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LYAPUNOV INEQUALITIES FOR TWO–PARAMETRIC QUANTUM HAMILTONIAN SYSTEMS AND THEIR APPLICATIONS

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Abstract. This paper deals with study of the two-parametric quantum Hamiltonian systems. The main objective in our study is Lyapunov inequalities of the two-parametric quantum Hamiltonian systems. In this paper, we first define two-parametric quantum analogous of the Leibniz rule, Cauchy-Schwarz and Holder inequalities and consequently as theoretical part of our main results, by the use of new Leibniz rule and Cauchy-Schwarz inequality on the considered Hamiltonian systems we obtain corresponding Lyapunov inequalities. Applicability of the obtained Lyapunov inequalities is examined by presenting a disconjugacy and at the same time a nonexistence criterion for the related Hamiltonian systems.

1. Introduction

As we know, origin of the quantum calculus turns to the eighteenth century when scientists such as L. Euler were studying the wave theory of light based on their experimental observations. Besides, at the beginning of the twentieth century, M. Planck proposed a new theory about nature of light that a few years later accepted as theory of quantum physics. According to the Planck's theory, light has a discrete nature consisting of energy particles that he called them quanta. But, it was F. H. Jackson that in the first decade of twentieth century formulated the quantum calculus systematically. According to the Jackson's quantum calculus that is known as q-calculus, for real-valued function $f : q^{\mathbb{N}_0} \to \mathbb{R}$, q-difference operator is defined as

$$(D_q f)(t) := \frac{f(t) - f(qt)}{(1-q)t}, \quad 0 < q < 1,\ t \in q^{\mathbb{N}_0} = \{1, q, q^2, \ldots\},\ \mathbb{N}_0 := 0, 1, 2, 3, \ldots .$$

Consequently, $D_q^n f(t) := D_q \left(D_q^{n-1} f \right)(t)$, $n \in \mathbb{Z}^+$, where by convention, $D_q^0 f(t) := f(t)$. Also, q-integral or Jackson integral is given by

$$(I_q f)(t) = \int_0^t f(t) d_q t := (1-q)t \sum_{n=0}^{\infty} q^n f(q^n t),$$

such that the infinite series in the right hand side is assumed to be convergent.

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In q-calculus, the q-bracket and q-factorial are defined as

$$[n]_q := \frac{1-q^n}{1-q}, \qquad [n]_q! := \prod_{k=1}^{n} [k]_q, \quad n \in \mathbb{Z}^+, \ [0]_q! = 1,$$

respectively. In this case, $D_q x^n := [n]_q x^{n-1}$. More details about q-calculus can be found in references [13]–[16], [19].

On the other hand, history of the Lyapunov inequalities turns to the last decade of the nineteenth century where the Russian mathematician A. M. Lyapunov was studying stability of the second order differential equations with nonconstant ω-periodic coefficient,

$$y'' + q(t)y = 0, \quad -\infty < t < +\infty, \ (q(t+\omega) = q(t)). \tag{1.1}$$

Indeed, the cornerstone of the Lyapunov inequalities has been founded in frame of the following stability criterion for the differential equation (1.1) as follows.

THEOREM 1.1. (cf. [18]) *If the function q takes only positive or zero values(without being identically zero), and if further it satisfies the condition*

$$\omega \int_0^{\omega} q(t) \leqslant 4,$$

then roots of the characteristic equation corresponding to (1.1) will always be complex and their modulus are equal to 1.

By means of the well known Floquet theory, one can derive that the outcome of Theorem 1.1 is equivalent to the stability of the differential equation (1.1). Nowadays, the inequality

$$\omega \int_0^{\omega} q(t)dt > 4, \tag{1.2}$$

is known as Lyapunov inequality.

Over 125 years, considerable number of mathematicians have been indicated this fact that Lyapunov inequalities of differential equations provide not only stability criteria but also propose a wide range of criteria to assessment concepts such as disconjugacy, nonexistence, oscillatory properties, upper bound estimation for real zeros of the nontrivial solutions and lower bound estimation for eigenvalues of the certain classes of the eigenvalue problems. To more consultation on these applications we refer the followers to the research papers [3]–[8], [10], [11], [17], [20]–[24] and references cited therein.

P. Hartman [12] in 1964 started the systematic investigation on Lyapunov inequalities by presenting the following theorem.

THEOREM 1.2. [12] *If the boundary value problem*

$$\begin{cases} y''(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = 0 = y(b), \end{cases} \tag{1.3}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{1.4}$$

Hartman, applied the Lyapunov inequality (1.4) to establish some upper bound estimations for maximum number of real zeros of nontrivial solutions of the boundary value problem (1.3).

