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# PERIODIC SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFERENTIAL SYSTEMS

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*Abstract.* In this paper, we establish some existence and uniqueness results for periodic solutions for a class of fractional differential equations with the Caputo fractional derivative. The arguments are based upon the Banach contraction principle, and Schaefer's fixed point theorem.

## 1. Introduction

Recently, fractional differential equations [11, 15, 16, 17, 19] have been studied intensively [1, 2, 3, 4, 5, 9, 21]. The mathematical modeling of many real world phenomena based on fractional order operators is regarded as better and improved than the one depending on integer-order operators. In particular, fractional calculus has played a significant role in the recent development of special functions and integral transforms, signal processing, control theory, bioengineering and biomedical, viscoelasticity, finance, stochastic processes, wave and diffusion phenomena, plasma physics, social sciences, etc. The motivation for this work arises from both the development of the theory of fractional calculus itself and its wide application to various fields of science, such as physics, chemistry, biological, electromagnetic of complex media, robotics, economics, etc. Much attention has been paid to the existence and uniqueness of the solutions of fractional dynamic systems [6, 7, 8, 10, 14] on account of the fact that existence is the fundamental problem and a necessary condition for considering some other properties for fractional dynamic systems, such as controllability, stability, etc. In [18] the authors provide some existence results for the dynamical system

$$^{c}D^{\alpha}_{0^+}x(t) - A\,^{c}D^{\beta}_{0^+}x(t) = f(t,x(t));\;\; t \in J := [0,T], \tag{1.1}$$

$$x(0) = x_0, \quad x'(0) = x'_0. \tag{1.2}$$

Motivated by the previous work [18], the purpose of this paper is to establish some existence and uniqueness results for the following functional fractional differential equation

$$^{c}D^{\alpha}_{0^+}x(t) - A\,^{c}D^{\beta}_{0^+}x(t) = f\left(t, x(t), {}^{c}D^{\beta}_{0^+}x(t), {}^{c}D^{\alpha}_{0^+}x(t)\right);\;\; t \in J := [0,T], \tag{1.3}$$

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with the periodic conditions

$$x(0) = x(T), \quad x'(0) = x'(T), \tag{1.4}$$

where $T > 0$, ${}^cD_0^\alpha$ is the Caputo fractional derivative of order $\alpha$, $0 < \beta \leqslant 1 < \alpha \leqslant 2$, $1 + \beta < \alpha$, $f : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given continuous function, and $A$ is an $\mathbb{R}^{n\times n}$ invertible matrix.

The present paper is organized as follows: In Section 2, some notations are introduced and we recall some concepts of preliminaries about fractional calculus and auxiliary results. The main results are presented in Section 3; by applying the Banach fixed point theorem and Schaefer's fixed point theorem. In the last Section, we give an illustrative example.

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J,\mathbb{R}^n)$ we denote the Banach space of continuous functions from $J$ into $\mathbb{R}^n$ with the norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in J\},$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

Denote $L^1(J,\mathbb{R}^n)$ the Banach space of measurable functions $u : J \longrightarrow \mathbb{R}^n$ that are Lebesgue integrable with norm

$$\|x\|_{L^1} = \int_0^T \|x(t)\| dt.$$

Let $AC(J,\mathbb{R}^n)$ be the space of absolutely continuous valued functions on $J$, and set

$$AC^n(J) = \{x : J \to \mathbb{R}^n : x, x', x'', \ldots, x^{(n-1)} \in C(J,\mathbb{R}^n) \text{ and } x^{(n-1)} \in AC(J,\mathbb{R}^n)\}.$$

By

$$C^1(J,\mathbb{R}^n) = \{x : J \to \mathbb{R}^n \text{ where } x' \in C(J,\mathbb{R}^n)\},$$

we denote the Banach space with the norm

$$\|x\|_1 = \max\{\|x\|_\infty, \|x'\|_\infty\}.$$

DEFINITION 1. ([15]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $h \in L^1((a,b],\mathbb{R}^n)$ is given by

$$I_{a^+}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds, \ t \in (a,b],$$

where $\Gamma$ is the Euler gamma function defined by

$$\Gamma(\xi) = \int_0^{+\infty} t^{\xi-1} e^{-t} dt; \ \xi > 0.$$

DEFINITION 2. ([15]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $h \in L^1((a,b],\mathbb{R}^n)$ is given by

$$D^{\alpha}_{a^+}h(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{h(s)}{(t-s)^{\alpha-n+1}}ds; \;\; t \in (a,b],$$

provided the right hand side is well defined for almost every $t \in (a,b)$, where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the real number $\alpha$.

DEFINITION 3. ([15]) The Caputo fractional of order $\alpha > 0$ of a function $h \in AC^n((a,b],\mathbb{R}^n)$ is defined via the above Riemann-Liouville derivatives by

$${}^cD^{\alpha}_{a^+}h(t) = \left(D^{\alpha}_{a^+}\left[h(t) - \sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(t-a)^k\right]\right)(t); \;\; t \in (a,b].$$

DEFINITION 4. ([15]) For a function $h \in AC^n(J)$, the Caputo fractional-order derivative of order $\alpha$ of $h$ is defined by

$$\left({}^cD^{\alpha}_{0^+}h\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{h^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,$$

LEMMA 1. ([15]) *From the definition of fractional integrals and Caputo derivatives, we have*

$$I^{\alpha}_{0^+}\left({}^cD^{\alpha}_{0^+}h(t)\right) = h(t) - \sum_{k=0}^{n-1}\frac{h^{(k)}(0)}{k!}t^k; \;\; t>0, \;\; n-1<\alpha<n.$$

*Especially, when* $1<\alpha<2$, *then we have*

$$I^{\alpha}_{0^+}\left({}^cD^{\alpha}_{0^+}h(t)\right) = h(t) - h(0) - th'(0).$$

LEMMA 2. ([15]) *Let* $\alpha > 0$ *and* $h \in C(J,\mathbb{R}^n)$. *Then*

$$\left({}^cD^{\alpha}_{0^+}I^{\alpha}_{0^+}h(t)\right) = h(t),$$

*holds on* $J$.

