Differential
Equations
& Applications
Volume 10, Number 2 (2018), 239–250
doi:10.7153/dea-2018-10-17

BOX–COUNTING DIMENSION OF OSCILLATORY SOLUTIONS TO THE EMDEN–FOWLER EQUATION

TAKANAO KANEMITSU AND SATOSHI TANAKA

(*Communicated by Darko Žubrinić*)

Abstract. The box-counting dimension of graphs of oscillatory solutions to the Emden-Fowler equation is studied. The half-linear equation is also considered.

1. Introduction

In this paper, we study the box-counting dimension of graphs of the oscillatory solutions to the Emden-Fowler equation

$$y'' + f(x)|y|^{\gamma-1}y = 0, \quad x \in (0,x_0], \tag{1.1}$$

where $\gamma > 0$, $\gamma \neq 1$ and

$$f \in C^2(0,x_0],\ f(x) > 0,\ f'(x) < 0,\ x \in (0,x_0],\ \lim_{x\to+0} f(x) = \infty. \tag{1.2}$$

The main result of the paper is stated in Theorem 1.1 below. A function $\lambda x^{-\sigma}$, $\lambda > 0$, $\sigma > 0$ is a typical example of $f(x)$ satisfying (1.2).

A solution y of (1.1) is said to be *oscillatory near* $x = 0$ if there exists $\{z_n\}_{n=1}^{\infty}$ such that $y(z_n) = 0$ for $n \in \mathbf{N}$ and $z_n \to 0$ as $n \to \infty$. Otherwise, it is said to be *nonoscillatory near* $x = 0$.

A study of the oscillatory solutions to the Emden-Fowler equation (1.1) has a long history. See, for example, [1, 2, 6, 7, 8, 10, 17]. For instance, the following result is well-known. (Consider the transformation $Y(t) = ty(t^{-1})$ on (1.1) and apply the celebrated results in [1] and [2].)

THEOREM A. *Let $\gamma > 0$, $\gamma \neq 1$, $f \in C(0,x_0]$ and $f(x) > 0$ for $x \in (0,x_0]$. Then every solution $y \in C^2(0,x_0]$ of (1.1) is oscillatory near $x = 0$ if and only if $x^{\gamma^*} f(x) \notin L^1(0,x_0]$, where $\gamma^* = \max\{\gamma, 1\}$.*

J. S. W. Wong [18] studied the rectifiability of oscillatory solutions to (1.1). A solution y of (1.1) is said to be *rectifiable* (resp. *nonrectifiable*) *oscillatory near* $x = 0$ if y is oscillatory near $x = 0$ and the length of y is finite (resp. infinite), that is,

$$\int_0^{x_0} \sqrt{1 + |y'(x)|^2}\,dx < \infty \quad (\text{resp. } = \infty).$$

Mathematics subject classification (2010): 34C10, 28A80.

Keywords and phrases: Oscillatory solution, box-counting dimension, Minkowski-Bouligand dimension, Minkowski content, Emden-Fowler equation, half-linear equation.

This work was supported by JSPS KAKENHI Grant Number 26400182.

Paper DEA-10-17

THEOREM B ([18, Theorem 1]). *Let* $\gamma > 0$ *and* $\gamma \neq 1$. *Assume that* $f \in C^2(0,x_0]$, $f(x) > 0$ *for* $x \in (0,x_0]$, $\lim_{x\to+0} f(x) = \infty$, $f^{-1}f'_+ \in L^1(0,x_0]$ *and* $(f^{-(\gamma^*+2)/(\gamma^*+1)}f')'$ $\in L^1(0,x_0]$, *where* $f'_+(x) = \max\{f'(x),0\}$ *and* $\gamma^* = \max\{\gamma,1\}$. *Then every nontrivial solution* y *of* (1.1) *is oscillatory near* $x = 0$ *and the following properties hold*:

(i) *y is rectifiable if* $f^{1/(\gamma+3)} \in L^1(0,x_0]$;

(ii) *y is nonrectifiable if* $f^{1/(\gamma+3)} \notin L^1(0,x_0]$.

Proposition 1.1 below shows that the box-counting dimension of rectifiable curves is 1. In this paper, we will obtain the box-counting dimension of nonrectifiable oscillatory solutions to (1.1). For linear differential equations, it has been studied by Pašić [11], Kwong, Pašić and Wong [9], and Pašić and Tanaka [12]. However, to the authors' knowledge, there is no study about it for the Emden-Fowler equations.

Let $\Gamma \subset \mathbf{R}^2$ be a bounded set. We define the *box-counting dimension* (*Minkowski-Bouligand dimension*) of Γ by

$$\dim_{\mathrm{B}} \Gamma = 2 - \lim_{\varepsilon\to+0} \frac{\log|\Gamma_\varepsilon|}{\log \varepsilon},$$

provided the limit exists, where Γ_ε denotes the ε-neighborhood of Γ defined by

$$\Gamma_\varepsilon = \{(x,y) \in \mathbf{R}^2 : d((x,y),\Gamma) \leqslant \varepsilon\},$$