As stated, we are interested in the planar Hamiltonian systems via integral inequalities. By a Hamiltonian system, we mean a dynamical system described by a scalar function $H(u,v)$ so-called Hamiltonian function. In this case, the Hamiltonian systems is defined by the following Hamiltonian equations:

$$\frac{\partial H}{\partial u} := \frac{dv}{dt}, \quad \frac{\partial H}{\partial v} := -\frac{du}{dt}, \quad u,v : \mathbb{R}^N \to \mathbb{R},\ N \in \mathbb{Z}^+.$$

Regarding this Hamiltonian system, $2N$-dimensional vector $r(t) := (u,v)$ is the solution of initial value problem defined by the above equations with the initial value $r(0) := r_0 \in \mathbb{R}^{2N}$. Maybe the main advantage of the Hamiltonian systems in comparison with the other differential systems is their better description of the evolution process in physical systems even if corresponding initial value problem cannot be solved analytically. Using the Hamiltonian systems, we can study dynamical systems related to the chaos, general relativity, planetary systems and electromagnetic fields for instance. So, studying the Lyapunov inequalities of the Hamiltonian systems and their applications can be considered as an important research field that links theory of the mathematical inequalities to theory of differential equations. In this way, we mention some of the selected inspiring research papers concerning the planar Hamiltonian systems as follows.

We begin with the reference [10]. In this paper, the authors first consider the linear Hamiltonian system

$$x' = a(t)x + b(t)u, \quad u' = -c(t)x - a(t)u, \qquad t \in \mathbb{R}, \tag{1.5}$$

such that a, b and c are real-valued piece-wise continuous functions on $\mathbb{R}$. Relying on some assumptions on the nontrivial solution (x,u), corresponding Lyapunov inequality of the Hamiltonian system (1.5) is obtained as

$$\int_\alpha^\beta |a(t)|dt + \left\{\int_\alpha^\beta b(t)dt . \int_\alpha^\beta c_+(t)dt\right\}^{\frac{1}{2}} \geqslant 2, \quad \alpha,\beta \in \mathbb{R},\ \alpha < \beta, \tag{1.6}$$

in which $c_+(t) = \max\{c(t),0\}$. In second phase, they considered the discrete Hamiltonian system

$$\Delta x(t) = \phi(t)x(t+1) + \psi(t)u(t), \quad \Delta u(t) = -\rho(t)x(t+1) - \phi(t)u(t), \quad t \in \mathbb{Z}. \tag{1.7}$$

Similar to the previous phase, they obtained the following Lyapunov inequalities for the Hamiltonian system (1.7).

$$\sum_{t=\alpha}^{\beta-2}|\phi(t)|+\left\{\sum_{t=\alpha}^{\beta-1}\psi(t).\sum_{t=\alpha}^{\beta-2}\rho_+(t)\right\}^{\frac{1}{2}}\geqslant 2, \tag{1.8}$$

$$\sum_{t=\alpha}^{\beta-2}|\phi(t)|+\left\{\sum_{t=\alpha}^{\beta-2}\psi(t).\sum_{t=\alpha}^{\beta-2}\rho_+(t)\right\}^{\frac{1}{2}}\geqslant 1, \tag{1.9}$$

such that $\alpha,\beta\in\mathbb{Z}$ and $\alpha<\beta-2$. As applications to the Lyapunov inequalities (1.6), (1.8) and (1.9), stability and disconjugacy of the Hamiltonian systems (1.5) and (1.7) have been established.

In continuation, the author in [24] considered the discrete Hamiltonian system (1.7) and using the concept of discrete exponential

$$e_p(n,s):=\begin{cases}\prod_{r=s}^{n-1}(1+p(r)) & ; \quad s\leqslant n,\\ \prod_{r=n}^{s-1}\dfrac{1}{1+p(r)}, & ; \quad s>n,\end{cases}$$

in which $n,s\in\mathbb{Z}$ and $p:\mathbb{Z}\to\mathbb{R}$ is a function such that $1+p(n)\neq 0$, $n\in\mathbb{Z}$, obtained the Lyapunov inequality

$$\sum_{t=\alpha}^{\beta-1}\psi(t).\sum_{t=\alpha}^{\beta-2}\rho_+(t)\geqslant 4\exp\left(-\sum_{t=\alpha}^{\beta-2}|\ln(|1-\phi(t)|)|\right). \tag{1.10}$$

He used this Lyapunov inequality to establish stability and disconjugacy of the Hamiltonian system (1.7).

Also, the authors in [23] theoretically studied the nonlinear Hamiltonian system

$$x'(t)=\alpha(t)x(t)+\beta(t)|y(t)|^{\mu-2}y(t),\quad y'(t)=-\gamma(t)|x(t)|^{\nu-2}x(t)-\alpha(t)y(t), \tag{1.11}$$

such that $\frac{1}{\mu}+\frac{1}{\nu}=1$ and α,β and γ are locally Lebesgue integrable real-valued functions on $\mathbb{R}$. Corresponding to this Hamiltonian system, the author obtained sequence of Lyapunov inequalities that we just mention one of them as follows:

$$\int_a^b\frac{\xi(t)\eta(t)}{\xi(t)+\eta(t)}\gamma^+(t)dt\geqslant 1,\quad a,b\in\mathbb{R},\ a<b, \tag{1.12}$$

where,

$$\xi(t)=\left[\int_a^t\beta(\tau)\exp\left(\mu\int_\tau^t\alpha(s)ds\right)d\tau\right]^{\frac{\nu}{\mu}},$$

$$\eta(t)=\left[\int_t^b\beta(\tau)\exp\left(-\mu\int_t^\tau\alpha(s)ds\right)d\tau\right]^{\frac{\nu}{\mu}}.$$

Furthermore, to see much more investigations about impulsive Hamiltonian systems, non-impulsive Hamiltonian systems, non-integer impulsive linear systems or matrix approach Lyapunov inequalities for Hamiltonian systems we refer the interested followers to the references [1]–[9], [11], [17], [22] and references cited therein.