LEMMA 3. ([12]) *Let* $0<\beta<1<\alpha<2$, *and* $x \in C^1(J,\mathbb{R}^n)$. *Then we have*

$$I^{\alpha}_{0^+}\left({}^cD^{\beta}_{0^+}x(t)\right) = I^{\alpha-\beta}_{0^+}(x(t)) - \frac{x(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.$$

We state the following generalization of Gronwall's lemma for singular kernels.

LEMMA 4. ([20]) *Let* $\upsilon : [0,T] \longrightarrow [0,+\infty)$ *be a real function and* $\omega(\cdot)$ *is a nonnegative, locally integrable function on* $[0,T]$. *Assume that there are constants a* $>0$ *and* $0<\alpha\leqslant 1$ *such that*

$$\upsilon(t) \leqslant \omega(t) + a\int_0^t (t-s)^{-\alpha}\upsilon(s)\,ds.$$

*Then, there exists a constant* $K = K(\alpha)$ *such that*

$$\upsilon(t) \leqslant \omega(t) + Ka\int_0^t (t-s)^{-\alpha}\omega(s)\,ds, \;\; \textit{for every } t \in [0,T].$$

THEOREM 1. ([13]) (Banach's fixed point theorem) *Let $C$ be a non-empty closed subset of a Banach space $X$, then any contraction mapping $T$ of $C$ into itself has a unique fixed point.*

THEOREM 2. [13] (Schaefer's fixed point theorem) *Let $X$ be a Banach space, and $N : X \longrightarrow X$ completely continuous operator.*

*If the set $\mathscr{E} = \{y \in X : \ y = \lambda Ny, \text{ for some } \lambda \in (0,1)\}$ is bounded, then $N$ has fixed points.*

## 3. Existence of solutions

LEMMA 5. *For any $x \in C^1(J,\mathbb{R}^n)$ and $0 < \beta \leqslant 1$, we have*

$$\|{}^cD^{\beta}_{0+}x\|_\infty \leqslant \frac{T^{1-\beta}}{\Gamma(2-\beta)}\|x'\|_\infty, \text{ and so } \|{}^cD^{\beta}_{0+}x\|_\infty \leqslant \frac{T^{1-\beta}}{\Gamma(2-\beta)}\|x\|_1.$$

*Proof.* Obviously, when $\beta = 1$, the conclusions are true. So, we only consider the case $0 < \beta < 1$. In fact, by Definition 4, for any $x \in C^1(J,\mathbb{R}^n)$ and $t \in J$, we have

$$\begin{aligned}
|{}^cD^{\beta}_{0+}x(t)| &= \frac{1}{\Gamma(1-\beta)}\left|\int_0^t (t-s)^{-\beta}x'(s)ds\right| \\
&\leqslant \|x'\|_\infty \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}ds \\
&= \frac{t^{1-\beta}}{\Gamma(2-\beta)}\|x'\|_\infty \\
&\leqslant \frac{T^{1-\beta}}{\Gamma(2-\beta)}\|x'\|_\infty \\
&\leqslant \frac{T^{1-\beta}}{\Gamma(2-\beta)}\|x\|_1. \quad \square
\end{aligned}$$

LEMMA 6. *Let $h \in C(J,\mathbb{R}^n)$. The function $x \in C^1(J,\mathbb{R}^n)$ is a periodic solution of the fractional differential problem*

$$^cD^{\alpha}_{0+}x(t) - A\,{}^cD^{\beta}_{0+}x(t) = h(t), \ t \in J, \tag{3.1}$$

$$x(0) = x(T), \quad x'(0) = x'(T), \tag{3.2}$$

*if and only if, $x$ is a solution of the fractional integral equation*

$$\begin{aligned}
x(t) = &\left(1 - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)x(0) + tx'(0) + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds,
\end{aligned} \tag{3.3}$$

*with*

$$x(0) = T^{1+\beta-\alpha}(\alpha-\beta-1)\int_0^T (T-s)^{\alpha-\beta-2}x(s)ds + \frac{T^{1+\beta-\alpha}\Gamma(\alpha-\beta)A^{-1}}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}h(s)ds, \tag{3.4}$$

*and*

$$\begin{aligned} x'(0) = & \frac{A}{(\alpha-\beta)\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}x(s)ds \\ & + \frac{1}{(\alpha-\beta)\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}h(s)ds \\ & - \frac{A}{T\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}x(s)ds \\ & - \frac{1}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds. \end{aligned} \tag{3.5}$$

*Proof.* Let $x \in C^1(J,\mathbb{R}^n)$ be a solution of (3.1)–(3.2). Then, we have

$$I^{\alpha}_{0^+}({}^cD^{\alpha}_{0^+}x(t) - A\,{}^cD^{\beta}_{0^+}x(t)) = (I^{\alpha}_{0^+}h)(t)$$

From Lemma 1, we get

$$x(t) = x(0) + tx'(0) - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}x(0) + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds.$$

Applying conditions (3.2), we obtain (3.4) and (3.5). Thus, $x$ is solution of the integral equation (3.3).

Conversely, assume that $x$ satisfies the fractional integral equation (3.3), and using the fact that ${}^cD^{\alpha}_{0^+}$ is the left inverse of $I^{\alpha}_{0^+}$, and the fact that ${}^cD^{\alpha}_{0^+}C = 0$, where $C$ is a constant, we get

$${}^cD^{\alpha}_{0^+}x(t) - A\,{}^cD^{\beta}_{0^+}x(t) = h(t), \quad \text{for each } t \in J.$$

Also, we can easily show that

$$x(0) = x(T) \text{ and } x'(0) = x'(T).$$

We are now in a position to state and prove our existence result for the problem (1.3)–(1.4) based on Banach's fixed point, we need to give the following hypothesis:

(H) There exist constants $L_1, L_2 > 0$ and $0 < L_3 < 1$ such that

$$\|f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w})\| \leqslant L_1\|u-\bar{u}\| + L_2\|v-\bar{v}\| + L_3\|w-\bar{w}\|$$

for any $u,v,w,\bar{u},\bar{v},\bar{w} \in \mathbb{R}^n$ and $t \in J$.

