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ON LINEAR AND NONLINEAR FRACTIONAL HADAMARD BOUNDARY VALUE PROBLEMS

SOUGATA DHAR

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Abstract. We establish new Lyapunov-type inequalities for linear Hadamard fractional differential equations with pointwise boundary conditions. Furthermore, we employ the contraction mapping principle to obtain the criterion of the existence of a unique solution for a nonlinear fractional Hadamard type boundary value problem.

1. Introduction

For the second-order linear differential equation

$$x'' + q(t)x = 0 \quad \text{on } (a,b) \tag{1.1}$$

with $q \in C([a,b],\mathbb{R})$, the following result is known as the Lyapunov inequality, see [15, 2].

THEOREM 1.1. *Assume Eq.* (1.1) *has a nontrivial solution* $x(t)$ *satisfying* $x(a) = x(b) = 0$ *and* $x(t) \neq 0$ *for* $t \in (a,b)$. *Then*

$$\int_a^b |q(t)|\, dt > \frac{4}{b-a}. \tag{1.2}$$

It was first noticed by Wintner [18] and later by several other authors that inequality (1.2) can be improved by replacing $|q(t)|$ by $q_+(t) := \max\{q(t), 0\}$, the nonnegative part of $q(t)$, to become

$$\int_a^b q_+(t)dt > \frac{4}{b-a}. \tag{1.3}$$

The Lyapunov inequality was extended by Hartman [11, Chapter XI] to the more general equation

$$(r(t)x')' + q(t)x = 0, \tag{1.4}$$

where $q, r \in C([a,b],\mathbb{R})$ such that $r(t) > 0$ for $t \in [a,b]$, as follows:

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THEOREM 1.2. *Assume Eq.* (1.4) *has a nontrivial solution* $x(t)$ *satisfying* $x(a) = x(b) = 0$ *and* $x(t) \neq 0$ *for* $t \in (a,b)$. *Then*

$$\int_a^b q_+(t)\,dt > \frac{4}{\int_a^b r^{-1}(t)\,dt}.$$

These Lyapunov inequalities have been used as an important tool in oscillation, disconjugacy, control theory, eigenvalue problems, and many other areas of differential equations. Due to their importance in applications, they have been extended in various directions by many authors. For more on Lyapunov-type inequalities, we refer the reader to [5, 6] for the higher order linear case, and [3, 4] for higher order half-linear case and the references cited therein.

Recently, the fractional differential equations have gained considerable importance and attention for their applications in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physical, mechanics, chemistry, aerodynamics etc. Due to its useful applications in the boundary value problems (BVPs), a subsequent search for the Lyapunov-type inequalities has also began in the direction of fractional calculus. Ferreira first obtained Lyapunov-type inequalities for fractional differential equations with pointwise boundary conditions (BCs) in [9, 8]. The former is with Riemann-Liouville fractional derivative and the latter is with Caputo fractional derivative. In this paper, without further mention, we let $1 < \alpha \leqslant 2$ and denote ${}^{RL}D^{\alpha}_{a+}x, {}^{C}D^{\alpha}_{a+}x$ and ${}^{H}D^{\alpha}_{a+}x$ as the Riemann-Liouville, Caputo and Hadamard fractional derivative of a function $x(t)$, respectively. We summarize the main results from [9, 8] in the following theorem.

THEOREM 1.3. *(a) Consider*

$$ {}^{RL}D^{\alpha}_{a+}x + q(t)x = 0. \tag{1.5}$$

Assume $q \in C([a,b],\mathbb{R})$ *and Eq.* (1.5) *has a nontrivial solution* $x(t)$ *satisfying* $x(a) = x(b) = 0$. *Then*

$$\int_a^b |q(t)|dt > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.6}$$

(b) Consider

$$ {}^{C}D^{\alpha}_{a+}x + q(t)x = 0. \tag{1.7}$$

Assume $q \in C([a,b],\mathbb{R})$ *and Eq.* (1.7) *has a nontrivial solution* $x(t)$ *satisfying* $x(a) = x(b) = 0$. *Then*

$$\int_a^b |q(t)|dt > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{1.8}$$

In [7, Theorem 2.3], the authors improved (1.6) by replacing $q(t)$ with $q_+(t)$. It is clear when $\alpha = 2$, the results lead to the classical Lyapunov inequality (1.2). For more on Lyapunov-type inequalities in the fractional setting, we refer the reader to [7, 12, 17] and the references cited therein.

