1 still holds if

$$C_M \beta d^2 < \frac{D_0(1 + \nu)}{4}.$$

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