Set

$$R_1 = \left(\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+1\right)M_1 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha}L_1\Gamma(2-\beta)+T^{1-\beta+\alpha}(L_3\|A\|+L_2)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_3)} + TM_2,$$

$$R_2 = M_2 + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1 + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{T^{\alpha-1}L_1\Gamma(2-\beta)+T^{\alpha-\beta}(L_3\|A\|+L_2)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_3)},$$

where

$$M_1 = 1 + \frac{T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|\left[L_1\Gamma(2-\beta)+T^{1-\beta}(L_3\|A\|+L_2)\right]}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_3)},$$

and

$$M_2 = \frac{T^{\alpha-1}(2\alpha-\beta)\left[L_1\Gamma(2-\beta)+T^{1-\beta}(L_3\|A\|+L_2)\right]}{(\alpha-\beta)\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_3)} + \frac{2T^{\alpha-\beta-1}\|A\|}{\Gamma(\alpha-\beta+1)}. \quad \square$$

THEOREM 3. *Assume that* $(H)$ *holds. If*

$$\max\{R_1,R_2\} < 1, \tag{3.6}$$

*then the problem* (1.3)–(1.4) *has a unique solution on* $J$.

*Proof.* Transform the problem (1.3)–(1.4) into a fixed point problem. Consider the operator $N : C^1(J,\mathbb{R}^n) \to C^1(J,\mathbb{R}^n)$ defined by

$$(Nx)(t) = \left(1-\frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)B_1 + tB_2 + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds, \tag{3.7}$$

with

$$B_1 = T^{1+\beta-\alpha}(\alpha-\beta-1)\int_0^T (T-s)^{\alpha-\beta-2}x(s)ds + \frac{T^{1+\beta-\alpha}\Gamma(\alpha-\beta)A^{-1}}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}g(s)ds,$$

and

$$B_2 = \frac{A}{(\alpha-\beta)\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}x(s)ds + \frac{1}{(\alpha-\beta)\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}g(s)ds - \frac{A}{T\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}x(s)ds$$

$$-\frac{1}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g(s)ds,$$

where $g \in C(J,\mathbb{R}^n)$ be such that

$$g(t) = f\left(t, x(t), {}^cD^{\beta}_{0^+}x(t), g(t) + A\,{}^cD^{\beta}_{0^+}x(t)\right).$$

For every $x \in C^1(J,\mathbb{R}^n)$ and each $t \in J$, we have

$$\begin{aligned}(Nx)'(t) = B_2 - \frac{(\alpha-\beta)At^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}B_1 + \frac{A}{\Gamma(\alpha-\beta-1)}\int_0^t (t-s)^{\alpha-\beta-2}x(s)ds \\ + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}g(s)ds.\end{aligned} \tag{3.8}$$

It is clear that $(Nx)' \in C(J,\mathbb{R}^n)$, consequently, $N$ is well defined.

Clearly, the fixed points of operator $N$ are solutions of problem (1.3)–(1.4).

Let $x, y \in C^1(J,\mathbb{R}^n)$. Then for $t \in J$, we have

$$\begin{aligned}\|(Nx)(t) - (Ny)(t)\| \leqslant \left(1 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\|B_1 - \tilde{B}_1\| + T\|B_2 - \tilde{B}_2\| \\ + \frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}\|x(s)-y(s)\|ds \\ + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|g(s)-h(s)\|ds,\end{aligned}$$

with

$$\begin{aligned}\tilde{B}_1 = T^{1+\beta-\alpha}(\alpha-\beta-1)\int_0^T (T-s)^{\alpha-\beta-2}y(s)ds \\ + \frac{T^{1+\beta-\alpha}\Gamma(\alpha-\beta)A^{-1}}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}h(s)ds,\end{aligned}$$

and

$$\begin{aligned}\tilde{B}_2 = \frac{A}{(\alpha-\beta)\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}y(s)ds \\ + \frac{1}{(\alpha-\beta)\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}h(s)ds - \frac{A}{T\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}y(s)ds \\ - \frac{1}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds,\end{aligned}$$

where $h \in C(J,\mathbb{R}^n)$ be such that

$$h(t) = f\left(t, y(t), {}^cD^{\beta}_{0^+}y(t), h(t) + A\,{}^cD^{\beta}_{0^+}y(t)\right).$$

From $(H)$, for each $t \in J$ we have

$$\|g(t) - h(t)\| \leqslant L_1\|x(t) - y(t)\| + L_2\left\|{}^cD^{\beta}_{0^+}x(t) - {}^cD^{\beta}_{0^+}y(t)\right\|$$

$$
\begin{aligned}
&+L_3\left\|g(t)+A{}^cD^{\beta}_{0^+}x(t)-h(t)-A{}^cD^{\beta}_{0^+}y(t)\right\| \\
&\leqslant L_1\|x(t)-y(t)\|+L_2\left\|{}^cD^{\beta}_{0^+}x(t)-{}^cD^{\beta}_{0^+}y(t)\right\| \\
&+L_3\|g(t)-h(t)\|+L_3\|A\|\left\|{}^cD^{\beta}_{0^+}x(t)-{}^cD^{\beta}_{0^+}y(t)\right\| \\
&\leqslant L_1\|x(t)-y(t)\|+L_3\|g(t)-h(t)\|+(L_3\|A\|+L_2)\left\|{}^cD^{\beta}_{0^+}(x(t)-y(t))\right\|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|g(t)-h(t)\| &\leqslant \frac{L_1}{1-L_3}\|x(t)-y(t)\|+\frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD^{\beta}_{0^+}(x(t)-y(t))\right\| \\
&\leqslant \frac{L_1}{1-L_3}\|x-y\|_\infty+\frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD^{\beta}_{0^+}(x-y)\right\|_\infty.
\end{aligned}
$$