Besides the variety of investigations on the Hamiltonian systems, to the best of our knowledge there is no any research work about Lyapunov inequalities of quantum Hamiltonian systems in the literature. This absence inspired us to initiate this investigation about linear Hamiltonian systems. In this way, motivated by the above mentioned research works and discussions, we consider the two-parametric quantum Hamiltonian system

$$\begin{cases} (D_{p,q}u)(t) = a(t)u(pt) + b(t)v(qt), \\ \\ (D_{p,q}v)(t) = -c(t)u(pt) - a(t)v(qt), \end{cases} \qquad 0 < q < p \leqslant 1, \tag{1.13}$$

such that $D_{p,q}$ stands for two parametric quantum difference, a, b and c are real-valued functions and $b(t) > 0$. The corresponding Hamiltonian function is given by

$$H(u,v) := \frac{c(t)u^2}{p+q} + a(t)uv + \frac{b(t)v^2}{p+q}. \tag{1.14}$$

Having a nontrivial solution in hand and some hypotheses about zeros of this solution, we will obtain relevant Lyapunov inequality for quantum Hamiltonian system (1.13). Afterward, applicability of the obtained Lyapunov inequality will be examined by presenting some disconjugacy and nonexistence criteria to estimate qualitative dynamics of the Hamiltonian system (1.13).

At the end of this section we state the organization of the paper. In Section 2, we present a brief overview on two-parametric quantum calculus particularly those parts that will be needed in this paper. In this section we present some technical tools such as Cauchy-Schwarz and Holder inequalities for two-parametric quantum calculus. In Section 3, first in the theoretical step we obtain Lyapunov inequality of quantum Hamiltonian system (1.13) and then in applied step we examine ability of the obtained Lyapunov inequality to establish qualitative behavior of Hamiltonian system (1.13) as stated above. For simplicity, from now on we call the two-parametric quantum differences as (p,q)-differences as well as the other two-parametric quantum elements.

2. Preliminaries

As mentioned previously, we begin this section with fundamental concepts of the (p,q)-calculus.

DEFINITION 2.1. ([19]) Assume $0 < q < p \leqslant 1$ and $f : \mathbb{R}+ \to \mathbb{R}$. The (p,q)-difference of f is defined by

$$(D_{p,q}f)(t) := \frac{f(pt) - f(qt)}{(p-q)t}, \quad t \in \mathbb{R}_+. \tag{2.1}$$

In this case, we define $D^n_{p,q}f := D_{p,q}\left(D^{n-1}_{p,q}f\right)$, $n \in \mathbb{Z}^+$. By convention we define $D^0_{p,q}f := f$. Note that, taking $p = 1$, the (p,q)-difference $D_{p,q}$ reduces to the classic q-difference D_q.

In the sequel, we present (p,q)-Jackson integrals that generalize the classic Jackson integrals.

DEFINITION 2.2. ([19]) Assume $0 < q < p \leqslant 1$. Then the (p,q)-Jackson integral of real-valued function f is defined by

$$(I_{p,q}f)(t) = \int_0^t f(s)d_{p,q}s := (p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}t\right). \tag{2.2}$$

Note that, taking $p = 1$ gives us the classic Jackson integral.

Next, we define the (p,q)-bracket and (p,q)-factorial as follows,

$$[n]_{p,q} := \frac{p^n - q^n}{p-q}, \qquad [n]_{p,q}! := \prod_{k=1}^{n}[k]_{p,q}!, \quad n \in \mathbb{Z}^+, \ [0]_{p,q}! := 1, \tag{2.3}$$

respectively, More details can be found in [19] and cited bibliography therein. We notice that as in the classic q-calculus, $D_{p,q}x^n := [n]_{p,q}x^{n-1}$, $n \in \mathbb{Z}^+$.

As an instant application of this power rule, we show here that why the function $H(u,v)$ defined by (1.14) is called the Hamiltonian of the (p,q)-Hamiltonian system (1.13). To this aim, by the use of (2.3), we get that

$$D_{p,q}x^2 := [2]_{p,q}x := (p+q)x.$$

Now, considering $H(u,v)$ in (1.14) it follows that

$$\frac{\partial_{p,q}H}{\partial v} := a(t)u + \frac{p+q}{p+q}b(t)v = D_{p,q}u, \quad \frac{\partial_{p,q}H}{\partial u} := c(t)u + \frac{p+q}{p+q}a(t)v = -D_{p,q}v.$$

This ensures that $H(u,v)$ is the Hamiltonian of the Hamiltonian system (1.13). Let us mention that by $\dfrac{\partial_{p,q}H}{\partial v}$ we mean the partial (p,q)-difference of H with respect to v that is defined as

$$\frac{\partial_{p,q}H}{\partial v} := \frac{H(u,v(pt)) - H(u,v(t))}{(p-q)t}.$$

In this position we are going to present a theorem including some of the basic properties of (p,q)-calculus.