It is to be noted that most of the work in the literature regarding Lyapunov-type inequalities, in the sense of fractional derivative, involves either the Riemann-Liouville or the Caputo definition. To the best of our knowledge, there is only one reference [16], where the authors studied the problem with Hadamard fractional derivative. We summarize the main results from [16] in the following theorem.

THEOREM 1.4. *Consider*

$$ {}^{H}D^{\alpha}_{a^+}x - q(t)x = 0. \tag{1.9}$$

Assume $q \in C([a,b],\mathbb{R})$ *and* (1.9) *has a nontrivial solution* $x(t)$ *satisfying* $x(1) = x(e) = 0$. *Then*

$$\int_1^e |q(t)|dt > \frac{\Gamma(\alpha)e^{\lambda}}{[\lambda(1-\lambda)]^{\alpha-1}}, \tag{1.10}$$

where $\lambda = \frac{1}{2}[2\alpha - 1 - \sqrt{4(\alpha-1)^2+1}]$.

We note that the authors in [16] choose the interval $[1,e]$ instead of a general $[a,b]$ with $a<b$ in order to avoid some complicated calculations in their proof. In this paper, we remove the restriction and provide some variation of Lyapunov-type inequalities. Furthermore, we consider a nonlinear fractional BVP in the form

$$ {}^{H}D^{\alpha}_{a^+}x + f(t,x(t)) = 0, \quad x(a) = 0,\ x(b) = k_2 \tag{1.11}$$

where $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ satisfies a uniform Lipschitz condition with respect to the second variable, i.e.,

$$|f(t,x) - f(t,y)| \leqslant K|x-y|$$

for all (t,x), $(t,y) \in [a,b]\times\mathbb{R}$ and $K>0$ is the Lipschitz constant. We use the contraction mapping theorem to establish the existence of a unique solution for BVP (1.11) for a class of functions f. This result is an extension of Kelly and Peterson [14, Theorem 7.7]. Similar extensions were done by Ferreira [10] for the Riemann-Liouville fractional derivative operator and Ahmed [1] for the Caputo fractional derivative operator. We believe that our results are new and provide useful supplemental tools in the study of this type of problems.

This paper is organized as follows. After this introduction, we recall some basic definitions of fractional calculus and prove some auxiliary lemmas in Section 2. Section 3 contains the main results regarding the Lyapunov-type inequalities and the existence of a unique solution of the nonlinear BVP (1.11).

2. Background materials and preliminaries

For the convenience of the reader, here we present the necessary definitions and lemmas from fractional calculus theory in the sense of Hadamard. These results can be found in the monograph [13].

DEFINITION 2.1. The Hadamard fractional integral of order $\alpha > 0$ of a function $x(t)$ for all $t>a>0$ is defined by

$$ {}^{H}I^{\alpha}_{a^+}x = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{x(s)}{s}ds,$$

where $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt$ is the gamma function, provided the right side is pointwise defined on $\mathbb{R}^+$.

DEFINITION 2.2. The Hadamard fractional derivative of order $\alpha>0$ of a function $x(t)$ for all $t>a>0$ is defined by

$$^{H}D^{\alpha}_{a^+}x=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\ln\frac{t}{s}\right)^{n-\alpha-1}\frac{x(s)}{s}ds,$$

where $n=\lfloor\alpha\rfloor+1$ with $\lfloor\alpha\rfloor$ is the integer part of α.

REMARK 2.1. As a basic example, we show for $\lambda>-1$,

$$^{H}D^{\alpha}_{a^+}\left(\ln\frac{t}{a}\right)^{\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(n+\lambda+1-\alpha)}\left(t\frac{d}{dt}\right)^n\left(\ln\frac{t}{a}\right)^{n+\lambda-\alpha}.$$

In particular, $^{H}D^{\alpha}_{a^+}\left(\ln\frac{t}{a}\right)^{\alpha-m}=0$, $m=1,2,\dots,n$, where n is the smallest integer greater than or equal to α.