Then, by using Lemma 5, we get

$$
\begin{aligned}
\|g(t)-h(t)\| &\leqslant \frac{L_1}{1-L_3}\|x-y\|_1+\frac{T^{1-\beta}(L_3\|A\|+L_2)}{\Gamma(2-\beta)(1-L_3)}\|x-y\|_1 \\
&= \frac{L_1\Gamma(2-\beta)+T^{1-\beta}(L_3\|A\|+L_2)}{\Gamma(2-\beta)(1-L_3)}\|x-y\|_1.
\end{aligned}
\tag{3.9}
$$

Using (3.9) we obtain

$$
\begin{aligned}
\|B_1-\tilde{B}_1\| &\leqslant \left[\frac{T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|\left[L_1\Gamma(2-\beta)+T^{1-\beta}(L_3\|A\|+L_2)\right]}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_3)}+1\right]\|x-y\|_1 \\
&= M_1\|x-y\|_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\|B_2-\tilde{B}_2\| &\leqslant \left[\frac{T^{\alpha-1}(2\alpha-\beta)\left[L_1\Gamma(2-\beta)+T^{1-\beta}(L_3\|A\|+L_2)\right]}{(\alpha-\beta)\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_3)}\right. \\
&\quad \left.+\frac{2T^{\alpha-\beta-1}\|A\|}{\Gamma(\alpha-\beta+1)}\right]\|x-y\|_1 \\
&= M_2\|x-y\|_1.
\end{aligned}
$$

Therefore, for each $t\in J$ we have

$$
\begin{aligned}
\|(Nx)(t)-(Ny)(t)\| &\leqslant \left[\left(\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+1\right)M_1+TM_2+\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right. \\
&\quad \left.+\frac{T^{\alpha}L_1\Gamma(2-\beta)+T^{1-\beta+\alpha}(L_3\|A\|+L_2)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_3)}\right]\|x-y\|_1 \\
&= R_1\|x-y\|_1.
\end{aligned}
$$

On the other hand, for each $t\in J$ we have

$$
\|(Nx)'(t)-(Ny)'(t)\| \leqslant \|B_2-\tilde{B}_2\|+\frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}\|B_1-\tilde{B}_1\|
$$

$$+\frac{\|A\|}{\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}\|x(s)-y(s)\|ds$$
$$+\frac{1}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\|g(s)-h(s)\|ds.$$

Using (3.9) we get

$$\begin{aligned}\|(Nx)'(t)-(Ny)'(t)\| \leqslant & \Bigg[M_2+\frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1+\frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \\ & +\frac{T^{\alpha-1}L_1\Gamma(2-\beta)+T^{\alpha-\beta}(L_3\|A\|+L_2)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_2)}\Bigg]\|x-y\|_1 \\ = & R_2\|x-y\|_1.\end{aligned}$$

Thus

$$\|N(x)-N(y)\|_1 \leqslant \max\{R_1,R_2\}\|x-y\|_1.$$

By (3.6), the operator $N$ is a contraction. Hence, by the Banach contraction principle, the operator $N$ has a unique fixed point which is the unique solution of (1.3)–(1.4).

Our second result is based on Schaefer's fixed point theorem. Set

$$\begin{aligned}R = & 1+\frac{2\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+\frac{2\|A\|(2T^{\alpha-\beta}+T^{\alpha-\beta-1})}{\Gamma(\alpha-\beta+1)} \\ & +D\left(\frac{(2\alpha-\beta)(T^{\alpha}+T^{\alpha-1})}{(\alpha-\beta)\Gamma(\alpha+1)}+\frac{T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|+2T^{\alpha-1}}{\Gamma(\alpha)}+\frac{T^{\alpha}(2\alpha-\beta)}{(\alpha-\beta)\Gamma(\alpha+1)}\right),\end{aligned}$$

where

$$D=\frac{L_1}{1-L_3}+\frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}. \quad \square$$

THEOREM 4. *Assume that $(H)$ holds. If $R<1$, then the problem (1.3)–(1.4) has at least one solution on $J$.*

*Proof.* Let $N$ be the operator defined in (3.7). The proof will be given in several steps.

*Step* 1: *$N$ is continuous.*
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C^1(J,\mathbb{R}^n)$. Then for each $t\in J$,

$$\begin{aligned}\|(Nx)(t)-(Nx_n)(t)\| \leqslant & \left(\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+1\right)\|B_1-\tilde{B_{n_1}}\|+T\|B_2-\tilde{B_{n_2}}\| \\ & +\frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}\|x(s)-x_n(s)\|ds \\ & +\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|g(s)-g_n(s)\|ds,\end{aligned}$$

where $\tilde{B_{n_1}}, \tilde{B_{n_2}} \in \mathbb{R}^n$, with

$$\begin{aligned}\tilde{B_{n_1}} &= T^{1+\beta-\alpha}(\alpha-\beta-1)\int_0^T (T-s)^{\alpha-\beta-2}x_n(s)ds \\ &\quad + \frac{T^{1+\beta-\alpha}\Gamma(\alpha-\beta)A^{-1}}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}g_n(s)ds,\end{aligned}$$

$$\begin{aligned}\tilde{B_{n_2}} &= \frac{A}{(\alpha-\beta)\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}x_n(s)ds \\ &\quad + \frac{1}{(\alpha-\beta)\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}g_n(s)ds \\ &\quad - \frac{A}{T\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}x_n(s)ds \\ &\quad - \frac{1}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g_n(s)ds,\end{aligned}$$

and $g_n \in C(J,\mathbb{R}^n)$ such that

$$g_n(t) = f\left(t, x_n(t), {}^cD^{\beta}_{0^+}x_n(t), g_n(t) + A\,{}^cD^{\beta}_{0^+}x_n(t)\right).$$

