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FACTORIZATION TECHNIQUES FOR THE NONLINEAR MODEL OF QUASI–STATIONARY PROCESSES IN CRYSTALLINE SEMICONDUCTORS

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Abstract. We consider the question of global existence and asymptotics of small solutions of a certain pseudoparabolic equation in one dimension. This model is motivated by the wave equation for media with a strong spatial dispersion, which appear in the nonlinear theory of the quasi-stationary processes in the electric media. We develop the factorization technique to study the large time asymptotics of solutions.

1. Introduction

We consider the Cauchy problem for the nonlinear pseudoparabolic equation

$$\partial_t \left(u - a_1\partial_x^2 u\right) + a_2\partial_x u + a_3\partial_x^3 u + a_4\partial_x^5 u = \partial_x u^3,$$

where $a_1 > 0,\ a_2, a_3, a_4 \in \mathbb{R}$. This model is motivated by the question of global existence of solutions of the wave equation for media with a strong spatial dispersion, which appears in the nonlinear theory of the quasi-stationary processes in the electric media (see [19]). For example, this equation describes the creation, propagation, and collapse of the so-called electric domains in semiconductors. This equation also can be considered as a higher order Benjamin-Bona-Mahony equation (see [5]) and a modified Sobolev type equation (see [4]). Local existence and blow up phenomena were studied in [1], [2], [3], [14], [13]. Here we are interested in the global existence and large time asymptotic behavior of solutions. Changing the independent variables we can reduce the equation to the case of $a_1 = 1$ and $a_2 = 0$. Thus we consider the equation of the form $\partial_t \left(u - \partial_x^2 u\right) - a\partial_x^3 u + b\partial_x^5 u = \partial_x u^3$ with $0 < a < 325b$. Then the pseudoparabolic equation can also be rewritten in the form including the pseudodifferential operator as follows

$$\begin{cases} u_t + i\Lambda\left(-i\partial_x\right)u = \langle i\partial_x\rangle^{-2}\partial_x u^3,\ (t,x) \in \mathbb{R}^2, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where the pseudodifferential operator $\Lambda(-i\partial_x)$ is defined by the symbol

$\Lambda(\xi) = \frac{a\xi^3+b\xi^5}{1+\xi^2}$, here the parameters $a,\ b>0$. Note that the symbol $\Lambda(\xi)$ has the following properties $\Lambda'(\xi) = \frac{3a\xi^2+a\xi^4+5b\xi^4+3b\xi^6}{(\xi^2+1)^2} = O(\xi^2)$. Also $\Lambda''(\xi) = 2\frac{3a\xi+(10b-a)\xi^3+9b\xi^5+3b\xi^7}{(\xi^2+1)^3} \geqslant C\xi$ for all $\xi>0$ (here we use the condition $0<a<325b$). So that $\Lambda'(\xi)$ grows monotonically for $\xi>0$ and $\Lambda(\xi)=a\xi^3+O(\xi^5)$ as $\xi\to 0$ and $\Lambda(\xi)=b\xi^3+O(\xi)$ as $\xi\to\infty$. Thus the coefficient a in the dispersion relation $\Lambda(\xi)$ determines the asymptotic behavior of the symbol $\Lambda(\xi)$ at the origin, whereas the coefficient b determines behavior of the symbol $\Lambda(\xi)$ at infinity. Also we need the information $\partial_\xi^k\Lambda(\xi)=O(\xi^{3-k})$ for all $\xi>0,\ k=0,1,2,3,$ and $\Lambda''(\xi)\geqslant C\xi$.

Specifically we are interested in the case of $\int_{\mathbb{R}} u_0(x)\,dx=0$. Then by equation (1.1) we get the conservation law $\int_{\mathbb{R}} u(t,x)\,dx = \int_{\mathbb{R}} u_0(x)\,dx = 0$ for all $t>0$. We also assume that the initial data $u_0(x)$ and the solution $u(t,x)$ are real-valued functions. As far as we know the global existence and the large time asymptotics of solutions to the Cauchy problem for the nonlinear pseudoparabolic equation (1.1) was not studied previously. In the present paper we fill this gap, developing the factorization techniques originated in papers [12], [7], [8], [9], [10], [11], [15], [16], [17].