In fact, for $\lambda>-1$,

$$^{H}D^{\alpha}_{a^+}\left(\ln\frac{t}{a}\right)^{\lambda}=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\ln\frac{t}{s}\right)^{n-\alpha-1}\left(\ln\frac{s}{a}\right)^{\lambda}\frac{ds}{s}. \tag{2.1}$$

Introducing the change of variable as $u=(\ln\frac{s}{a})/\ln(\frac{t}{a})$, we obtain from (2.1) that

$$\begin{aligned}^{H}D^{\alpha}_{a^+}\left(\ln\frac{t}{a}\right)^{\lambda}&=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\left(\ln\frac{t}{a}\right)^{n+\lambda-\alpha}\int_0^1u^{\lambda}(1-u)^{n-\alpha-1}du\\&=\frac{\Gamma(\lambda+1)}{\Gamma(n+\lambda+1-\alpha)}\left(t\frac{d}{dt}\right)^n\left(\ln\frac{t}{a}\right)^{n+\lambda-\alpha}.\end{aligned}$$

Hence,

$$^{H}D^{\alpha}_{a^+}\left(\ln\frac{t}{a}\right)^{\alpha-m}=\frac{\Gamma(\alpha+1-m)}{\Gamma(n+1-m)}\left(t\frac{d}{dt}\right)^n\left(\ln\frac{t}{a}\right)^{n-m}=0\quad\text{for } m=1,2,\dots,n.$$

From Definition 2.2 and Remark 2.1, we obtain the following lemma.

LEMMA 2.1. *Assume that $x\in C(a,b)$ and $\alpha>0$. Then the equality $\left({}^{H}D^{\alpha}_{a^+}x\right)(t)=0$ is valid if and only if*

$$x(t)=\sum_{i=1}^n c_i\left(\ln\frac{t}{a}\right)^{\alpha-i},$$

where $n=\lfloor\alpha\rfloor+1$ with $\lfloor\alpha\rfloor$ is the integer part of α and $c_i\in\mathbb{R}$ for $i=1,2,\dots,n$ are arbitrary constants.

The proof is straightforward hence omited.

It is well known that for all $x(t)\in C(a,b)$, we have $\left({}^{H}D^{\alpha}_{a^+}\left({}^{H}I^{\alpha}_{a^+}x\right)\right)(t)=x(t)$. Then using Lemma 2.1, we derive the following law of composition.

LEMMA 2.2. *Assume that $x(t) \in C(a,b)$ and $\alpha > 0$. Then*

$$\left({}^H I_{a^+}^{\alpha}\left({}^H D_{a^+}^{\alpha} x\right)\right)(t) = x(t) + \sum_{i=1}^{n} c_i \left(\ln \frac{t}{a}\right)^{\alpha-i},$$

where $n = \lfloor \alpha \rfloor + 1$ with $\lfloor \alpha \rfloor$ is the integer part of α and $c_i \in \mathbb{R}$ for $i = 1,2,\ldots,n$ are arbitrary constants.

In the following, we present the Green's function of fractional BVP in the sense of Hadamard fractional derivative.

LEMMA 2.3. *Assume that $h \in C(a,b)$ and $1 < \alpha \leqslant 2$. Then the unique solution of the BVP*

$$ {}^H D_{a^+}^{\alpha} x + h(t) = 0, \quad x(a) = x(b) = 0 \tag{2.2}$$

is

$$x(t) = \int_a^b G(t,s)h(s)ds, \tag{2.3}$$

where

$$G(t,s) = \frac{1}{s\Gamma(\alpha)} \begin{cases} \dfrac{\left(\ln \frac{t}{a}\right)^{\alpha-1}\left(\ln \frac{b}{s}\right)^{\alpha-1}}{\left(\ln \frac{b}{a}\right)^{\alpha-1}} - \left(\ln \frac{t}{s}\right)^{\alpha-1}, & a \leqslant s \leqslant t \leqslant b, \\ \dfrac{\left(\ln \frac{t}{a}\right)^{\alpha-1}\left(\ln \frac{b}{s}\right)^{\alpha-1}}{\left(\ln \frac{b}{a}\right)^{\alpha-1}}, & a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{2.4}$$