From $(H)$ for each $t \in J$ we have

$$\begin{aligned}\|g(t)-g_n(t)\| &\leqslant L_1\|x(t)-x_n(t)\| + L_2\left\|g(t)+A\,{}^cD^{\beta}_{0^+}x(t) - g_n(t) - A\,{}^cD^{\beta}_{0^+}x_n(t)\right\| \\ &\quad + L_3\left\|{}^cD^{\beta}_{0^+}x(t) - {}^cD^{\beta}_{0^+}x_n(t)\right\| \\ &\leqslant L_1\|x(t)-x_n(t)\| + L_2\|g(t)-g_n(t)\| \\ &\quad + L_2\|A\|\left\|{}^cD^{\beta}_{0^+}x(t) - {}^cD^{\beta}_{0^+}x_n(t)\right\| \\ &\quad + L_3\left\|{}^cD^{\beta}_{0^+}x(t) - {}^cD^{\beta}_{0^+}x_n(t)\right\| \\ &\leqslant L_1\|x(t)-x_n(t)\| + L_2\|g(t)-g_n(t)\| \\ &\quad + (L_2\|A\| + L_3)\left\|{}^cD^{\beta}_{0^+}(x(t)-x_n(t))\right\|.\end{aligned}$$

Thus

$$\begin{aligned}\|g(t)-g_n(t)\| &\leqslant \frac{L_1}{1-L_3}\|x(t)-x_n(t)\| \\ &\quad + \frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD^{\beta}_{0^+}(x(t)-x_n(t))\right\| \\ &\leqslant \frac{L_1}{1-L_3}\|x-x_n\|_\infty + \frac{L_3\|A\|+L_2}{1-L_3}\left\|{}^cD^{\beta}_{0^+}(x-x_n)\right\|_\infty.\end{aligned}$$

Then, by using Lemma 5, we obtain

$$\begin{aligned}\|g(t)-g_n(t)\| &\leqslant \frac{L_1}{1-L_3}\|x-x_n\|_1+\frac{T^{1-\beta}(L_3\|A\|+L_2)}{\Gamma(2-\beta)(1-L_3)}\|x-x_n\|_1\\ &=\frac{L_1\Gamma(2-\beta)+T^{1-\beta}\left(L_3\|A\|+L_2\right)}{\Gamma(2-\beta)(1-L_3)}\|x-x_n\|_1.\end{aligned}\tag{3.10}$$

Hence,

$$\begin{aligned}\|B_1-\tilde{B_{n_1}}\| &\leqslant \left[1+\frac{T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|\left[L_1\Gamma(2-\beta)+T^{1-\beta}\left(L_3\|A\|+L_2\right)\right]}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_3)}\right]\\ &\quad\times\|x-x_n\|_1\\ &=M_1\|x-x_n\|_1,\end{aligned}$$

and

$$\begin{aligned}\|B_2-\tilde{B_{n_2}}\| \leqslant &\left[\frac{T^{\alpha-1}(2\alpha-\beta)\left[L_1\Gamma(2-\beta)+T^{1-\beta}\left(L_3\|A\|+L_2\right)\right]}{(\alpha-\beta)\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_3)}\right.\\ &\left.+\frac{2T^{\alpha-\beta-1}\|A\|}{\Gamma(\alpha-\beta+1)}\right]\|x-x_n\|_1\\ =\ &M_2\|x-x_n\|_1.\end{aligned}$$

Therefore, for each $t\in J$ we get

$$\begin{aligned}\|(Nx)(t)-(Nx_n)(t)\| \leqslant &\left[\left(\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+1\right)M_1+TM_2+\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right.\\ &\left.+\frac{T^{\alpha}L_1\Gamma(2-\beta)+T^{1-\beta+\alpha}\left(L_3\|A\|+L_2\right)}{\Gamma(\alpha+1)\Gamma(2-\beta)(1-L_3)}\right]\|x-x_n\|_1\\ =\ &R_1\|x-x_n\|_1.\end{aligned}$$

On the other hand, for each $t\in J$ we have

$$\begin{aligned}\|(Nx)'(t)-(Nx_n)'(t)\| \leqslant\ &\|B_2-\tilde{B_{n_2}}\|+\frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}\|B_1-\tilde{B_{n_1}}\|\\ &+\frac{\|A\|}{\Gamma(\alpha-\beta-1)}\int_0^T(T-s)^{\alpha-\beta-2}\|x(s)-x_n(s)\|ds\\ &+\frac{1}{\Gamma(\alpha-1)}\int_0^T(T-s)^{\alpha-2}\|g(s)-g_n(s)\|ds.\end{aligned}$$

Using (3.10) we get

$$\begin{aligned}\|(Nx)'(t)-(Nx_n)'(t)\| &\leqslant \Bigg[M_2+\frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_1+\frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\\ &\quad+\frac{T^{\alpha-1}L_1\Gamma(2-\beta)+T^{\alpha-\beta}\left(L_3\|A\|+L_2\right)}{\Gamma(\alpha)\Gamma(2-\beta)(1-L_3)}\Bigg]\|x-x_n\|_1\\ &= R_2\|x-x_n\|_1.\end{aligned}$$

Thus, $\|Nx-Nx_n\|_1\to 0$ as $n\to\infty$, which implies that the operator $N$ is continuous.