We denote the Lebesgue space by $\mathbf{L}^p=\{\phi\in\mathbf{S}';\|\phi\|_{\mathbf{L}^p}<\infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p}=(\int|\phi(x)|^p\,dx)^{\frac{1}{p}}$ for $1\leqslant p<\infty$ and $\|\phi\|_{\mathbf{L}^\infty}=\operatorname{ess.sup}_{x\in\mathbf{R}}|\phi(x)|$ for $p=\infty$. The weighted Sobolev space is $\mathbf{H}_p^{k,s}=\left\{\varphi\in\mathbf{S}';\|\phi\|_{\mathbf{H}_p^{k,s}}=\left\|\langle x\rangle^s\langle i\partial_x\rangle^k\phi\right\|_{\mathbf{L}^p}<\infty\right\}$, for $k,s\in\mathbf{R},\ 1\leqslant p\leqslant\infty$, where $\langle x\rangle=\sqrt{1+x^2}$, $\langle i\partial_x\rangle=\sqrt{1-\partial_x^2}$, $\mathbf{S}'$ stands for the space of tempered distributions (the (continuous) dual of the Schwartz space $\mathbf{S}$). We also use the notations $\mathbf{H}^{k,s}=\mathbf{H}_2^{k,s}$, $\mathbf{H}^k=\mathbf{H}^{k,0}$ shortly, if it does not cause any confusion. Let $\mathbf{C}(\mathbf{I};\mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C. We denote by $\mathscr{F}\phi$ or $\hat{\phi}$ the Fourier transform of the function ϕ by $\mathscr{F}\phi=\hat{\phi}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\phi(x)dx$, then the inverse Fourier transformation $\mathscr{F}^{-1}$ is given by $\mathscr{F}^{-1}\phi=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\phi(\xi)d\xi$.

Define the stationary point $\mu(x)$ as the root of the equation $\Lambda'(\mu)=x$ for all $x>0$. The Heaviside function $\theta(x)$ is defined by $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x\leqslant 0$.

We now state our main result.

THEOREM 1.1. *Assume that the initial data $u_0\in\mathbf{H}^{1,1}$ are real-valued with a sufficiently small norm $\|u_0\|_{\mathbf{H}^{1,1}}\leqslant\varepsilon$ and $\int_{\mathbb{R}} u_0(x)\,dx=0$. Then there exists a unique global solution $u\in\mathbf{C}([0,\infty);\mathbf{H}^1)$ of the Cauchy problem (1.1) satisfying the time decay estimate $\|u(t)\|_{\mathbf{L}^\infty}\leqslant C\varepsilon t^{-\frac{1}{2}}$. Moreover there exists a unique modified final state*

$W_+ \in \mathbf{L}^\infty$ *such that the asymptotics*

$$u(t) = \frac{2}{\sqrt{t}} Re e^{-it\left(\Lambda\left(\mu\left(\frac{x}{t}\right)\right) - \frac{x}{t}\mu\left(\frac{x}{t}\right)\right)} \frac{\theta(x)}{\sqrt{i\Lambda''\left(\mu\left(\frac{x}{t}\right)\right)}} W_+\left(\mu\left(\frac{x}{t}\right)\right)$$
$$\times \exp\left(\frac{3i\mu\left(\frac{x}{t}\right)}{\left|\Lambda''\left(\mu\left(\frac{x}{t}\right)\right)\right|}\left|W_+\left(\mu\left(\frac{x}{t}\right)\right)\right|^2 \log t\right) + O\left(t^{-\frac{1}{2}-\delta}\right) \tag{1.2}$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in \mathbb{R}$, *where* $\delta > 0$ *is a small constant.*