$d((x,y),\Gamma)$ denotes the Euclidean distance from (x,y) to Γ, and $|\Gamma_\varepsilon|$ denotes the two-dimensional Lebesgue measure of Γ_ε. For any $d \geqslant 0$, the *lower d-dimensional Minkowski content* of Γ and the *upper d-dimensional Minkowski content* of Γ are defined by

$$\mathscr{M}^d_*(\Gamma) := \liminf_{\varepsilon\to+0} \varepsilon^{-(2-d)}|\Gamma_\varepsilon| \quad \text{and} \quad \mathscr{M}^{*d}(\Gamma) := \limsup_{\varepsilon\to+0} \varepsilon^{-(2-d)}|\Gamma_\varepsilon|,$$

respectively. If $\mathscr{M}^d_*(\Gamma) = \mathscr{M}^{*d}(\Gamma)$, then it is said to be the *d-dimensional Minkowski content* of Γ and denoted by $\mathscr{M}^d(\Gamma)$. More details on the above definitions can be found in Falconer [5] and Tricot [16]. It is easy to see that if $0 < \mathscr{M}^d_*(\Gamma) \leqslant \mathscr{M}^{*d}(\Gamma) < \infty$, then $\dim_{\mathrm{B}} \Gamma = d$. We have the following well-known result, see e.g. Tricot [16, §9.1, Theorem].

PROPOSITION 1.1. *Let* $\Gamma \subset \mathbf{R}^2$ *be a simple curve of finite length. Then*

$$\lim_{\varepsilon\to+0} \frac{|\Gamma_\varepsilon|}{2\varepsilon} = \mathrm{length}(\Gamma),$$

where $\mathrm{length}(\Gamma)$ *denotes the length of* Γ.

Therefore, the box-counting dimension of rectifiable curves is 1.

For each $y \in C(0,x_0]$, we define the graph $\Gamma(y)$ of y by

$$\Gamma(y) = \{(x,y(x)) : 0 < x \leqslant x_0\}.$$

The main result of this paper is as follows.

THEOREM 1.1. *Let $\gamma > 0$ and $\gamma \neq 1$. Assume that* (1.2) *holds,* $(f^{-(\gamma^*+2)/(\gamma^*+1)}f')' \in L^1(0,x_0]$ *and* $f^{1/(\gamma+3)} \notin L^1(0,x_0]$, *where* $\gamma^* = \max\{\gamma, 1\}$. *Assume, moreover, that there exists* $d \in (1,2)$ *such that*

$$\liminf_{x\to+0} [f(x)]^{\frac{2(2-d)}{\gamma+3}} \int_0^x [f(\xi)]^{-\frac{1}{\gamma+3}} d\xi > 0,$$
$$\limsup_{x\to+0} \left[x[f(x)]^{-\frac{1}{\gamma+3}}\right]^{\frac{d-1}{2-d}} \int_x^{x_0} [f(\xi)]^{\frac{1}{\gamma+3}} d\xi < \infty.$$

Then every nontrivial solution y of (1.1) *is oscillatory near $x=0$ and* $0 < \mathscr{M}_*^d(\Gamma(y)) \leqslant \mathscr{M}^{*d}(\Gamma(y)) < \infty$ *and hence* $\dim_B \Gamma(y) = d$.

We will prove Theorem 1.1 in Section 3.

Applying Theorem 1.1 to

$$y'' + \lambda x^{-\sigma}|y|^{\gamma-1}y = 0, \quad x \in (0,x_0], \tag{1.3}$$

we have the following corollary.

COROLLARY 1.1. *Let $\lambda > 0$, $\gamma > 0$, $\gamma \neq 1$ and $\sigma > \gamma+3$. Then every nontrivial solution y of* (1.3) *is oscillatory near $x = 0$,* $0 < \mathscr{M}_*^d(\Gamma(y)) \leqslant \mathscr{M}^{*d}(\Gamma(y)) < \infty$ *and* $\dim_B \Gamma(y) = d$, *where* $d = \frac{3}{2} - \frac{\gamma+3}{2\sigma}$.

REMARK 1.1. Corollary 1.1 answers the open problem raised by J. S. W. Wong [18]. Corollary 1.1 with $\gamma = 1$ was obtained by Pašić [11, Theorem 1.5]. From Theorem A, it follows that if $\sigma \leqslant \gamma^* + 1$, then (1.3) has a nonoscillatory solution near $x = 0$. Theorem B implies that if $\sigma > \gamma^* + 1$, then every nontrivial solution y of (1.3) is oscillatory near $x = 0$ and we have: y is rectifiable when $\gamma^* + 1 < \sigma < \gamma + 3$; y is nonrectifiable when $\sigma \geqslant \gamma + 3$. (See also [18, Corollary].) Proposition 1.1 shows that $\dim_B \Gamma(y) = 1$ when $\gamma^* + 1 < \sigma < \gamma + 3$. Consequently, we do not know the box-counting dimension of oscillatory solutions for the case where $\sigma = \gamma + 3$. We formulate the conjecture as follows. We note that Pašić [11, Theorem 1.5] proved that $\dim_B \Gamma(y) = 1$ for every nontrivial solution y of (1.3) with $\gamma = 1$ and $\sigma = \gamma + 3 = 4$.

CONJECTURE 1.1. *If $\lambda > 0$, $\gamma > 0$, $\gamma \neq 1$ and $\sigma = \gamma + 3$, then* $\dim_B \Gamma(y) = 1$ *for every nontrivial solution y of* (1.3).