THEOREM 2.3. ([19]) *Assume* $0 < q < p \leqslant 1$. *If* $\lambda, c_i \in \mathbb{R}$, $i = 1,2$ *and* $f,g : \mathbb{R}_+ \to \mathbb{R}$, *then*

(P_1) $D_{p,q}\lambda = 0$;

(P_2) $D_{p,q}(c_1f + c_2g)(t) = c_1D_{p,q}f(t) + c_2D_{p,q}g(t)$;

(P_3) $(D_{p,q}fg)(t) = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t)$;

(P_4) $\left(D_{p,q}\dfrac{f}{g}\right)(t) = \dfrac{g(qt)D_{p,q}f(t) - f(qt)D_{p,q}g(t)}{g(pt)g(qt)}, \quad g(t) \not\equiv 0$;

(P_5) $\displaystyle\int_a^b f(t)d_{p,q}t = \int_0^b f(t)d_{p,q}t - \int_0^a f(t)d_{p,q}t$;

(P_6) $\displaystyle D_{p,q}\int_a^b f(t)d_{p,q}t = f(t)$;

(P_7) $\displaystyle\int_a^b D_{p,q}f(t)d_{p,q}t = f(b) - f(a)$.

Next, we state and prove (p,q)-approaches of the Cauchy-Schwarz and Holder inequalities. The Cauchy-Schwarz inequality will be played key role to extract Lyapunov inequality of the (p,q)-Hamiltonian system (1.13).

LEMMA 2.4. ((p,q)-Cauchy-Schwarz inequality) *Assume $f,g : \mathbb{R}_+ \to \mathbb{R}$ are two (p,q)-Jackson integrable functions. Then, the (p,q)-Cauchy-Schwarz inequality*

$$\left|\int_0^t f(s)g(s)d_{p,q}s\right| \leqslant \sqrt{\int_0^t f^2(s)d_{p,q}s}.\sqrt{\int_0^t g^2(s)d_{p,q}s}, \tag{2.4}$$

holds.

Proof. According to Definition 2.2, we have

$$\begin{aligned}\left|\int_0^t f(s)g(s)d_{p,q}s\right| &= (p-q)t\left|\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}t\right)g\left(\frac{q^n}{p^{n+1}}t\right)\right| \\ &= (p-q)t\left|\sum_{n=0}^{\infty}\sqrt{\frac{q^n}{p^{n+1}}}f\left(\frac{q^n}{p^{n+1}}t\right)\cdot\sqrt{\frac{q^n}{p^{n+1}}}g\left(\frac{q^n}{p^{n+1}}t\right)\right|.\end{aligned}$$

Now, applying the discrete Cauchy-Schwarz inequality on the recent inequality, we get that

$$\begin{aligned}\left|\int_0^t f(s)g(s)d_{p,q}s\right| &\leqslant (p-q)t\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f^2\left(\frac{q^n}{p^{n+1}}t\right)\right)^{\frac{1}{2}}\cdot\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}g^2\left(\frac{q^n}{p^{n+1}}t\right)\right)^{\frac{1}{2}} \\ &= \sqrt{(p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f^2\left(\frac{q^n}{p^{n+1}}t\right)}.\sqrt{(p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}g^2\left(\frac{q^n}{p^{n+1}}t\right)} \\ &= \sqrt{\int_0^t f^2(t)d_{p,q}t}.\sqrt{\int_0^t g^2(t)d_{p,q}t}.\end{aligned}$$

This completes the proof. $\square$

LEMMA 2.5. ((p,q)-Holder inequality) *Assume* $\alpha,\beta\in(1,+\infty)$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *and* $f,g:\mathbb{R}_+\to\mathbb{R}$ *are two* (p,q)*-Jackson integrable functions. Then, the* (p,q)*-Holder inequality*

$$\int_0^t |f(s)||g(s)|d_{p,q}s \leqslant \left(\int_0^t |f(s)|^{\alpha} d_{p,q}s\right)^{\frac{1}{\alpha}} \left(\int_0^t |g(s)|^{\beta} d_{p,q}s\right)^{\frac{1}{\beta}}, \qquad (2.5)$$

is satisfied.

Proof. Using Definition 2.2, we get

$$\begin{aligned}
&\int_0^t |f(s)||g(s)|d_{p,q}s \\
&= (p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|f\left(\frac{q^n}{p^{n+1}}t\right)\right|\left|g\left(\frac{q^n}{p^{n+1}}t\right)\right| \\
&= (p-q)t\sum_{n=0}^{\infty}\left\{\left(\frac{q^n}{p^{n+1}}\right)^{\frac{1}{\alpha}}\left|f\left(\frac{q^n}{p^{n+1}}t\right)\right|\right\}\left\{\left(\frac{q^n}{p^{n+1}}\right)^{\frac{1}{\beta}}\left|g\left(\frac{q^n}{p^{n+1}}t\right)\right|\right\}.
\end{aligned}$$