Section 2 is devoted to the factorization techniques. In Section 3 we find the asymptotics for the decomposition operator $\mathscr{V}(t)\phi = \sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\phi(\xi)\,d\xi$ and the conjugate operator $\mathscr{V}^*(t)\phi = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\left|\Lambda''(\eta)\right|d\eta$. Section 4 and Section 5 are devoted to the estimates for the commutators with operators $\mathscr{V}$ and $\mathscr{V}^*$, respectively. Then in Section 6 we obtain a priori estimates for the solutions in the following norm

$$\|u\|_{\mathbf{X}_T} = \sup_{t\in[1,T]} \left\|\xi^{-1}\left(1+\xi^2\right)\mathscr{F}\mathscr{U}(-t)u(t)\right\|_{\mathbf{L}^\infty}$$
$$+ \sup_{t\in[1,T]} t^{-\gamma}\left(\|\mathscr{J}u(t)\|_{\mathbf{H}^1} + \|u(t)\|_{\mathbf{H}^1}\right),$$

where $\gamma > 0$ is small. Finally, Section 7 is devoted to the proof of Theorem 1.1.

2. Factorization techniques

Define the free evolution group $\mathscr{U}(t) = \mathscr{F}^{-1}e^{-it\Lambda(\xi)}\mathscr{F}$. We write

$$u = \mathscr{U}(t)\mathscr{F}^{-1}\phi = \mathscr{D}_t\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\,d\xi,$$

where $\mathscr{D}_t\phi = |t|^{-\frac{1}{2}}\phi\left(\frac{x}{t}\right)$ is the dilation operator. In the integral $\int_{\mathbb{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\,d\xi$, there are two stationary points $\xi = \pm\mu(x)$ which are obtained as the roots of the equation $\Lambda'(\xi) = x$ for all $x > 0$. Also $\Lambda''(\xi) > 0$ for all $\xi > 0$ and $\Lambda(-\xi) = -\Lambda(\xi)$. Since $\Lambda'(\xi)$ is monotone in the domain $\xi \in \mathbb{R}_+$, there exists a unique stationary point $\xi = \mu(x)$, such that $\Lambda'(\mu(x)) = x$ for $x > 0$. We extend $\mu(x)$ for all $x \in \mathbb{R}$ by the odd continuation $\mu(x) = -\mu(-x)$. We have $\Lambda'(\xi) = O\left(\xi^2\right)$. Hence $\mu(x) = O\left(x^{\frac{1}{2}}\right)$ as $x \to 0$ and as $x \to \infty$, i.e. $\mu(x) = O\left(x^{\frac{1}{2}}\right)$. Thus we have $\frac{\mu}{|\mu|}\Lambda'(\mu(x)) = x$ for all $x \in \mathbb{R}$ and define the scaling operator $(\mathscr{B}\phi)(x) = \phi(\mu(x))$.

Since $u = \mathscr{U}(t)\mathscr{F}^{-1}\phi$ is a real-valued function, we have $\phi(-\xi) = \overline{\phi(\xi)}$, hence

$$u = \mathscr{U}(t)\mathscr{F}^{-1}\phi = 2\mathrm{Re}\mathscr{D}_t\mathscr{B}M\mathscr{V}\phi,$$

where $M = e^{-it\left(\frac{\eta}{|\eta|}\Lambda(\eta)-|\eta|\Lambda'(\eta)\right)}$, the phase function $S(\xi,\eta) = \Lambda(\xi) - \frac{\eta}{|\eta|}\Lambda(\eta) - \frac{\eta}{|\eta|}\Lambda'(\eta)(\xi-\eta)$ and the decomposition operator

$$\mathscr{V}(t)\phi = \sqrt{\frac{|t|}{2\pi}}\int_0^{\infty} e^{-itS(\xi,\eta)}\phi(\xi)\,d\xi.$$