J. S. W. Wong [18] gave the asymptotic behavior of oscillatory solutions to (1.1) as the following form

$$y(x) = [f(x)]^{-\frac{1}{\gamma+3}}[V(x)]^{\frac{1}{\gamma+1}} S(\varphi(x)),$$
$$y'(x) = -[f(x)]^{\frac{1}{\gamma+3}}[V(x)]^{\frac{1}{2}} S(\varphi(x)),$$

where $V \in C^1(0,x_0]$ satisfies $\lim_{x\to+0} V(x) = V_0$ for some $V_0 > 0$ and $S \in C(\mathbf{R})$ is a periodic function having zeros. See Proposition 3.1 below. In this paper, we give a sufficient condition such that $\mathscr{M}_*^d(\Gamma(y)) > 0$ for each chirp-like function

$$y(x) = a(x)v(x)S(\varphi(x)), \quad x \in (0,x_0], \tag{1.4}$$

where $a \in C^1(0,x_0]$, $v \in C(0,x_0]$, $\varphi \in C^2(0,x_0]$ and $S \in C^1(\mathbf{R})$ satisfy

$$a(x) > 0,\ a'(x) \geqslant 0,\ x \in (0,x_0], \tag{1.5}$$

$$\lim_{x\to+0} v(x) = v_0 \quad \text{for some } v_0 > 0, \quad v(x) > 0,\ x \in (0,x_0], \tag{1.6}$$

$$\lim_{x\to+0} \varphi(x) = \infty,\ \varphi(x) > 0 \ \text{ and } \ \varphi'(x) < 0,\ x \in (0,x_0], \tag{1.7}$$

$$|S(t+\tau)| = |S(t)|,\ t \in \mathbf{R} \quad \text{for some } \tau > 0, \tag{1.8}$$

$$S(\tau_0) = 0,\ S(t) \neq 0,\ t \in (\tau_0, \tau_0+\tau) \quad \text{for some } \tau_0 \in \mathbf{R}. \tag{1.9}$$

Namely, we will prove the following result.

THEOREM 1.2. *Assume that $a \in C^1(0,x_0]$, $v \in C(0,x_0]$ $\varphi \in C^1(0,x_0]$ and $S \in C^1(\mathbf{R})$ satisfy (1.5)–(1.9), and that there exists $\Phi \in C^1(0,x_0]$ such that*

$$0 < \liminf_{x\to+0} \frac{-\varphi'(x)}{\Phi(x)} \leqslant \limsup_{x\to+0} \frac{-\varphi'(x)}{\Phi(x)} < \infty, \tag{1.10}$$

$$\Phi(x) > 0,\ \Phi'(x) < 0,\ x \in (0,x_0], \quad \limsup_{x\to+0} \left([\Phi(x)]^{-1}\right)' < \infty. \tag{1.11}$$

Assume, moreover, that there exists $d \in (1,2)$ such that

$$\liminf_{x\to+0} [\Phi(x)]^{2-d} \int_0^x a(\xi)d\xi > 0. \tag{1.12}$$

Then $\mathscr{M}_*^d(\Gamma(y)) > 0$ for each chirp-like function (1.4).

Theorem 1.2 with $v(x) \equiv 1$ and $\Phi(x) \equiv -\varphi'(x)$ has been obtained in [13]. The proof of Theorem 1.2 will be given in Section 2.

Using the result in [13, Lemma 28], we can obtain the following consequence, which will be shown in Section 2.

PROPOSITION 1.2. *Let $y \in C^1(0,x_0]$ be bounded on $(0,x_0]$. Assume that*

$$\limsup_{x\to+0} \left[x \sup_{\xi\in(0,x]} |y(\xi)|\right]^{\frac{d-1}{2-d}} \int_x^{x_0} |y'(\xi)|d\xi < \infty \tag{1.13}$$

for some $d \in (1,2)$. Then $\mathscr{M}^{*d}(\Gamma(y)) < \infty$.

Combining these results, we can obtain Theorem 1.1.

Next we consider the half-linear equation

$$(|y'|^{p-2}y')' + f(x)|y|^{p-2}y = 0, \quad x \in (0,x_0], \tag{1.14}$$

where $p > 1$ and f satisfies (1.2). There are a lot of oscillatory results for (1.14). See, for example, Došlý and P. Řehák [3]. Pašić and J. S. W. Wong [14] established the following result.

THEOREM C ([14, Theorem 6]). *Let* $p > 1$. *Assume that* (1.2) *holds and*

$$f^{-\theta}(f^{\theta-(1/p)})'' \in L^1(0,x_0] \quad \textit{for some } \theta \in (0,1/p), \tag{1.15}$$

$$0 < \liminf_{x\to+0} x^\sigma f(x) \leqslant \limsup_{x\to+0} x^\sigma f(x) < \infty \quad \textit{for some } \sigma > p.$$

Then every nontrivial solution y *of* (1.14) *is oscillatory near* $x = 0$ *and the following properties hold*:

(i) *if* $p < \sigma < p^2$, *then* $0 < \mathscr{M}^1_*(\Gamma(y)) \leqslant \mathscr{M}^{*1}(\Gamma(y)) < \infty$ *and* $\dim_{\mathrm{B}} \Gamma(y) = 1$;

(ii) *if* $\sigma > p^2$, *then* $0 < \mathscr{M}^d_*(\Gamma(y)) \leqslant \mathscr{M}^{*d}(\Gamma(y)) < \infty$ *and* $\dim_{\mathrm{B}} \Gamma(y) = d$, *where* $d = 1 + \frac{1}{p} - \frac{p}{\sigma}$.

Pašić and J. S. W. Wong [14] gave the asymptotic behavior of oscillatory solutions of (1.14). Therefore, in the same way as (1.1), we can obtain the box-counting dimension of graphs of oscillatory solutions as follows.