Proof. We apply Lemma 2.2 to reduce equation of (2.2) to an equivalent integral equation

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds + c_1 \left(\ln \frac{t}{a}\right)^{\alpha-1} + c_2 \left(\ln \frac{t}{a}\right)^{\alpha-2},$$

for some $c_1, c_2 \in \mathbb{R}$. Using the BCs, we have $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha)\left(\ln \frac{b}{s}\right)^{\alpha-1}} \int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds.$$

Therefore, the unique solution of BVP (2.2) is

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds + \frac{\left(\ln \frac{t}{a}\right)^{\alpha-1}}{\Gamma(\alpha)\left(\ln \frac{b}{s}\right)^{\alpha-1}} \int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\ &= \frac{1}{\Gamma(\alpha)} \left[\int_a^t \left\{ \frac{\left(\ln \frac{t}{a}\right)^{\alpha-1}\left(\ln \frac{b}{s}\right)^{\alpha-1}}{\left(\ln \frac{b}{a}\right)^{\alpha-1}} - \left(\ln \frac{t}{s}\right)^{\alpha-1} \right\} \frac{h(s)}{s} ds \right. \\ &\quad \left. + \int_t^b \frac{\left(\ln \frac{t}{a}\right)^{\alpha-1}\left(\ln \frac{b}{s}\right)^{\alpha-1}}{\left(\ln \frac{b}{a}\right)^{\alpha-1}} \frac{h(s)}{s} ds \right] \\ &= \int_a^b G(t,s)h(s)ds. \end{aligned}$$

The proof is complete. $\square$

LEMMA 2.4. *Let $G(t,s)$ be given by (2.4). Then*

$$\int_a^b G(t,s)ds \leqslant \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha \Gamma(\alpha+1)}\left(\ln\frac{b}{a}\right)^\alpha. \tag{2.5}$$

Proof. Using the expression of $G(t,s)$ in (2.4), it follows that

$$\int_a^b G(t,s)ds = \frac{1}{\Gamma(\alpha)}\left[\frac{\left(\ln\frac{t}{a}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}\int_a^b \left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\right]. \tag{2.6}$$

A simple calculation lead to

$$\frac{\left(\ln\frac{t}{a}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}\int_a^b \left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s} = \frac{1}{\alpha}\left(\ln\frac{t}{a}\right)^{\alpha-1}\ln\frac{b}{a}, \tag{2.7}$$

and

$$\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} = \frac{1}{\alpha}\left(\ln\frac{t}{a}\right)^{\alpha}. \tag{2.8}$$

Substituting (2.7) and (2.8) in (2.6) we see that

$$\begin{aligned}\int_a^b G(t,s)ds &= \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{a}\right)^{\alpha-1}\ln\frac{b}{t} \\ &\leqslant \frac{1}{\Gamma(\alpha+1)}\max_{t\in[a,b]}\left[\left(\ln\frac{t}{a}\right)^{\alpha-1}\ln\frac{b}{t}\right].\end{aligned} \tag{2.9}$$

Define $g(t) := \left(\ln\frac{t}{a}\right)^{\alpha-1}\ln\frac{b}{t}$. Clearly $g(a) = g(b) = 0$ and $g(t) > 0$ on (a,b). By Rolle's Theorem, there exists $t^* \in (a,b)$ such that $g(t^*) = \max g(t)$ for $t \in (a,b)$, i.e., $g'(t^*) = 0$. Note that

$$g'(t) = \frac{(\ln\frac{t}{a})^{\alpha-2}}{t}\left[(\alpha-1)\ln\frac{b}{t} - \ln\frac{t}{a}\right].$$

Using the facts that $1 < \alpha \leqslant 2$ and $0 < a < s < b$, it is easy to see that $g'(t) = 0$ only at $t = t^* = (ab^{\alpha-1})^{\frac{1}{\alpha}}$. Hence $g(t)$ has a unique maximum at t^* given by

$$\max_{t\in[a,b]} g(t) = g(t^*) = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\left(\ln\frac{b}{a}\right)^\alpha. \tag{2.10}$$

Using (2.10) in (2.9) we see that (2.5) holds. $\square$

We define $H(t,s) = sG(t,s)$. Here we prove three properties of the function H which will be essential to prove our main theorems.