Denote $\mathscr{A}_k = \overline{M}^k \frac{1}{t|\Lambda''(\eta)|}\partial_\eta M^k,\ k=0,1,\ M = e^{-it\left(\frac{\eta}{|\eta|}\Lambda(\eta)-|\eta|\Lambda'(\eta)\right)}$. We have $\mathscr{A}_1 = \mathscr{A}_0 + i\eta$, also $\mathscr{A}_1\mathscr{V} = \mathscr{V}i\xi$, $[i\eta,\mathscr{V}] = -\mathscr{A}_0\mathscr{V}$ therefore we obtain the commutator $\partial_\eta\mathscr{V} = -t|\Lambda''(\eta)|[i\eta,\mathscr{V}]$. Since $\partial_\xi S(\xi,\eta) = \Lambda'(\xi) - \frac{\eta}{|\eta|}\Lambda'(\eta)$, then we get

$it\left[\frac{\eta}{|\eta|}\Lambda'(\eta),\mathscr{V}\right]\phi = -\mathscr{V}(t)\partial_\xi\phi$. Also we need the representation for the inverse evolution group $\mathscr{F}\mathscr{U}(-t)$ for $\xi \geqslant 0$

$$\mathscr{F}\mathscr{U}(-t)\phi = e^{it\Lambda(\xi)}\mathscr{F}\phi = \mathscr{V}^*\overline{M}\mathscr{B}^{-1}\mathscr{D}_t^{-1}\phi,$$

where $\mathscr{D}_t^{-1}\phi = |t|^{\frac{1}{2}}\phi(xt)$, $\left(\mathscr{B}^{-1}\phi\right)(\eta) = \phi\left(\frac{\eta}{|\eta|}\Lambda'(\eta)\right)$, and the conjugate operator

$$\mathscr{V}^*(t)\phi = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\left|\Lambda''(\eta)\right|d\eta.$$

We have $i\xi\mathscr{V}^*\phi = \mathscr{V}^*\mathscr{A}_1\phi$. Hence $[i\xi,\mathscr{V}^*] = \mathscr{V}^*\mathscr{A}_0$.

Define the new dependent variable $\widehat{\varphi} = \mathscr{F}\mathscr{U}(t)u(t)$. Since $\mathscr{F}\mathscr{U}(-t)\mathscr{L} = \partial_t\mathscr{F}\mathscr{U}(-t)$, with $\mathscr{L} = \partial_t + i\Lambda(-i\partial_x)$, applying the operator $\mathscr{F}\mathscr{U}(-t)$ to equation (1.1) we get $\partial_t\widehat{\varphi} = \mathscr{F}\mathscr{U}(-t)\partial_x\left(u^3\right) = i\xi\mathscr{F}\mathscr{U}(-t)u^3$. Then since $u = 2\mathrm{Re}\mathscr{D}_t\mathscr{B}M\psi$, $\psi = \mathscr{V}\widehat{\varphi}$, we find

$$\begin{aligned}\mathscr{F}\mathscr{U}(-t)u^3 &= \mathscr{V}^*\overline{M}\mathscr{B}^{-1}\mathscr{D}_t^{-1}\left(2\mathrm{Re}\mathscr{D}_t\mathscr{B}M\psi\right)^3 = |t|^{-1}\mathscr{V}^*\overline{M}\left(M\psi+\overline{M}\overline{\psi}\right)^3\\ &= |t|^{-1}\left(\mathscr{V}^*M^2\psi^3 + 3\mathscr{V}^*|\psi|^2\psi + 3\mathscr{V}^*\overline{M}^2|\psi|^2\overline{\psi} + \mathscr{V}^*\overline{M}^4\overline{\psi}^3\right).\end{aligned}$$