THEOREM 1.3. *Let* $p > 1$. *Assume that* (1.2) *and* (1.15) *hold. Assume, moreover, that there exists* $d \in (1,2)$ *such that*

$$\liminf_{x\to+0} [f(x)]^{\frac{2-d}{p}} \int_0^x [f(\xi)]^{-\frac{1}{pp'}} d\xi > 0,$$
$$\limsup_{x\to+0} \left[x[f(x)]^{-\frac{1}{pp'}}\right]^{\frac{d-1}{2-d}} \int_x^{x_0} [f(\xi)]^{\frac{1}{p^2}} d\xi < \infty,$$

where p' *is a positive number with* $\frac{1}{p} + \frac{1}{p'} = 1$, *that is,* $p' = p/(p-1)$. *Then every nontrivial solution* y *of* (1.14) *is oscillatory near* $x = 0$ *and* $0 < \mathscr{M}^d_*(\Gamma(y)) \leqslant \mathscr{M}^{*d}(\Gamma(y)) < \infty$ *and hence* $\dim_{\mathrm{B}} \Gamma(y) = d$.

The proof of Theorem 1.3 will be given in Section 3.

EXAMPLE 1.1. We consider the equation

$$(|y'|^{p-2}y')' + \lambda x^{-\sigma}|y|^{p-2}y = 0, \quad x \in (0,x_0], \tag{1.16}$$

where $p > 1$, $\lambda > 0$ and $\sigma \in \mathbf{R}$. Theorem 1.3 implies that if $\sigma > p^2$, then every nontrivial solution y of (1.16) is oscillatory near $x = 0$ and $0 < \mathscr{M}^d_*(\Gamma(y)) \leqslant \mathscr{M}^{*d}(\Gamma(y)) < \infty$ and $\dim_{\mathrm{B}} \Gamma(y) = d$, where $d = 1 + \frac{1}{p} - \frac{p}{\sigma}$.

2. Lower and upper Minkowski contents

In this section we prove Theorem 1.2. To this end, we need the following result obtained in [9, pp. 2350].

LEMMA 2.1. *Let $y \in C(0,x_0]$ be a bounded function on $(0,x_0]$ and let $a_n \in (0,x_0]$ be a decreasing sequence of consecutive zeros of $y(x)$ such that $a_n \to 0$. Assume that there exists $\varepsilon_0 > 0$ and $k : (0,\varepsilon_0) \to \mathbf{N}$ such that*

$$a_n - a_{n+1} \leqslant \varepsilon \quad \textit{for all } n \geqslant k(\varepsilon) \textit{ and } \varepsilon \in (0,\varepsilon_0). \tag{2.1}$$

Then

$$|\Gamma_\varepsilon(y)| \geqslant \sum_{n \geqslant k(\varepsilon)} \max_{x \in [a_{n+1},a_n]} |y(x)|(a_n - a_{n+1}), \quad \varepsilon \in (0,\varepsilon_0). \tag{2.2}$$

Proof of Theorem 1.2. By (1.10) and (1.11), there exist $x_1 \in (0,x_0]$, $C_1 > 0$ and $C_2 > 0$ such that

$$0 < C_1\Phi(x) \leqslant -\varphi'(x) \leqslant C_2\Phi(x), \quad x \in (0,x_1]. \tag{2.3}$$

Let $y(x)$ be a chirp function given by (1.4). Set $a_n = \varphi^{-1}(\tau_0 + n\tau)$ for all sufficiently large $n \in \mathbf{N}$, where φ^{-1} is the inverse function of φ. From (1.7), it follows that a_n is strictly decreasing and $a_n \to 0$ as $n \to \infty$. Then $y(a_n) = 0$ and $y(x) \neq 0$ on (a_{n+1},a_n) for all sufficiently large $n \in \mathbf{N}$. We take $N \in \mathbf{N}$ such that $a_N \leqslant x_1$. By the mean value theorem, for each $n \geqslant N$, there exists $c_n \in (\tau_0 + n\tau, \tau_0 + (n+1)\tau)$ such that

$$a_n - a_{n+1} = \varphi^{-1}(\tau_0 + n\tau) - \varphi^{-1}(\tau_0 + (n+1)\tau) = \frac{-\tau}{\varphi'(\varphi^{-1}(c_n))}.$$

Since $\Phi(\varphi^{-1}(t))$ is increasing, by (2.3), we observe that

$$\frac{-\tau}{\varphi'(\varphi^{-1}(c_n))} \leqslant \frac{\tau}{C_1\Phi(\varphi^{-1}(c_n))} \leqslant \frac{\tau}{C_1\Phi(\varphi^{-1}(\tau_0 + n\tau))} = \frac{\tau}{C_1\Phi(a_n)}$$

for $n \geqslant N$. In the same way, we have

$$\frac{-\tau}{\varphi'(\varphi^{-1}(c_n))} \geqslant \frac{\tau}{C_2\Phi(a_{n+1})}, \quad n \geqslant N.$$

Consequently, we obtain

$$\frac{\tau}{C_2\Phi(a_{n+1})} \leqslant a_n - a_{n+1} \leqslant \frac{\tau}{C_1\Phi(a_n)}, \quad n \geqslant N. \tag{2.4}$$

Note that $1/\Phi(a_n)$ is decreasing. Moreover, $1/\Phi(a_n) \to 0$ as $n \to \infty$. Indeed, by (2.4), we have

$$0 < \frac{1}{\Phi(a_{n+1})} \leqslant \frac{C_2}{\tau}(a_n - a_{n+1}) \to 0 \quad \text{as } n \to \infty.$$

Set

$$\varepsilon_0 = \frac{\tau}{C_1\Phi(a_{N+1})}.$$

For each $\varepsilon \in (0,\varepsilon_0)$, let $k(\varepsilon)$ be the smallest positive integer satisfying

$$\frac{\tau}{C_1\Phi(a_{k(\varepsilon)})} \leqslant \varepsilon. \tag{2.5}$$

From (2.4), it follows that $k(\varepsilon)$ satisfies (2.1). Therefore, Lemma 2.1 implies that (2.2) holds.