Now, applying the discrete Holder inequality on the recent inequality, yields

$$\begin{aligned}
&\int_0^t |f(s)||g(s)|d_{p,q}s \\
&\leqslant \left((p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|f\left(\frac{q^n}{p^{n+1}}t\right)\right|^{\alpha}\right)^{\frac{1}{\alpha}}\cdot\left((p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|g\left(\frac{q^n}{p^{n+1}}t\right)\right|^{\beta}\right)^{\frac{1}{\beta}} \\
&= \left(\int_0^t |f(t)|^{\alpha} d_{p,q}t\right)^{\frac{1}{\alpha}}\left(\int_0^t |g(t)|^{\beta} d_{p,q}t\right)^{\frac{1}{\beta}}.
\end{aligned}$$

This completes the proof. □

REMARK 2.6. If we concentrate on (p,q)-analogous of the Cauchy-Schwarz and Holder inequalities (2.4) and (2.5), respectively, we figure out this fact that if we consider the Jackson integral $\int_a^t z(s)d_{p,q}s$ instead $\int_0^t z(s)d_{p,q}s$, so Definition 2.2 and the property (P_5) in Theorem 2.3 imply that

$$\int_a^t z(s)d_{p,q}s := \int_0^t z(s)d_{p,q}s - \int_0^a z(s)d_{p,q}s,$$

and consequently, it follows that

$$\int_a^t z(s)d_{p,q}s := (p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}z\left(\frac{q^n}{p^{n+1}}t\right)-(p-q)a\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}z\left(\frac{q^n}{p^{n+1}}a\right).$$

Now, we present a counter example to show that why totality of the Cauchy-Schwarz inequality (2.4) is broken when we take the lower bound a nonzero. Let

$$f(s):=c,\ c\in\mathbb{R},\quad g(s):=\frac{1}{\sqrt{\frac{p^{n+1}}{q^n}s}},\ \ n=0,1,2,\dots\ .$$

It is easy to check that,

$$\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+2}}\left(tf\left(\frac{q^n}{p^{n+1}}t\right)g\left(\frac{q^n}{p^{n+1}}t\right)-af\left(\frac{q^n}{p^{n+1}}a\right)g\left(\frac{q^n}{p^{n+1}}a\right)\right)\right)^2 := \frac{c^2(\sqrt{t}-\sqrt{a})^2}{p^2(p-q)^2}$$

$$\sum_{n=0}^{\infty}\frac{q^n}{p^{n+2}}\left(tf^2\left(\frac{q^n}{p^{n+1}}t\right)-af^2\left(\frac{q^n}{p^{n+1}}a\right)\right) := \frac{c^2(t-a)}{p(p-q)},$$

$$\sum_{n=0}^{\infty}\frac{q^n}{p^{n+2}}\left(tg^2\left(\frac{q^n}{p^{n+1}}t\right)-ag^2\left(\frac{q^n}{p^{n+1}}a\right)\right) := 0.$$

In this case, the Cauchy-Schwarz inequality (2.4) holds for the lower bound $a \neq 0$, if and only if

$$\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+2}}\left(tf\left(\frac{q^n}{p^{n+1}}t\right)g\left(\frac{q^n}{p^{n+1}}t\right)-af\left(\frac{q^n}{p^{n+1}}a\right)g\left(\frac{q^n}{p^{n+1}}a\right)\right)\right)^2 \leqslant 0,$$

that is this inequality is not true unless for the trivial case $t = a$, $a \neq 0$. Now, if we take, $a = 0$, then we get that

$$\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+2}}\left(tf\left(\frac{q^n}{p^{n+1}}t\right)g\left(\frac{q^n}{p^{n+1}}t\right)\right)\right)^2 := \frac{c^2t}{p^2(p-q)^2}$$

$$\sum_{n=0}^{\infty}\frac{q^n}{p^{n+2}}\left(tf^2\left(\frac{q^n}{p^{n+1}}t\right)\right) := \frac{c^2t}{p(p-q)},$$

$$\sum_{n=0}^{\infty}\frac{q^n}{p^{n+2}}\left(tg^2\left(\frac{q^n}{p^{n+1}}t\right)\right) := \frac{1}{p(p-q)}.$$

As is expected, in this case the Cauchy-Schwarz inequality (2.4) is satisfied. This is the main reason that in this paper, we restrict ourselves to the Jackson integrals with the lower bound 0.

Here, we define so-called (p,q)-basic numbers as follows.

DEFINITION 2.7. [19] By (p,q)-basic number that we show it by the notation $[n]_{p,q}$, we mean

$$[n]_{p,q} := \frac{p^n - q^n}{p-q} = \sum_{k=0}^{n-1} p^{n-k-1}q^k. \tag{2.6}$$

Accordingly, the (p,q)-factorial is defined by

$$[n]_{p,q}! := \prod_{k=1}^{n}[k]_{p,q}, \quad [0]_{p,q}! := 1. \tag{2.7}$$

Consequently, we define so-called (p,q)-shifted factorials as follows.