By the definition of the operator $\mathscr{V}^*(t)$ we obtain

$\mathscr{V}^*(t)M^k\phi = e^{it\Omega_{k+1}}\mathscr{D}_{k+1}\mathscr{V}^*((k+1)t)\phi$, where $\Omega_{k+1} = \Lambda(\xi) - (k+1)\Lambda\left(\frac{\xi}{k+1}\right)$ for $k \neq -1$. Note that $\Omega_3 = \Omega_{-3}$ and $\Omega_1 = \Omega_{-1} = 0$. We denote below $\Omega = \Omega_3$. Therefore we find

$$\begin{aligned}|t|\mathscr{F}\mathscr{U}(-t)u^3 &= e^{it\Omega}\mathscr{D}_3\mathscr{V}^*(3t)\psi^3 + 3\mathscr{V}^*(t)|\psi|^2\psi\\ &\quad + 3\mathscr{D}_{-1}\mathscr{V}^*(-t)|\psi|^2\overline{\psi} + e^{it\Omega}\mathscr{D}_{-3}\mathscr{V}^*(-3t)\overline{\psi}^3. \qquad (2.1)\end{aligned}$$

Hence we get the following equation for the new dependent variable $\widehat{\varphi}$

$$\begin{aligned}\partial_t\widehat{\varphi} &= \frac{i\xi}{|t|\langle\xi\rangle^2}e^{it\Omega}\mathscr{D}_3\mathscr{V}^*(3t)\psi^3 + \frac{3i\xi}{|t|\langle\xi\rangle^2}\mathscr{V}^*(t)|\psi|^2\psi\\ &\quad + \frac{3i\xi}{|t|\langle\xi\rangle^2}\mathscr{D}_{-1}\mathscr{V}^*(-t)|\psi|^2\overline{\psi} + \frac{i\xi}{|t|\langle\xi\rangle^2}e^{it\Omega}\mathscr{D}_{-3}\mathscr{V}^*(-3t)\overline{\psi}^3,\end{aligned}$$

where $\langle\xi\rangle=\sqrt{1+\xi^2}$. Finally we mention some important identities. Define the free evolution group $\mathscr{U}(t)=\mathscr{F}^{-1}e^{-it\Lambda(\xi)}\mathscr{F}$. The operator

$$\begin{aligned}\mathscr{J}&=\mathscr{U}(t)x\mathscr{U}(-t)=\mathscr{F}^{-1}e^{-it\Lambda(\xi)}i\partial_\xi e^{it\Lambda(\xi)}\mathscr{F}\\&=\mathscr{F}^{-1}\left(i\partial_\xi-t\Lambda'(\xi)\right)\mathscr{F}=x-t\Lambda'(-i\partial_x),\end{aligned}$$

plays a crucial role in the large time asymptotic estimates. Note that $\mathscr{J}=x-t\Lambda'(-i\partial_x)$ commutes with $\mathscr{L}=\partial_t+i\Lambda(-i\partial_x)$, i.e. $[\mathscr{J},\mathscr{L}]=0$. Note that the symbol $\Lambda(\xi)=\xi^3\frac{1+b(a\xi)^2}{1+(a\xi)^2}$ satisfies the identity $\xi\partial_\xi\Lambda-a\partial_a\Lambda=3\Lambda$. Hence we have the commutator relation $\left[\widehat{\mathscr{P}},e^{-it\Lambda(\xi)}\right]=0$, with $\widehat{\mathscr{P}}=3t\partial_t-\xi\partial_\xi+a\partial_a$. Thus to avoid the derivative loss we also use the operators

$$\mathscr{P}=3t\partial_t+\partial_x x+a\partial_a$$

and

$$\mathscr{I}=\partial_a+it\partial_a\Lambda(-i\partial_x).$$

Also the commutator relations $[\mathscr{L},\mathscr{I}]=0$ and $[\mathscr{L},\mathscr{P}]=3\mathscr{L}$ hold. Using $u(t)=\mathscr{U}(t)\mathscr{F}^{-1}\widehat{\varphi}=\mathscr{D}_tM\mathscr{V}\widehat{\varphi}=\mathscr{F}^{-1}e^{-it\Lambda(\xi)}\widehat{\varphi}$, we get

$$\begin{aligned}\mathscr{P}u&=(3t\partial_t+\partial_x x+a\partial_a)\mathscr{F}^{-1}e^{-it\Lambda(\xi)}\widehat{\varphi}=\mathscr{F}^{-1}\widehat{\mathscr{P}}e^{-it\Lambda(\xi)}\widehat{\varphi}\\&=\mathscr{F}^{-1}\left[\widehat{\mathscr{P}},e^{-it\Lambda(\xi)}\right]\widehat{\varphi}+\mathscr{F}^{-1}e^{-it\Lambda(\xi)}\widehat{\mathscr{P}}\widehat{\varphi}=\mathscr{U}(t)\mathscr{F}^{-1}\widehat{\mathscr{P}}\widehat{\varphi}.\end{aligned}$$