By (1.11), there exists $L > 0$ such that

$$\frac{-\Phi'(x)}{[\Phi(x)]^2} = \left([\Phi(x)]^{-1}\right)' \leqslant L, \quad x \in (0,x_0]. \tag{2.6}$$

Set $\psi(t) = \log\Phi(\varphi^{-1}(t))$ for $t \geqslant \varphi(x_0)$. By the mean value theorem, for each $t \geqslant \varphi(x_1)$, there exists $c \in (t, t+2\tau)$ such that

$$\psi(t+2\tau) - \psi(t) = 2\tau\psi'(c),$$

that is,

$$\log\Phi(\varphi^{-1}(t+2\tau)) - \log\Phi(\varphi^{-1}(t)) = \frac{2\tau[-\Phi'(\varphi^{-1}(c))]}{\Phi(\varphi^{-1}(c))[-\varphi'(\varphi^{-1}(c))]}.$$

Therefore, by (2.3) and (2.6), we have

$$\begin{aligned}\log\frac{\Phi(\varphi^{-1}(t+2\tau))}{\Phi(\varphi^{-1}(t))} \leqslant \frac{2\tau[-\Phi'(\varphi^{-1}(c))]}{\Phi(\varphi^{-1}(c))[C_1\Phi(\varphi^{-1}(c))]} &= 2\tau C_1^{-1}\frac{-\Phi'(\varphi^{-1}(c))}{[\Phi(\varphi^{-1}(c))]^2} \\ &\leqslant 2\tau C_1^{-1}L,\end{aligned}$$

which implies that

$$\Phi(\varphi^{-1}(t)) \geqslant C_3\Phi(\varphi^{-1}(t+2\tau)), \quad t \geqslant \varphi(x_1), \tag{2.7}$$

where $C_3 = e^{-2\tau C_1^{-1}L}$.

By the definition of $k(\varepsilon)$ and (2.5), we find that

$$\frac{\tau}{C_1\Phi(a_{k(\varepsilon)-1})} > \varepsilon.$$

Hence, from (2.7), it follows that

$$\begin{aligned}\tau\varepsilon^{-1} > C_1\Phi(a_{k(\varepsilon)-1}) &= C_1\Phi(\varphi^{-1}(\tau_0 + (k(\varepsilon)-1)\tau)) \\ &\geqslant C_1C_3\Phi(\varphi^{-1}(\tau_0 + (k(\varepsilon)-1)\tau + 2\tau)) \\ &= C_1C_3\Phi(\varphi^{-1}(\tau_0 + (k(\varepsilon)+1)\tau)) \\ &= C_1C_3\Phi(a_{k(\varepsilon)+1}),\end{aligned}$$

which means that

$$\Phi(a_{k(\varepsilon)+1}) < C_4\varepsilon^{-1}, \quad \varepsilon \in (0,\varepsilon_0), \tag{2.8}$$

where $C_4 = \tau/(C_1C_3)$.

Set

$$M = \max_{t\in\mathbf{R}} |S(t)| = \max_{t\in[0,\tau]} |S(t)|.$$

We can take $v_1 > 0$ such that $v(x) \geqslant v_1$ for $x \in (0,x_0]$ in view of (1.6). Since $a(x)$ is nondecreasing on $(0,x_0]$ by (1.5), we find that

$$\begin{aligned}\max_{x\in[a_{n+1},a_n]} |y(x)| &= \max_{x\in[a_{n+1},a_n]} a(x)v(x)|S(\varphi(x))|\\ &\geqslant v_1 a(a_{n+1}) \max_{x\in[a_{n+1},a_n]} |S(\varphi(x))| = v_1 M a(a_{n+1}).\end{aligned} \tag{2.9}$$

From (2.4) and (2.9), it follows that

$$\max_{x\in[a_{n+1},a_n]} |y(x)|(a_n - a_{n+1}) \geqslant C_5 \frac{a(a_{n+1})}{\Phi(a_{n+1})} = C_5 \frac{a\big(\varphi^{-1}(\tau_0 + (n+1)\tau)\big)}{\Phi\big(\varphi^{-1}(\tau_0 + (n+1)\tau)\big)}$$

for $n \geqslant N$, where $C_5 = \tau v_1 M C_2^{-1}$. Since $a(\varphi^{-1}(t))/\Phi(\varphi^{-1}(t))$ is nonincreasing, using (2.3), we observe that

$$\begin{aligned}\sum_{n\geqslant k(\varepsilon)} \max_{x\in[a_{n+1},a_n]} |y(x)|(a_n - a_{n+1}) &\geqslant C_5 \sum_{n\geqslant k(\varepsilon)} \frac{a\big(\varphi^{-1}(\tau_0 + (n+1)\tau)\big)}{\Phi\big(\varphi^{-1}(\tau_0 + (n+1)\tau)\big)}\\ &\geqslant C_5 \int_{k(\varepsilon)}^{\infty} \frac{a\big(\varphi^{-1}(\tau_0 + (t+1)\tau)\big)}{\Phi\big(\varphi^{-1}(\tau_0 + (t+1)\tau)\big)}\, dt\\ &= \tau^{-1} C_5 \int_0^{\varphi^{-1}(\tau_0+(k(\varepsilon)+1)\tau)} \frac{a(\xi)}{\Phi(\xi)}[-\varphi'(\xi)]d\xi\\ &= C_6 \int_0^{a_{k(\varepsilon)+1}} a(\xi)\, d\xi,\end{aligned} \tag{2.10}$$

where $C_6 = \tau^{-1}C_1C_5$. By (1.12), there exists $C_7 > 0$ such that

$$\int_0^x a(\xi)\, d\xi \geqslant C_7[\Phi(x)]^{-(2-d)}, \quad x \in (0,x_0]. \tag{2.11}$$