LEMMA 2.5. *The function $H(t,s)$ defined above satisfies the following conditions:*

1. $H(t,s) \geqslant 0$ *on* $[a,b] \times [a,b]$.
2. $\max_{t\in[a,b]} H(t,s) = H(s,s)$.
3. $H(s,s)$ *has a unique maximum given by*

$$\max_{s\in[a,b]} H(s,s) = H\left(\sqrt{ab}, \sqrt{ab}\right) = \frac{\left(\ln\frac{b}{a}\right)^{\alpha-1}}{4^{\alpha-1}\Gamma(\alpha)}.$$

Proof. (1) We define two functions as

$$h_1(t,s) = \frac{\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}} - \left(\ln\frac{t}{s}\right)^{\alpha-1}, \quad a \leqslant s \leqslant t \leqslant b;$$

and

$$h_2(t,s) = \frac{\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}, \quad a \leqslant t \leqslant s \leqslant b.$$

It is clear that $h_2(t,s) \geqslant 0$. Now, regarding the function $h_1(t,s)$, we observe that

$$\begin{aligned} h_1(t,s) &= \frac{\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}} - \left(\ln\frac{t}{s}\right)^{\alpha-1} \\ &= \frac{\left(\ln\frac{t}{a}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}(\ln b - \ln s)^{\alpha-1} - \frac{\left(\ln\frac{t}{a}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}\left\{\ln b - \left(\ln a + \frac{\ln\frac{s}{a}\ln\frac{b}{a}}{\ln\frac{t}{a}}\right)\right\}^{\alpha-1}. \end{aligned}$$

Note that

$$\begin{aligned} \ln a + \frac{\ln\frac{s}{a}\ln\frac{b}{a}}{\ln\frac{t}{a}} \geqslant \ln s &\Leftrightarrow \frac{\ln a \ln\frac{t}{a} + \ln\frac{s}{a}\ln\frac{b}{a}}{\ln\frac{t}{a}} \geqslant \ln s \\ &\Leftrightarrow \ln s \ln\frac{b}{t} \geqslant \ln a \ln\frac{b}{t} \\ &\Leftrightarrow s \geqslant a, \end{aligned}$$

and hence $h_1(t,s) \geqslant 0$. This concludes the proof of (1).

(2) To prove $\max_{t\in[a,b]} H(t,s) = H(s,s)$, we only need to differentiate $h_1(t,s)$ with respect to t for every fixed s. Then, a similar analysis as discussed in the proof of (1) shows that $h_1(t,s)$ is a decreasing function of t and $h_2(t,s)$ is an increasing function of t. We leave the details to the reader.

(3) Let

$$H(s,s) := h(s) = \frac{1}{\Gamma(\alpha)}\left[\frac{\ln\frac{s}{a}\ln\frac{b}{s}}{\ln\frac{b}{a}}\right]^{\alpha-1}, \quad s \in [a,b].$$

Clearly $h(a) = h(b) = 0$ and $h(s) > 0$ on (a,b). By Rolle's Theorem, there exists $s^* \in (a,b)$ such that $h(s^*) = \max h(s)$ on (a,b), i.e., $h'(s^*) = 0$. Note that

$$h'(s) = \frac{\alpha-1}{s\Gamma(\alpha)}\left[\frac{\ln\frac{s}{a}\ln\frac{b}{s}}{\ln\frac{b}{a}}\right]^{\alpha-2}\left[\frac{\ln ab - \ln s^2}{\ln\frac{b}{a}}\right].$$