Note that

$$\begin{aligned}\mathscr{I}u&=\mathscr{I}\mathscr{U}(t)\mathscr{F}^{-1}\widehat{\varphi}=\mathscr{F}^{-1}(\partial_a+it\partial_a\Lambda(\xi))E\widehat{\varphi}\\&=\mathscr{F}^{-1}E\partial_a\widehat{\varphi}=\mathscr{U}(t)\mathscr{F}^{-1}\partial_a\widehat{\varphi}.\end{aligned}$$

Hence $\|\mathscr{I}u\|_{\mathbf{L}^2}=\|\partial_a\widehat{\varphi}\|_{\mathbf{L}^2}$. Also we have the identity $\mathscr{P}=3t\mathscr{L}+\partial_x\mathscr{J}+a\mathscr{I}$. Also we have $\partial_a\Lambda(\xi)=\frac{2a(b-1)\xi^5}{\left(1+(a\xi)^2\right)^2}=O\left(\{\xi\}^5\langle\xi\rangle\right)$, where $\{\xi\}=\xi\langle\xi\rangle^{-1}$.

3. Estimates for the uniform norm

3.1. Kernels

Define the kernel

$$A_j(t,\eta)=\theta(\eta)\sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\Theta\left(\xi\eta^{-1}\right)\xi^j d\xi,$$

where the cut off function $\Theta(z)\in\mathbf{C}^2(\mathbb{R})$ is such that $\Theta(z)=0$ for $z\leqslant\frac{1}{3}$ or $z\geqslant 3$ and $\Theta(z)=1$ for $\frac{2}{3}\leqslant z\leqslant\frac{3}{2}$ and the Heaviside function $\theta(\eta)=1$ for $\eta>0$ and $\theta(\eta)=0$ for $\eta\leqslant 0$. We change $\xi=\eta y$, then we get

$$A_j(t,\eta)=\eta^{1+j}\theta(\eta)\sqrt{\frac{|t|}{2\pi}}\int_{\frac{1}{3}}^3 e^{-it\eta^3G(y,\eta)}\Theta(y)y^j dy.$$

where $S(\eta y,\eta) = \Lambda(\eta y) - \Lambda(\eta) - \eta\Lambda'(\eta)(y-1) \equiv \eta^3 G(y,\eta)$. To compute the asymptotics of the kernel $A_j(t,\eta)$ for large t we apply the stationary phase method (see [6])

$$\int_{\mathbb{R}} e^{i\lambda g(y)} f(y)\,dy = e^{i\lambda g(y_0)} f(y_0)\sqrt{\frac{2\pi}{\lambda\,|g''(y_0)|}}\, e^{i\frac{\pi}{4}\operatorname{sgn} g''(y_0)} + O\left(\lambda^{-\frac{3}{2}}\right) \tag{3.1}$$

for $\lambda \to +\infty$, where the stationary point y_0 is defined by the equation $g'(y_0) = 0$. By virtue of formula (3.1) with $g(y) = -G(y,\eta)$, $f(y) = \Theta(y)\,y^j$, $y_0 = 1$, we get

$$A_j(t,\eta) = \frac{t^{\frac{1}{6}-\frac{j}{3}}\left(\eta t^{\frac{1}{3}}\right)^{1+j}}{\sqrt{\langle t\eta^3\rangle\frac{\Lambda''(\eta)}{\eta}}}\, e^{-i\frac{\pi}{4}\operatorname{sgn} t} + O\left(t^{\frac{1}{2}}\eta^{1+j}\left\langle t\eta^3\right\rangle^{-1}\right)$$