Combining (2.2), (2.10), (2.11) and (2.8), we obtain

$$\begin{aligned}|\Gamma_\varepsilon(y)| \geqslant \sum_{n\geqslant k(\varepsilon)} \max_{x\in[a_{n+1},a_n]} |y(x)|(a_n - a_{n+1}) &\geqslant C_6C_7\big[\Phi(a_{k(\varepsilon)+1})\big]^{-(2-d)}\\ &\geqslant C_6C_7\big[C_4\varepsilon^{-1}\big]^{-(2-d)}\\ &= c_1\varepsilon^{2-d}, \quad \varepsilon \in (0,\varepsilon_0),\end{aligned}$$

where $c_1 = C_6C_7C_4^{-(2-d)}$. Consequently, $\mathscr{M}_*^d(\Gamma(y)) \geqslant c_1 > 0$. $\square$

Now we give a proof of Proposition 1.2. That is a corollary of the following lemma obtained in [13, Lemma 28].

LEMMA 2.2. *Let $y \in C^1(0,x_0]$ be bounded on $(0,x_0]$. Assume that $y' \notin L^1(0,x_0]$ and (1.13) holds for some $d \in (1,2)$. Then there exists $c_2 > 0$ such that $|\Gamma_\varepsilon(y)| \leqslant c_2\varepsilon^{2-d}$ for $\varepsilon \in (0,1)$.*

Proof of Proposition 1.2. If $y' \notin L^1(0,x_0]$, then Lemma 2.2 implies that $\mathscr{M}^{*d}(\Gamma(y)) < \infty$. Now we assume that $y' \in L^1(0,x_0]$. Then y is rectifiable. Indeed, since $\sqrt{1+x^2} \leqslant 1+|x|$ for $x \in \mathbf{R}$, we have

$$\int_0^{x_0} \sqrt{1+|y'(x)|^2}dx \leqslant \int_0^{x_0} (1+|y'(x)|)dx < \infty.$$

Hence, by Proposition 1.1, we conclude that $\mathscr{M}^{*d}(\Gamma(y)) = 0 < \infty$. $\square$

3. Box-counting dimension of oscillatory solutions

In this section, we give proofs of Theorems 1.1 and 1.3.

According to Drábek and Manásevich [4] and Takeuchi [15], for each p, $q \in (1,\infty)$, we define the generalized sine function $\sin_{p,q}$ and the corresponding generalized π by the inverse function of

$$\sin_{p,q}^{-1} x := \int_0^x (1-s^q)^{-1/p} ds, \quad 0 \leqslant x \leqslant 1$$

and

$$\pi_{p,q} := 2\sin_{p,q}^{-1} 1 = 2\int_0^1 (1-s^q)^{-1/p} ds,$$

respectively. We note here that $\sin_{p,q}(\pi_{p,q}/2) = 1$. The function $\sin_{p,q} x$ is increasing in $[0,\pi_{p,q}/2]$ onto $[0,1]$. We extend it to $(\pi_{p,q}/2, \pi_{p,q}]$ by $\sin_{p,q}(\pi_{p,q}-x)$ and to the whole real line $\mathbf{R}$ as the odd $2\pi_{p,q}$-periodic continuation of the function. Then $\sin_{p,q} x$ is a solution of

$$(\phi_p(S'))' + \frac{q}{p'}\phi_q(S) = 0, \quad S(0) = 0,\ S'(0) = 1.$$

We define the generalized cosine function $\cos_{p,q}$ by $\cos_{p,q} x := (\sin_{p,q} x)'$ for $x \in \mathbf{R}$. Then we have

$$|\sin_{p,q} x|^p + |\cos_{p,q} x|^q = 1, \quad x \in \mathbf{R}.$$

For Emden-Fowler equations, we have the following asymptotic behavior result obtained by J. S. W. Wong [18].

PROPOSITION 3.1. *Let $\gamma > 0$ and $\gamma \neq 1$. Assume that $f \in C^2(0,x_0]$, $f(x) > 0$ for $x \in (0,x_0]$, $\lim_{x\to+0} f(x) = \infty$, $f^{-1}f'_+ \in L^1(0,x_0]$ and $(f^{-(\gamma^*+2)/(\gamma^*+1)}f')' \in L^1(0,x_0]$, where $f'_+(x) = \max\{f'(x),0\}$ and $\gamma^* = \max\{\gamma,1\}$. Then, for each nontrivial solution y of (1.1), there exist functions V, $\varphi \in C^1(0,x_0]$ such that*

$$y(x) = [f(x)]^{-\frac{1}{\gamma+3}}[V(x)]^{\frac{1}{\gamma+1}} \sin_{2,\gamma+1}(\varphi(x)), \tag{3.1}$$

$$y'(x) = -[f(x)]^{\frac{1}{\gamma+3}}[V(x)]^{\frac{1}{2}} \cos_{2,\gamma+1}(\varphi(x)), \tag{3.2}$$

$$\lim_{x\to+0} V(x) = V_0 \quad \textit{for some constant } V_0 > 0, \tag{3.3}$$

$$\lim_{x\to+0} \varphi(x) = \infty, \tag{3.4}$$

$$\varphi'(x) = -[f(x)]^{\frac{2}{\gamma+3}}[V(x)]^{\frac{\gamma-1}{2(\gamma+1)}} + \frac{1}{\gamma+1}\frac{f'(x)}{f(x)}\sin_{2,\gamma+1}(\varphi(x))\cos_{2,\gamma+1}(\varphi(x)) \tag{3.5}$$

for $x \in (0,x_0]$.