Using the facts that $1 < \alpha \leqslant 2$ and $0 < a < s < b$, it is easy to see that $h'(s) = 0$ only at $s = s^* = \sqrt{ab}$. Hence $h(s)$ has a unique maximum at s^* given by

$$\max_{s\in[a,b]} h(s) = h(s^*) = \frac{\left(\ln\frac{b}{a}\right)^{\alpha-1}}{4^{\alpha-1}\Gamma(\alpha)}.$$

The proof is now complete. $\square$

REMARK 2.2. Recall that $G(t,s) = \frac{1}{s}H(t,s)$. Since $H(t,s) \geqslant 0$ and $s \in [a,b] \subset \mathbb{R}^+$, we conclude that $G(t,s) \geqslant 0$ on $[a,b]\times[a,b]$.

3. Main results

In this section, we consider the following fractional differential equations

$$^{H}D^{\alpha}_{a^+}x + tq(t)x = 0, \quad 0 < a \leqslant t \leqslant b, \tag{3.1}$$

and

$$^{H}D^{\alpha}_{a^+}x + q(t)x = 0, \quad 0 < a \leqslant t \leqslant b, \tag{3.2}$$

where $q \in C(a,b)$; with the Dirichilet BC

$$x(a) = x(b) = 0. \tag{3.3}$$

We first present a Lyapunov-type inequality for BVP (3.1), (3.3).

THEOREM 3.1. *Assume Eq.* (3.1) *has a nontrivial solution $x(t)$ satisfying* (3.3) *and $x(t) \neq 0$ on (a,b). Then*

$$\int_a^b q_+(t)dt > \frac{4^{\alpha-1}\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}. \tag{3.4}$$

Proof. It follows from Lemma 2.3 with $h(t) = tq(t)x(t)$ that a solution to the fractional BVP (3.1), (3.3) satisfies the integral equation

$$x(t) = \int_a^b sG(t,s)q(s)x(s)ds = \int_a^b H(t,s)q(s)x(s)ds.$$

Without any loss of generaity we may assume $x(t) > 0$ on (a,b).

Define $m = \max_{t\in[a,b]} x(t)$. Using Lemma 2.5 and the facts that $0 \leqslant x(t) \leqslant m$ and $x(t) \not\equiv m$ on $[a,b]$ and $q(t) \leqslant q_+(t)$, we have

$$m < m\int_a^b H(t,s)q_+(s)ds \leqslant m\int_a^b H(s,s)q_+(s)ds \leqslant m\frac{\left(\ln\frac{b}{a}\right)^{\alpha-1}}{4^{\alpha-1}\Gamma(\alpha)}\int_a^b q_+(s)ds.$$

Cancelling m from both sides we see that (3.4) holds. $\square$

Next we present another Lyapunov-type inequality for BVP (3.2), (3.3).

THEOREM 3.2. *Assume Eq.* (3.2) *has a nontrivial solution* $x(t)$ *satisfying* (3.3) *and* $x(t) \neq 0$ *on* (a,b). *Then*

$$\int_a^b \frac{q_+(t)}{t} dt > \frac{4^{\alpha-1}\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}. \tag{3.5}$$

Proof. We define $p(t) = \frac{q(t)}{t}$ for $0 < a \leqslant t \leqslant b$. Then $p(t) \in C([a,b],\mathbb{R})$ and is well-defined. Hence BVP (3.2), (3.3) becomes

$$^{H}D^{\alpha}_{a^+}x + q(t)x = 0, \quad x(a) = x(b) = 0. \tag{3.6}$$

Applying Theorem 3.1 for BVP (3.6) we have

$$\int_a^b p_+(t)dt > \frac{4^{\alpha-1}\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}.$$

Substituting $p(t)$ we see that the conclusion follows. □

REMARK 3.1. We note that, for $\alpha = 2$, $^{RL}D^{\alpha}_{a^+}x = {}^{C}D^{\alpha}_{a^+}x = x''$. This is why inequalities (1.6) and (1.7) reduces to the classical Lyapunov-type inequality (1.2). However, the same does not hold for $^{H}D^{\alpha}_{a^+}x$. In fact, From Definition 2.2, it is clear that $\alpha = 2$ implies $^{H}D^{\alpha}_{a^+}x = t^2x''$. Hence equations (3.1) and (3.2) becomes

$$tx'' + q(t)x = 0, \quad 0 < a \leqslant t \leqslant b,$$

and

$$t^2x'' + q(t)x = 0, \quad 0 < a \leqslant t \leqslant b,$$

respectively. In this case, we obtain a variation of the classical case from (3.4) and (3.5).