for $t\eta^3 \to +\infty$. Also since $\Lambda''(\eta) = O(\eta)$ we obtain the estimate $\left|A_j(t,\eta)\right| \leqslant C\eta^{j-\frac{1}{2}}$. Next changing $\eta = \xi y$, we get

$$A_\alpha^*(t,\xi) = \xi^{1-\alpha}\theta(\xi)\sqrt{\frac{|t|}{2\pi}}\int_{\frac{1}{3}}^{3} e^{it\xi^3\widetilde{G}(y,\xi)}\Theta(y)\left|\Lambda''(\xi y)\right| y^{-\alpha}\,dy$$

where $S(\xi,\xi y) = \Lambda(\xi) - \Lambda(\xi y) - \xi\Lambda'(\xi y)(1-y) = \xi^3\widetilde{G}(y,\xi)$. Then by virtue of formula (3.1) with $g(y) = \widetilde{G}(y,\xi)$, $f(y) = \Theta(y)\,|\Lambda''(\xi y)|\,y^{-\alpha}$, $y_0 = 1$, we obtain

$$A_\alpha^*(t,\xi) = \xi^{2-\alpha} t^{\frac{1}{2}}\sqrt{\frac{\Lambda''(\xi)}{\langle t\xi^3\rangle\,\xi}}\, e^{i\frac{\pi}{4}\operatorname{sgn} t} + O\left(t^{\frac{1}{2}}\xi^{2-\alpha}\left\langle t\xi^3\right\rangle^{-1}\right)$$

for $t\xi^3 \to +\infty$. Since $\Lambda''(\xi) = O(\{\xi\})$ we find the estimate $|A_\alpha^*(t,\xi)| \leqslant C\xi^{\frac{1}{2}-\alpha}$.

3.2. Asymptotics for the operator $\mathscr{V}$

In the next lemma we estimate the operator $\mathscr{V}$ in the uniform norm. Define the cut off function $\chi_1(z) \in \mathbf{C}^2(\mathbb{R})$ such that $\chi_1(z) = 1$ for $z \leqslant 2$ and $\chi_1(z) = 0$ for $z \geqslant 3$ and $\chi_2(z) = 1 - \chi_1(z)$ and the operators

$$\mathscr{V}_j(t)\,\phi = \sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\phi(\xi)\,\chi_j\left(\xi\,|\eta|^{-1}\right)d\xi,$$

so that we have the following representation $\mathscr{V}(t)\,\phi = \mathscr{V}_1(t)\,\phi + \mathscr{V}_2(t)\,\phi$. Define the norm

$$\|\phi\|_{\mathbf{Z}} = \left\|\xi^{-1}\phi\right\|_{\mathbf{L}^\infty(\mathbb{R}_+)} + \left\|\langle\xi\rangle\,\partial_\xi\phi\right\|_{\mathbf{L}^2(\mathbb{R}_+)} + \|\langle\xi\rangle\,\phi\|_{\mathbf{L}^2(\mathbb{R}_+)}.$$

Denote $\delta_1(0) = \frac{1}{22}$, $\delta_1(k) = \frac{1}{4}$ if $k \geqslant 1$, $\delta_2(0) = \frac{1}{14}$, $\delta_2(1) = \frac{5}{14}$ and $\delta_2(k) = \frac{1}{2}$ if $k \geqslant 2$.

LEMMA 3.1. *Let $\phi(0)=0$. Then the following estimates are valid for all $t \geqslant 1$*

$$\left\| \langle \eta \rangle^{\frac{7}{4}-j} \{\eta\}^{k-j} \left(\mathscr{V}_1 \xi^j \phi - A_j(t,\eta)\phi(\eta) \right) \right\|_{\mathbf{L}^\infty(\mathbb{R}_+)} \leqslant C t^{-\delta_1(k)} \|\phi\|_{\mathbf{Z}},$$