*Step* 2: *$N$ maps bounded sets into bounded sets in $C^1(J,\mathbb{R}^n)$.*

Indeed, it is enough to show that for any $\eta^*>0$, there exists a positive constant $\ell$ such that for each $x\in B_{\eta^*}=\{x\in C^1(J,\mathbb{R}^n):\|x\|_1\leqslant\eta^*\}$, we have $\|N(x)\|_1\leqslant\ell$. We have for each $t\in J$,

$$\begin{aligned}\|g(t)\| &= \left\|f\left(t,x(t),g(t)+A{}^cD^{\beta}_{0^+}x(t),{}^cD^{\beta}_{0^+}x(t)\right)-f(t,0,0,0)+f(t,0,0,0)\right\|\\ &\leqslant L_1\|x(t)\|+L_3\|g(t)+A{}^cD^{\beta}_{0^+}x(t)\|+L_2\|{}^cD^{\beta}_{0^+}x(t)\|+\|f(t,0,0,0)\|\\ &\leqslant L_1\|x\|_\infty+L_3\|g(t)\|+(L_3\|A\|+L_2)\|D^{\beta}_{0^+}x\|_\infty+f^*,\end{aligned}$$

where

$$f^*=\sup_{t\in J}\|f(t,0,0,0)\|.$$

Thus

$$\|g(t)\|\leqslant\frac{L_1}{1-L_3}\|x\|_\infty+\frac{L_3\|A\|+L_2}{1-L_3}\|D^{\beta}_{0^+}x\|_\infty+\frac{f^*}{1-L_3}.$$

Then, by using Lemma 5, we have

$$\begin{aligned}\|g(t)\| &\leqslant \frac{L_1}{1-L_3}\|x\|_\infty+\frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\|x'\|_\infty+\frac{f^*}{1-L_3}\\ &\leqslant \frac{L_1}{1-L_3}\|x\|_1+\frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\|x\|_1+\frac{f^*}{1-L_3}\\ &\leqslant \frac{L_1}{1-L_3}\eta^*+\frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\eta^*+\frac{f^*}{1-L_3}:=M_1,\end{aligned}\tag{3.11}$$

which implies that

$$\begin{aligned}\|B_1\| &\leqslant T^{1+\beta-\alpha}(\alpha-\beta-1)\int_0^T(T-s)^{\alpha-\beta-2}\|x(s)\|ds\\ &\quad+\frac{T^{1+\beta-\alpha}\Gamma(\alpha-\beta)\|A^{-1}\|}{\Gamma(\alpha-1)}\int_0^T(T-s)^{\alpha-2}\|g(s)\|ds\\ &\leqslant \eta^*+\frac{T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|}{\Gamma(\alpha)}M_1:=M_2,\end{aligned}$$

and

$$\begin{aligned}\|B_2\| \leqslant & \frac{\|A\|}{(\alpha-\beta)\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}\|x(s)\|ds \\ & +\frac{1}{(\alpha-\beta)\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\|g(s)\|ds \\ & +\frac{\|A\|}{T\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}\|x(s)\|ds \\ & +\frac{1}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|g(s)\|ds \\ \leqslant & \frac{2\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}\eta^* + \frac{(2\alpha-\beta)T^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha+1)}M_1 := M_3.\end{aligned}$$

Thus (3.7) implies

$$\|(Nx)(t)\| \leqslant M_2 + TM_3 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}M_2 + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\eta^* + \frac{T^{\alpha}}{\Gamma(\alpha+1)}M_1 := \ell_1.$$

On the other hand we have, for each $t \in J$

$$\|(N'x)(t)\| \leqslant M_3 + \frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}M_2 + \frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\eta^* + \frac{T^{\alpha-1}}{\Gamma(\alpha)}M_1 := \ell_2.$$

Thus, $\|Nx\|_1 \leqslant \max\{\ell_1,\ell_2\} := \ell$.

*Step 3: $N$ maps bounded sets into equicontinuous sets of $C^1(J,\mathbb{R}^n)$.*
Let $t_1,t_2 \in J$, $t_1 < t_2$, $B_{\eta^*}$ be a bounded set of $C^1(J,\mathbb{R}^n)$ as in Step 2, and let $x \in B_{\eta^*}$. Then

$$\begin{aligned} & \|(Nx)(t_2)-(Nx)(t_1)\| \\ \leqslant & M_3(t_2-t_1) + \frac{\|A\|M_2}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}) \\ & +\frac{\|A\|\eta^*}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}ds \\ & +\frac{\|A\|\eta^*}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1}\right]ds \\ & +\frac{M_1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds + \int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds\right] \\ \leqslant & M_3(t_2-t_1) + \frac{\|A\|M_2}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}) \\ & +\frac{\|A\|\eta^*}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}) + \frac{M_1}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha}).\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Now from (3.8), we have

$$
\begin{aligned}
&\|(N'x)(t_2)-(N'x)(t_1)\| \\
\leqslant\ & \frac{(\alpha-\beta)\|A\|M_2}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}) \\
&+\frac{\|A\|\eta^*}{\Gamma(\alpha-\beta-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-2}ds \\
&+\frac{\|A\|\eta^*}{\Gamma(\alpha-\beta-1)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\beta-2}-(t_1-s)^{\alpha-\beta-2}\right]ds \\
&+\frac{M_1}{\Gamma(\alpha-1)}\left[\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}ds+\int_0^{t_1}\left[(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}\right]ds\right] \\
\leqslant\ & \frac{(\alpha-\beta)\|A\|M_2}{\Gamma(\alpha-\beta+1)}(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}) \\
&+\frac{\|A\|\eta^*}{\Gamma(\alpha-\beta)}(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1})+\frac{M_1}{\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1}).
\end{aligned}
$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Ascoli-Arzela theorem, we can conclude that $N: C^1(J,\mathbb{R}^n)\to C^1(J,\mathbb{R}^n)$ is completely continuous.