The following corollary is immediate from Theorems 3.1 and 3.2.

COROLLARY 3.1. *(a) Assume*

$$\int_a^b q_+(t)dt \leqslant \frac{4^{\alpha-1}\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}.$$

Then BVP (3.1), (3.3) *has only the trivial solution.*

(b) Assume

$$\int_a^b \frac{q_+(t)}{t} dt \leqslant \frac{4^{\alpha-1}\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}.$$

Then BVP (3.2), (3.3) *has only the trivial solution.*

Now we extend the result of [14, Theorem 7.7] to the fractional Hadamard case. We consider the following nonlinear BVP

$$ {}^{H}D^{\alpha}_{a^+}x + f(t,x(t)) = 0, \quad x(a)=0,\ x(b)=k, \tag{3.7}$$

for some $k \in \mathbb{R}$.

THEOREM 3.3. *Assume $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to the second variable on $[a,b]\times\mathbb{R}$ with Lipschitz constant K; that is*

$$|f(t,x_1) - f(t,x_2)| \leqslant K|x_1 - x_2|, \tag{3.8}$$

for all $(t,x_1),(t,x_2) \in [a,b]\times\mathbb{R}$. If

$$\ln\left(\frac{b}{a}\right) < \left[\frac{\alpha^{\alpha}\Gamma(\alpha+1)}{K(\alpha-1)^{\alpha-1}}\right]^{\frac{1}{\alpha}} \tag{3.9}$$

then BVP (3.7) has a unique solution on $[a,b]$.

Proof. Let $\mathscr{B}$ be the Banach space of continuous functions defined on $[a,b]$ with norm

$$||x|| = \max_{t\in[a.b]} |x(t)|.$$

By Lemma 2.3, $x(t)$ is a solution of BVP (3.7) if and only if $x(t)$ satisfies the following integral equation

$$x(t) = k\left(\frac{\ln(\frac{t}{a})}{\ln(\frac{b}{a})}\right)^{\alpha-1} + \int_a^b G(t,s)f(s,x(s))ds.$$

Define the operator $T : \mathscr{B} \to \mathscr{B}$ by

$$Tx(t) = k\left(\frac{\ln(\frac{t}{a})}{\ln(\frac{b}{a})}\right)^{\alpha-1} + \int_a^b G(t,s)f(s,x(s))ds.$$

Then T is completely continuous. We claim that T has a unique fixed point in $\mathscr{B}$. In fact, for any $x_1,x_2 \in \mathscr{B}$, we have

$$|Tx_1(t) - Tx_2(t)| \leqslant \int_a^b |G(t,s)||f(s,x_1(s) - f(s,x_2(s)))|ds.$$

Since, $G(t,s) \geqslant 0$ on $[a,b]\times[a,b]$ and f satisfies (3.8), we have

$$|Tx_1(t) - Tx_2(t)| \leqslant K\int_a^b G(t,s)|x_1(s) - x_2(s)|ds \leqslant K||x_1 - x_2||\int_a^b G(t,s)ds.$$

From Lemma 2.4 it follows that

$$|Tx_1(t) - Tx_2(t)| \leqslant K\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}\Gamma(\alpha+1)}\left(\ln\frac{b}{a}\right)^{\alpha}||x_1 - x_2|| < ||x_1 - x_2||,$$

where we have used (3.9). Hence T is a contraction mapping on $\mathscr{B}$. By the contraction mapping theorem we get the desired result. □

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Sougata Dhar
Department of Mathematics & Statistics
The University of Maine
Orono, ME 04469, USA
e-mail: sougata.dhar@maine.edu