The following lemma was also obtained by J. S. W. Wong [18, Lemma 2]

LEMMA 3.1. *Assume that* $f \in C^2(0,x_0]$, $f(x) > 0$ *for* $x \in (0,x_0]$, $\lim_{x\to+0} f(x) = \infty$, *and* $(f^{-(\gamma^*+2)/(\gamma^*+1)}f')' \in L^1(0,x_0]$, *where* $\gamma^* = \max\{\gamma,1\}$. *Then*

$$\lim_{x\to+0} [f(x)]^{-\frac{\gamma+5}{\gamma+3}} f'(x) = 0.$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let y be a nontrivial solution of (1.1). Proposition 3.1 implies that there exist V, $\varphi \in C^1(0,x_0]$ such that (3.1)–(3.5) hold. Therefore, y is oscillatory near $x = 0$ and bounded on $(0,x_0]$. Set

$$a(x) = [f(x)]^{-\frac{1}{\gamma+3}}, \quad v(x) = [V(x)]^{\frac{1}{\gamma+1}}, \quad \Phi(x) = [f(x)]^{\frac{2}{\gamma+3}}.$$

From Lemma 3.1, (3.3) and (3.5), it follows that

$$\lim_{x\to+0} \frac{-\varphi'(x)}{\Phi(x)} = V_0^{\frac{\gamma-1}{2(\gamma+1)}} \tag{3.6}$$

and

$$\lim_{x\to+0} ([\Phi(x)]^{-1})' = -\frac{2}{\gamma+3}\lim_{x\to+0} [f(x)]^{-\frac{\gamma+5}{\gamma+3}} f'(x) = 0.$$

By (3.3), (3.4) and (3.6), there exists $x_1 \in (0,x_0]$ such that $v(x) > 0$, $\varphi(x) > 0$ and $\varphi'(x) < 0$ for $x \in (0,x_1]$. Applying Theorem 1.2, we have $\mathscr{M}_*^d(\Gamma(y)) > 0$.

Next we will show that $\mathscr{M}^{*d}(\Gamma(y)) < \infty$, by Proposition 1.2. From (3.1) and (3.3), it follows that

$$|y(x)| \leqslant C_1[f(x)]^{-\frac{1}{\gamma+3}}, \quad x \in (0,x_0]$$

for some $C_1 > 0$. Since $f'(x) < 0$ for $x \in (0,x_0]$, we have

$$\sup_{\xi\in(0,x]} |y(\xi)| \leqslant C_1[f(x)]^{-\frac{1}{\gamma+3}}, \quad x \in (0,x_0].$$

By (3.2), we obtain

$$|y'(x)| \leqslant C_2[f(x)]^{\frac{1}{\gamma+3}}, \quad x \in (0,x_0]$$

for some $C_2 > 0$. Hence, Proposition 1.2 shows that $\mathscr{M}^{*d}(\Gamma(y)) < \infty$. □

Finally, we give a proof of Theorem 1.3. To this end, we need the following Proposition 3.2 and Lemma 3.2, which were established by Pašić and Wong [14].

PROPOSITION 3.2. *Let $p>1$. Assume that (1.2) and (1.15) hold. Then, for each nontrivial solution y of (1.14), there exist functions V, $\varphi \in C^1(0,x_0]$ satisfying (3.3), (3.4) and*

$$y(x)=(p-1)^{\frac{1}{pp'}}[f(x)]^{-\frac{1}{pp'}}[V(x)]^{\frac{1}{p}}\sin_{p,p}(\varphi(x)), \tag{3.7}$$

$$y'(x)=-(p-1)^{\frac{1}{p^2}}[f(x)]^{\frac{1}{p^2}}[V(x)]^{\frac{1}{p}}\cos_{p,p}(\varphi(x)), \tag{3.8}$$

$$\varphi'(x)=-(p-1)^{-\frac{1}{p}}[f(x)]^{\frac{1}{p}}+\frac{1}{p}\frac{f'(x)}{f(x)}\sin_{p,p}(\varphi(x))|\cos_{p,p}(\varphi(x))|^{p-2}\cos_{p,p}(\varphi(x)) \tag{3.9}$$

for $x\in(0,x_0]$.