DEFINITION 2.8. [19] By (p,q)-shifted factorial that we show it with the notation $((a,b);(p,q))_n$, we mean

$$((a,b);(p,q))_n := \begin{cases} 1; & n=0, \\ \prod_{k=0}^{n-1}\left(p^k a - q^k b\right); & n=1,2,\ldots \end{cases} \tag{2.8}$$

Two particular cases are mentioned as below:

$$((p,q);(p,q))_n := \begin{cases} 1; & n=0, \\ \prod_{k=0}^{n-1}\left(p^{k+1} - q^{k+1}\right) = (p-q)^n [n]_{p,q}!, & n=1,2,\ldots \end{cases} \tag{2.9}$$

$$((1,a);(1,q))_n := (a;q)_n = \begin{cases} 1; & n=0, \\ \prod_{k=0}^{n-1}\left(1 - q^k a\right); & n=1,2,\ldots \end{cases} \tag{2.10}$$

Also, $(a;q)_\infty := \lim_{n\to\infty}(a;q)_n$.

Using the above definitions, now we can define the (p,q)-trigonometric functions.

DEFINITION 2.9. The (p,q)-basic *sine* and *cosine* functions are defined by

$$\sin_{p,q} t := \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} \frac{t^{2n+1}}{(p-q)^{2n+1}}, \tag{2.11}$$

$$\cos_{p,q} t := \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} \frac{t^{2n}}{(p-q)^{2n}}. \tag{2.12}$$

A direct calculation indicates that,

$$D_{p,q}\sin_{p,q}(t) := \frac{1}{p-q}\cos_{p,q}(t), \quad D_{p,q}\cos_{p,q}(t) := \frac{-1}{p-q}\sin_{p,q}(t). \tag{2.13}$$

REMARK 2.10. Let us note that setting $p := 1$ in all of the above definitions, lemmas and theorems, gives us the corresponding results in the classic one-parametric q-calculus.

At the and of this section we mention this point that in what follows, we work in frame of the Banach space $(\mathfrak{B}; \|.\|)$, such that

$$\mathfrak{B} := \left\{u \middle|\, u : \mathbb{R}_+ \to \mathbb{R}\right\}, \quad \|u\| := \max_{t\in\mathbb{R}_+} |u(t)|.$$

3. Main results

This section contains two parts. In the first part that makes theoretical body of the our investigation, we obtain Lyapunov inequalities of the (p,q)-Hamiltonian system (1.13). To this aim, we have the following theorem.

THEOREM 3.1. *Assume (u,v) is a nontrivial solution of the (p,q)-Hamiltonian system (1.13) such that $u(0)=0$ and $u(\beta)=0$, for $\beta\in\mathbb{R}_+$ with $0<\beta$ and $u(t)\neq 0$ for $t\in(0,\beta)$. Then, the Lyapunov inequality of this Hamiltonian system is given by*

$$\int_0^\beta |a(t)|d_{p,q}t+\sqrt{\int_0^\beta b(t)d_{p,q}t}.\sqrt{\int_0^\beta c_+(t)d_{p,q}t}\geqslant 2. \tag{3.1}$$

Proof. The proof begins with the (p,q)-Leibniz rule (P_3) in Theorem 2.3. If we multiply the first equation of the (p,q)-Hamiltonian system (1.13) by $v(qt)$ and second equation by $u(pt)$, then, adding the resulting equations gives us the following equation:

$$(D_{p,q}uv)(t)=b(t)v^2(qt)-c(t)u^2(pt). \tag{3.2}$$

(p,q)-integrating of both sides of the equation (3.2) from $t=0$ to $t=\beta$, and then applying the property (P_7) in Theorem 2.3, we get the following equality:

$$\int_0^\beta b(t)v^2(qt)d_{p,q}t=\int_0^\beta c(t)u^2(pt)d_{p,q}t. \tag{3.3}$$

Now, if we separate the integrating interval $[0,\beta]$ into the two subintervals $[0,t]$ and $[t,\beta]$, then take (p,q)-integral from both sides of the first equation in (p,q)-Hamiltonian system (1.13), firstly from 0 to t, and then from t to β, after implying the boundary conditions $u(0)=0=u(\beta)$, we conclude that

$$\left.\begin{aligned} u(t)&=\int_0^t a(s)u(ps)d_{p,q}s+\int_0^t b(s)v(qs)d_{p,q}s,\\ -u(t)&=\int_t^\beta a(s)u(ps)d_{p,q}s+\int_t^\beta b(s)v(qs)d_{p,q}s.\end{aligned}\right\} \tag{3.4}$$

Thus we have

$$\begin{aligned} |u(t)|&\leqslant\int_0^t |a(s)||u(ps)|d_{p,q}s+\int_0^t b(s)|v(qs)|d_{p,q}s,\\ |u(t)|&\leqslant\int_t^\beta |a(s)||u(ps)|d_{p,q}s+\int_t^\beta b(s)|v(qs)|d_{p,q}s.\end{aligned} \tag{3.5}$$

Adding both sides of the recent inequalities, we get

$$2|u(t)|\leqslant\int_0^\beta |a(t)||u(pt)|d_{p,q}t+\int_0^\beta b(t)|v(qt)|d_{p,q}t. \tag{3.6}$$