*Step* 4: *A priori bounds.*

Now it remains to show that the set

$$E=\{x\in C^1(J,\mathbb{R}^n): x=\lambda N(x) \text{ for some } 0<\lambda<1\}$$

is bounded. Let $x\in E$, then $x=\lambda N(x)$ for some $0<\lambda<1$. Thus, for each $t\in J$ we have

$$
\begin{aligned}
x(t)=\ &\lambda B_1+\lambda tB_2-\frac{\lambda At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}B_1+\frac{\lambda A}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds \\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)ds.
\end{aligned}
$$

From (H), for each $t\in J$ we have

$$
\begin{aligned}
\|g(t)\|=\ &\left|f\left(t,x(t),{}^cD_{0^+}^{\beta}x(t),g(t)+A{}^cD_{0^+}^{\beta}x(t)\right)-f(t,0,0,0)+f(t,0,0,0)\right\| \\
\leqslant\ & L_1\|x(t)\|+L_3\|g(t)+A{}^cD_{0^+}^{\beta}x(t)\|+L_2\|{}^cD_{0^+}^{\beta}x(t)\|+\|f(t,0,0,0)\| \\
\leqslant\ & L_1\|x\|_\infty+L_3\|g(t)\|+(L_3\|A\|+L_2)\|D_{0^+}^{\beta}x\|_\infty+f^*.
\end{aligned}
$$

Thus

$$\|g(t)\|\leqslant\frac{L_1}{1-L_3}\|x\|_\infty+\frac{L_3\|A\|+L_2}{1-L_3}\|D_{0^+}^{\beta}x\|_\infty+\frac{f^*}{1-L_3}.$$

Then, by using Lemma 5, we have

$$\|g(t)\|\leqslant\frac{L_1}{1-L_3}\|x\|_\infty+\frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\|x'\|_\infty+\frac{f^*}{1-L_3}$$

$$\leqslant \left[\frac{L_1}{1-L_3}+\frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\right]\|x\|_1+\frac{f^*}{1-L_3}$$
$$\leqslant D\|x\|_1+\frac{f^*}{1-L_3},$$

which implies that

$$\begin{aligned}\|B_1\| &\leqslant T^{1+\beta-\alpha}(\alpha-\beta-1)\int_0^T (T-s)^{\alpha-\beta-2}\|x(s)\|ds\\ &\quad+\frac{T^{1+\beta-\alpha}\Gamma(\alpha-\beta)\|A^{-1}\|}{\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\|g(s)\|ds\\ &\leqslant \left[1+\left(\frac{T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|}{\Gamma(\alpha)}\right)\left(\frac{L_1}{1-L_3}+\frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\right)\right]\|x\|_1\\ &\quad+\frac{f^*T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|}{(1-L_3)\Gamma(\alpha)}\\ &\leqslant \left[1+\frac{T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|}{\Gamma(\alpha)}D\right]\|x\|_1+\frac{f^*T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|}{(1-L_3)\Gamma(\alpha)}\end{aligned}$$

and

$$\begin{aligned}\|B_2\| &\leqslant \frac{\|A\|}{(\alpha-\beta)\Gamma(\alpha-\beta-1)}\int_0^T (T-s)^{\alpha-\beta-2}|x(s)|ds\\ &\quad+\frac{1}{(\alpha-\beta)\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\|g(s)\|ds\\ &\quad+\frac{\|A\|}{T\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}\|x(s)\|ds+\frac{1}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|g(s)\|ds\\ &\leqslant \left[\frac{2\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}+\left(\frac{(2\alpha-\beta)T^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha+1)}\right)\left(\frac{L_1}{1-L_3}+\frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\right)\right]\|x\|_1\\ &\quad+\left[\frac{(2\alpha-\beta)T^{\alpha-1}f^*}{(1-L_3)(\alpha-\beta)\Gamma(\alpha+1)}\right]\\ &\leqslant \left[\frac{2\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}+\frac{(2\alpha-\beta)T^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha+1)}D\right]\|x\|_1+\left[\frac{(2\alpha-\beta)T^{\alpha-1}f^*}{(1-L_3)(\alpha-\beta)\Gamma(\alpha+1)}\right].\end{aligned}$$

Thus, for each $t\in J$, we have

$$\begin{aligned}\|x(t)\| &\leqslant \|B_1\|+T\|B_2\|+\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|B_1|+\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|x\|_1\\ &\quad+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\left[\frac{L_1}{1-L_3}+\frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\right]\|x\|_1+\frac{f^*T^{\alpha}}{(1-L_3)\Gamma(\alpha+1)}\\ &\leqslant \left[\left(\frac{T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|}{\Gamma(\alpha)}+\frac{(2\alpha-\beta)T^{\alpha}}{(\alpha-\beta)\Gamma(\alpha+1)}+\frac{T^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)D\right.\end{aligned}$$

$$+1+\frac{4\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\Bigg]\|x\|_1+\frac{T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|f^*}{(1-L_3)\Gamma(\alpha)}$$
$$+\frac{(2\alpha-\beta)T^{\alpha}f^*}{(1-L_3)(\alpha-\beta)\Gamma(\alpha+1)}$$
$$+\frac{T^{\alpha}f^*}{(1-L_3)(\alpha-\beta)\Gamma(\alpha)}+\frac{T^{\alpha}f^*}{(1-L_3)\Gamma(\alpha+1)}$$
$$\leqslant R\|x\|_1+\frac{(2\alpha-\beta)f^*(T^{\alpha-1}+2T^{\alpha})}{(1-L_3)(\alpha-\beta)\Gamma(\alpha+1)}+\frac{2T^{\alpha-1}f^*+T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|f^*}{(1-L_3)\Gamma(\alpha)}.$$

Thus,

$$\|x\|_\infty\leqslant R\|x\|_1+\frac{(2\alpha-\beta)f^*(T^{\alpha-1}+2T^{\alpha})}{(1-L_3)(\alpha-\beta)\Gamma(\alpha+1)}+\frac{2T^{\alpha-1}f^*+T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|f^*}{(1-L_3)\Gamma(\alpha)}. \tag{3.12}$$