LEMMA 3.2. *Let $p>1$. Assume that (1.2) and (1.15) hold. Then*

$$\lim_{x\to+0}[f(x)]^{-\frac{1}{p}-1}f'(x)=0.$$

Proof of Theorem 1.3. Let y be a nontrivial solution of (1.14). By Proposition 3.2, there exist V, $\varphi\in C^1(0,x_0]$ such that (3.3), (3.4), (3.7), (3.8), and (3.9) hold. Then y is oscillatory near $x=0$ and bounded on $(0,x_0]$. Set

$$a(x)=[f(x)]^{-\frac{1}{pp'}},\quad v(x)=(p-1)^{\frac{1}{pp'}}[V(x)]^{\frac{1}{p}},\quad \Phi(x)=[f(x)]^{\frac{1}{p}}.$$

By Lemma 3.2 and (3.9), we find that

$$\lim_{x\to+0}\frac{-\varphi'(x)}{\Phi(x)}=(p-1)^{-\frac{1}{p}} \tag{3.10}$$

and

$$\lim_{x\to+0}([\Phi(x)]^{-1})'=-\frac{1}{p}\lim_{x\to+0}[f(x)]^{-\frac{1}{p}-1}f'(x)=0.$$

From (3.3), (3.4) and (3.10), it follows that $v(x)>0$, $\varphi(x)>0$ and $\varphi'(x)<0$ on $(0,x_1]$ for some $x_1\in(0,x_0]$. Therefore, Theorem 1.2 implies that $\mathscr{M}_*^d(\Gamma(y))>0$.

By (3.7), there exists a constant $C_1>0$ such that

$$|y(x)|\leqslant C_1[f(x)]^{-\frac{1}{pp'}},\quad x\in(0,x_0].$$

Since $f'(x)<0$ for $x\in(0,x_0]$, we have

$$\sup_{\xi\in(0,x]}|y(\xi)|\leqslant C_1[f(x)]^{-\frac{1}{pp'}},\quad x\in(0,x_0].$$

From (3.8), it follows that

$$|y'(x)|\leqslant C_2[f(x)]^{\frac{1}{p^2}},\quad x\in(0,x_0]$$

for some $C_2>0$. Hence, Proposition 1.2 implies that $\mathscr{M}^{*d}(\Gamma(y))<\infty$. $\square$

Acknowledgement. The authors would like to thank the referee for carefully reading our manuscript and for giving positive, constructive and useful comments.

REFERENCES

[1] F. V. ATKINSON, *On second-order non-linear oscillations*, Pacific J. Math. **5** (1955), 643–647.

[2] S. BELOHOREC, *Oscillatory solutions of certain nonlinear differential equations of second order*, Mat. Fyz. Cas. Slovensk. Akad. Vied. **11** (1961), 250–255.

[3] O. DOŠLÝ AND P. ŘEHÁK, *Half-linear differential equations*, North-Holland Mathematics Studies **202**, Elsevier Science B. V., Amsterdam, 2005.

[4] P. DRÁBEK AND R. MANÁSEVICH, *On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian*, Differential Integral Equations **12** (1999), 773–788.

[5] K. FALCONER, *Fractal Geometry. Mathematical Fondations and Applications*, John Willey-Sons, 1999.

[6] I. T. KIGURADZE, *On the oscillatory and monotone solutions of ordinary differential equations*, Arch. Math. (Brno) **14** (1978), 21–44.

[7] I. T. KIGURADZE AND T. A. CHANTURIA, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Translated from the 1985 Russian original, Mathematics and its Applications (Soviet Series) **89**, Kluwer Academic Publishers Group, Dordrecht, 1993.

[8] T. KUSANO AND M. NAITO, *Unbounded nonoscillatory solutions of nonlinear ordinary differential equations of arbitrary order*, Hiroshima Math. J. **18** (1988), 361–372.

[9] M. K. KWONG, M. PAŠIĆ AND J. S. W. WONG, *Rectifiable oscillations in second-order linear differential equations*, J. Differential Equations **245** (2008), 2333–2351.

[10] M. NAITO, *A remark on the existence of slowly growing positive solutions to second order superlinear ordinary differential equations*, No. DEA Nonlinear Differential Equations Appl. **20** (2013), 1759–1769.

[11] M. PAŠIĆ, *Fractal oscillations for a class of second order linear differential equations of Euler type*, J. Math. Anal. Appl. **341** (2008), 211–223.

[12] M. PAŠIĆ AND S. TANAKA, *Fractal oscillations of self-adjoint and damped linear differential equations of second-order*, Appl. Math. Comput. **218** (2011), 2281–2293.

[13] M. PAŠIĆ AND S. TANAKA, *Fractal oscillations of chirp functions and applications to second-order linear differential equations*, Int. J. Differ. Equ. **2013**, Art. ID 857410, 11 pp.

[14] M. PAŠIĆ AND J. S. W. WONG, *Rectifiable oscillations in second-order half-linear differential equations*, Ann. Mat. Pura Appl. (4) **188** (2009), 517–541.

[15] S. TAKEUCHI, *Multiple-angle formulas of generalized trigonometric functions with two parameters*, J. Math. Anal. Appl. **444** (2016), 1000–1014.

[16] C. TRICOT, *Curves and Fractal Dimension*, Springer-Verlag, New York, 1995.

[17] J. S. W. WONG, *On the generalized Emden-Fowler equation*, SIAM Rev. **17** (1975), 339–360.

[18] J. S. W. WONG, *On rectifiable oscillation of Emden-Fowler equations*, Mem. Differential Equations Math. Phys. **42** (2007), 127–144.

(Received January 19, 2018)

Takanao Kanemitsu
Department of Applied Mathematics, Faculty of Science
Okayama University of Science
Ridaichou 1–1, Okayama 700–0005, Japan

Satoshi Tanaka
Department of Applied Mathematics, Faculty of Science
Okayama University of Science
Ridaichou 1–1, Okayama 700–0005, Japan
e-mail: tanaka@xmath.ous.ac.jp