Using the (p,q)-Cauchy-Schwarz inequality (2.4), we get

$$\begin{aligned}\int_0^\beta b(t)|v(qt)|d_{p,q}t &\leqslant \sqrt{\int_0^\beta b(t)d_{p,q}t}.\sqrt{\int_0^\beta b(t)v^2(qt)d_{p,q}t} \\ &\leqslant \sqrt{\int_0^\beta b(t)d_{p,q}t}.\sqrt{\int_0^\beta |c(t)|u^2(pt)d_{p,q}t}.\end{aligned}$$

So, we have

$$\int_0^\beta b(t)|v(qt)|d_{p,q}t \leqslant \sqrt{\int_0^\beta b(t)d_{p,q}t}.\sqrt{\int_0^\beta c_+(t)u^2(pt)d_{p,q}t}. \tag{3.7}$$

Now, applying the inequality (3.7) into the inequality (3.6), leads us to the inequality

$$2|u(t)| \leqslant \int_0^\beta |a(t)||u(pt)|d_{p,q}t + \sqrt{\int_0^\beta b(t)d_{p,q}t}.\sqrt{\int_0^\beta c_+(t)u^2(pt)d_{p,q}t}.$$

Here we take max-norm on both sides, to reach the (p,q)-Lyapunov inequality

$$\int_0^\beta |a(t)|d_{p,q}t + \sqrt{\int_0^\beta b(t)d_{p,q}t}.\sqrt{\int_0^\beta c_+(t)d_{p,q}t} \geqslant 2.$$

This completes the proof. $\square$

REMARK 3.2. By the use of Definition 2.2, one may represent the Lyapunov inequality (3.1) as infinite series corresponding to the (p,q)-Jackson integrals. To this aim, according to Definition 2.2 we expand the (p,q)-Jackson integrals in (3.1). Consequently, the Lyapunov inequality of the (p,q)-Hamiltonian system (1.13) takes itself the following form:

$$\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left|a\left(\frac{q^n}{p^{n+1}}\beta\right)\right| + \sqrt{\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} b\left(\frac{q^n}{p^{n+1}}\beta\right)}.\sqrt{\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} c_+\left(\frac{q^n}{p^{n+1}}\beta\right)} \geqslant \frac{2}{\beta(p-q)}. \tag{3.8}$$

But we are believed that this is not the end. Because we can apply another beautiful rule to reach new refinement of the (p,q)-Lyapunov inequality (3.8). Applying the Cauchy product of two infinite series

$$\sum_{n=0}^{\infty} \mu_n \sum_{n=0}^{\infty} \nu_n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mu_n \nu_{n-k},$$

into the (p,q)-Lyapunov inequality (3.8), we get the (p,q)-quantum representation

$$\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left|a\left(\frac{q^n}{p^{n+1}}\beta\right)\right| + \sqrt{\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{q^{2n-k}}{p^{2n-k+1}} b\left(\frac{q^n}{p^{n+1}}\beta\right) c_+\left(\frac{q^{n-k}}{p^{n-k+1}}\beta\right)} \geqslant \frac{2}{\beta(p-q)}. \tag{3.9}$$

COROLLARY 3.3. *If we set $p := 1$ in the Lyapunov inequalities (3.1) and (3.9), we get the following classic q-Lyapunov inequalities:*

$$\int_0^\beta |a(t)|d_qt + \sqrt{\int_0^\beta b(t)d_qt}.\sqrt{\int_0^\beta c_+(t)d_qt} \geqslant 2,$$

and

$$\sum_{n=0}^{\infty} q^n \Big| a(q^n\beta) \Big| + \sqrt{\sum_{n=0}^{\infty}\sum_{k=0}^{n} q^{2n-k} b(q^n\beta)\, c_+(q^{n-k}\beta)} \geqslant \frac{2}{\beta(1-q)}.$$

In this position we present an example to justify the Lyapunov inequality (3.1).

EXAMPLE 3.4. Consider the $(1,\sqrt{0.1})$-Hamiltonian system

$$D_{1,\sqrt{0.1}}u = v, \quad D_{1,\sqrt{0.1}}v = -u. \tag{3.10}$$

Actually, if we take $a(t) = 0$, $b(t) = c(t) = 1$, $p = 1$ and $q = \sqrt{0.1}$ in the Hamiltonian system (1.13), this setting gives us the Hamiltonian system (3.10) and the corresponding Hamiltonian

$$H(u,v) : \frac{u^2 - v^2}{1+\sqrt{0.1}}.$$

To find its solution, we apply Definition 2.9 to reach:

$$u(t) := (1-\sqrt{0.1})\sin_{1,\sqrt{0.1}}(t), \quad v(t) := (1-\sqrt{0.1})\cos_{1,\sqrt{0.1}}(t). \tag{3.11}$$

Unlike the classic trigonometric function $\sin(t)$, that has the exact real zeros $n\pi$, $n \in \mathbb{Z}$, exact real zeros of the $(1,\sqrt{q})$-trigonometric function $\sin_{1,\sqrt{q}}(t)$ have not been identified by now. Instead, its asymptotic real zeros for certain values of the quantum parameter q have been obtained for instance in [21] (Paper 11, page 203, Sec. 2.2, Tables of zeros and Sec. 2.3, Asymptotic of zeros, Eq. (2.13)) as follows:

$$\zeta_n = q^{\frac{1}{4}-n} - c_1(q) + o(1), \quad n \to \infty, \tag{3.12}$$

with

$$c_1(q) := \frac{1}{2}\frac{q^{\frac{1}{4}}}{1-q^{\frac{1}{2}}}\frac{(q;q^2)^2_\infty}{(q^2;q^2)^2_\infty}.$$