On the other hand, for each $t\in J$ we have

$$\|x'(t)\|\leqslant|B_2|+\frac{(\alpha-\beta)\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}|B_1|+\frac{\|A\|T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\|x\|_1$$
$$+\frac{T^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{L_1}{1-L_3}+\frac{(L_3\|A\|+L_2)T^{1-\beta}}{(1-L_3)\Gamma(2-\beta)}\right]\|x\|_1+\frac{f^*T^{\alpha-1}}{(1-L_3)\Gamma(\alpha)}$$
$$\leqslant\left[\left(\frac{(2\alpha-\beta)T^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha+1)}+\frac{2T^{\alpha-1}}{\Gamma(\alpha)}\right)D+\frac{2\|A\|T^{\alpha-\beta-1}(1+\alpha-\beta)}{\Gamma(\alpha-\beta+1)}\right]\|x\|_1$$
$$+\frac{(2\alpha-\beta)T^{\alpha-1}f^*}{(1-L_3)(\alpha-\beta)\Gamma(\alpha+1)}+\frac{2T^{\alpha-1}f^*}{(1-L_3)\Gamma(\alpha)}.$$

Hence

$$\|x'\|_\infty\leqslant R\|x\|_1+\frac{(2\alpha-\beta)f^*(T^{\alpha-1}+2T^{\alpha})}{(1-L_3)(\alpha-\beta)\Gamma(\alpha+1)}+\frac{2T^{\alpha-1}f^*+T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|f^*}{(1-L_3)\Gamma(\alpha)}. \tag{3.13}$$

From (3.12) and (3.13), we get

$$\|x\|_1\leqslant R\|x\|_1+\frac{(2\alpha-\beta)f^*(T^{\alpha-1}+2T^{\alpha})}{(1-L_3)(\alpha-\beta)\Gamma(\alpha+1)}+\frac{2T^{\alpha-1}f^*+T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|f^*}{(1-L_3)\Gamma(\alpha)}.$$

Since $R<1$, then

$$\|x\|_1\leqslant\frac{(2\alpha-\beta)f^*(T^{\alpha-1}+2T^{\alpha})}{(1-R)(1-L_3)(\alpha-\beta)\Gamma(\alpha+1)}+\frac{2T^{\alpha-1}f^*+T^{\beta}\Gamma(\alpha-\beta)\|A^{-1}\|f^*}{(1-R)(1-L_3)\Gamma(\alpha)}=\psi.$$

This shows that the set $E$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $N$ has a fixed point which is a solution of problem (1.3)–(1.4).

## 4. An example

Consider the following fractional differential equation

$$^{c}D^{\alpha}_{0^+}x(t) - A\,^{c}D^{\beta}_{0^+}x(t) = f\left(t, x(t), ^{c}D^{\beta}_{0^+}x(t), ^{c}D^{\alpha}_{0^+}x(t)\right); \;\; t \in [0,1], \tag{4.1}$$

with the periodic conditions

$$x(0) = x(1), \quad x'(0) = x'(1), \tag{4.2}$$

where $f : [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ such that, $f = (f_1, f_2)$ with

$$f_i(t,x,y,z) = \frac{c_i t^2}{1 + \|x\| + \|y\| + \|z\|}; \;\; i = 1,2,$$

$c_i > 0$, $x, y, z \in \mathbb{R}^2$, $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $\beta = \frac{1}{4}$ and $\alpha = \frac{3}{2}$. It is clear that the hypothesis $(H)$ is satisfied. A simple computations show that all conditions of Theorem 4 are satisfied for an appropriate choice of the constants $c_i$. It follows that the problem (4.1)–(4.2) has at least one solution defined on $[0,1]$.

### REFERENCES


[1] S. Abbas and M. Benchohra, *On the generalized Ulam-Hyers-Rassias stability for Darboux problem for partial fractional implicit differential equations*, Appl. Math. E-Notes **14** (2014), 20–28.

[2] S. Abbas, M. Benchohra, J. Graef and J. Henderson, *Implicit Fractional Differential and Integral Equations; Existence and Stability*, De Gruyter, Berlin, 2018.

[3] S. Abbas, M. Benchohra and G M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer-Verlag, New York, 2012.

[4] S. Abbas, M. Benchohra and G M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[5] R. P. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math. **109** (2010), 973–1033.

[6] M. Benchohra and S. Bouriah, *Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order*, Moroccan J. Pure Appl. Anal. **1** (1) (2017), 22–37.

[7] M. Benchohra, S. Bouriah, and J. J. Nieto, *Existence of periodic solutions for nonlinear implicit Hadamard fractional differential equations*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM **112** (2018), 25–35.

[8] M. Benchohra, S. Bouriah, J. E. Lazreg, and J. J. Nieto, *Nonlinear Implicit Hadamard fractional differential equations with delay in Banach space*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **55** (2016), 15–26.

[9] M. Benchohra and J. E. Lazreg, *Nonlinear fractional implicit differential equations*, Commun. Appl. Anal. **17** (2013), 471–482.

[10] G. Chai, *Existence results for anti-periodic boundary value problems of fractional differential equations*, Adv. Difference Equ. **2013**, 2013:53, 15 pp.

[11] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.

[12] X. L. Dind, *Controllability and optimality of linear time-invariant neutral control systems with different fractional orders*, Acta Math. Sci. **35B** (5) (2015), 1003–1013.

[13] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[14] T. Guo, W. Jiang, *Impulsive fractional functional differential equations*, Comput. Math. Appl. **64** (2012), 3414–4324.

[15] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differenatial Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B. V., Amsterdam, 2006.
[16] K. S. MILLER, B. ROSS, *An Introduction to the Fractional Calculus and Differential Equations*, Wiley, New York, 1993.
[17] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[18] J. SHENG AND W. JIANG, *Existence and uniqueness of the solution of fractional damped dynamical systems*, Adv. Difference Equ. **2017**, paper no. 16, 14 pp.
[19] V. E. TARASOV, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles*, Fields and Media, Springer, Berlin, 2011.
[20] H. YE, J. GAO AND Y. DING, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl. **328** (2007), 1075–1081.
[21] Y. ZHOU, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

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