Accordingly, if we choose the value $q := 0.1$, then (3.12) gives us the corresponding asymptotic real zero of $\sin_{1,\sqrt{0.1}}$, for $n = 1$ as $\zeta_1 = 5.2733546903306$. Hence, we get that

$$u(0) := 0, \quad u(\zeta_1) := 0.$$

Reviewing the assumptions of Theorem 3.1, since all of the assumptions are satisfied, it follows that the Lyapunov inequality

$$
\begin{aligned}
&\int_0^{\beta} |a(t)| d_{p,q}t + \sqrt{\int_0^{\beta} b(t) d_{p,q}t}.\sqrt{\int_0^{\beta} c_+(t) d_{p,q}t} \\
&= \int_0^{5.2733546903306} 1 d_{1,\sqrt{0.1}}t \\
&= (1-\sqrt{0.1}) 5.2733546903306 \sum_{n=0}^{\infty} (\sqrt{0.1})^n, \quad \text{(Definition (2.2))} \\
&= 5.2733546903306 > 2,
\end{aligned}
$$

is satisfied.

Having the (p,q)-Lyapunov inequalities (3.1) and (3.9) of the (p,q)-Hamiltonian system (1.13) in hand, enables us to estimate qualitative dynamics of this Hamiltonian system. To this aim, we present the second part of the main results consisting of some disconjugacy and nonexistence criteria for the nontrivial solutions of the (p,q)-Hamiltonian system (1.13) as follows.

DEFINITION 3.5. *Disconjugacy*. The (p,q)-Hamiltonian system (1.13) is said to be disconjugate on the interval $[0,\beta]$ provided that there is no real solution (u,v) of this system with u nontrivial and having at least two zeros on $[0,\beta]$. Otherwise, the (p,q)-Hamiltonian system (1.13) is said to be conjugate on $[0,\beta]$.

THEOREM 3.6. *Assume all assumptions of Theorem 3.1 are satisfied. If*

$$
\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left| a\left(\frac{q^n}{p^{n+1}}\beta\right) \right| + \sqrt{\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{q^{2n-k}}{p^{2n-k+1}} b\left(\frac{q^n}{p^{n+1}}\beta\right) c_+\left(\frac{q^{n-k}}{p^{n-k+1}}\beta\right)} < \frac{2}{\beta(p-q)}. \tag{3.13}
$$

then, the (p,q)-Hamiltonian system (1.13) is disconjugate on $[0,\beta]$.

Proof. Suppose on the contrary that the (p,q)-Hamiltonian system (1.13) is conjugate on the interval $[0,\beta]$. So, according to Definition 3.5, there exists a solution (u,v) of the (p,q)-Hamiltonian system (1.13) such that u is nontrivial and has at lest two zeros t_1 and t_2, $t_1 < t_2$ in the interval $[0,\beta]$ and $u(t) \neq 0$ for $t \in (t_1,t_2)$. Hence, Theorem 3.1 and Remark 3.2 for setting $t_1 = 0$ and $t_2 = \beta$ imply that

$$
\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left| a\left(\frac{q^n}{p^{n+1}}\beta\right) \right| + \sqrt{\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{q^{2n-k}}{p^{2n-k+1}} b\left(\frac{q^n}{p^{n+1}}\beta\right) c_+\left(\frac{q^{n-k}}{p^{n-k+1}}\beta\right)} \geqslant \frac{2}{\beta(p-q)}.
$$

Since this inequality contradicts the assumption (3.13), then the resulting contradiction ensures disconjugacy of the (p,q)-Hamiltonian system (1.13). This completes the proof. $\square$

The next application of the Lyapunov inequality (3.9), deals with non-existence of nontrivial solutions for the (p,q)-Hamiltonian system (1.13). Indeed, we show that this disconjugacy criterion (Theorem 3.6) at the same time can be considered as a nonexistence criterion.

THEOREM 3.7. *Assume the conditions of Theorem 3.6 hold. Then the (p,q)-Hamiltonian system (1.13) has no nontrivial solution on the interval $[0,\beta]$.*

Proof. Suppose on the contrary that there exists at least one nontrivial solution for the (p,q)-Hamiltonian system (1.13). If we denote this solution by (u,v), then u is a nontrivial with $u(0)=0=u(\beta)$ and $u(t)\neq 0$ for $t\in(0,\beta)$. Thus, according to Theorem 3.1 and Remark 3.2, we have

$$\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left|a\left(\frac{q^n}{p^{n+1}}\beta\right)\right|+\sqrt{\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{q^{2n-k}}{p^{2n-k+1}}b\left(\frac{q^n}{p^{n+1}}\beta\right)c_+\left(\frac{q^{n-k}}{p^{n-k+1}}\beta\right)}\geqslant\frac{2}{\beta(p-q)},$$

that contradicts the inequality (3.13). So, the (p,q)-Hamiltonian system (1.13) has no nontrivial solution on $[0,\beta]$. □

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