if $j \geqslant 0,\ k \geqslant 0$,

$$\left\| \langle \eta \rangle^{\frac{5}{2}-j} \{\eta\}^{k-j} \mathscr{V}_1 \xi^j \phi \right\|_{\mathbf{L}^\infty(\mathbb{R}_-)} \leqslant C t^{-\delta_2(k)} \|\phi\|_{\mathbf{Z}}$$

if $j \geqslant 0,\ k \geqslant 0$, and

$$\left\| \langle \eta \rangle^{\frac{5}{2}-j} \{\eta\}^{k-j} \mathscr{V}_2 \xi^j \phi \right\|_{\mathbf{L}^\infty(\mathbb{R})} \leqslant C t^{-\delta_2(k)} \|\phi\|_{\mathbf{Z}}$$

if $j = 0,1,2,\ k \geqslant j$.

Proof. We write $\mathscr{V}_1 \xi^j \phi - A_j \phi = \sum_{k=1}^{4} I_k$, where

$$I_1 = \sqrt{\frac{|t|}{2\pi}} \int_0^\infty e^{-itS(\xi,\eta)} \phi(\xi) \Theta\left(\xi \eta^{-1}\right) \chi_1(\xi t^\sigma) \xi^j d\xi$$

$$I_2 = -\phi(\eta) \sqrt{\frac{|t|}{2\pi}} \int_0^\infty e^{-itS(\xi,\eta)} \Theta\left(\xi \eta^{-1}\right) \chi_1(\xi t^\sigma) \xi^j d\xi$$

$$I_3 = \sqrt{\frac{|t|}{2\pi}} \int_0^\infty e^{-itS(\xi,\eta)} (\phi(\xi) - \phi(\eta)) \Theta\left(\xi \eta^{-1}\right) \chi_2(\xi t^\sigma) \xi^j d\xi$$

$$I_4 = \sqrt{\frac{|t|}{2\pi}} \int_0^\infty e^{-itS(\xi,\eta)} \phi(\xi) \left(\chi_1\left(\xi \eta^{-1}\right) - \Theta\left(\xi \eta^{-1}\right)\right) \chi_2(\xi t^\sigma) \xi^j d\xi$$

for $\eta > 0$, with $\sigma > 0$. We have

$$\langle \eta \rangle^{\frac{5}{2}-j} \{\eta\}^{k-j} (|I_1| + |I_2|) \leqslant C t^{\frac{1}{2}-\sigma(2+k)} \left\| \xi^{-1} \phi \right\|_{\mathbf{L}^\infty(\mathbb{R}_+)}.$$

For the summand I_3 we integrate by parts via the identity
$e^{-itS(\xi,\eta)} = H_1 \partial_\xi \left((\xi - \eta) e^{-itS(\xi,\eta)} \right)$ with $H_1 = \left(1 - it(\xi-\eta)\partial_\xi S(\xi,\eta)\right)^{-1}$, to get

$$I_3 = C t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \langle \xi \rangle (\phi(\xi) - \phi(\eta))$$
$$\times (\xi - \eta) \partial_\xi \left(H_1 \Theta\left(\xi \eta^{-1}\right) \xi^j \langle \xi \rangle^{-1} \chi_2(\xi t^\sigma) \right) d\xi$$
$$+ C t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} (\xi - \eta) H_1 \Theta\left(\xi \eta^{-1}\right) \xi^j \langle \xi \rangle^{-1} \chi_2(\xi t^\sigma) \partial_\xi (\langle \xi \rangle \phi(\xi)) d\xi.$$

Note that $\partial_\xi^k \Lambda(\xi) = O\left(\xi^{3-k}\right)$, $k = 0,1,2$, also $\partial_\xi S(\xi,\eta) = \Lambda'(\xi) - \frac{\eta}{|\eta|}\Lambda'(\eta)$. Hence we find the estimates

$$\left| H_1 \Theta\left(\xi \eta^{-1}\right) \xi^j \langle \xi \rangle^{-1} \chi_2(\xi t^\sigma) \right| + \left| (\xi - \eta) \partial_\xi \left(H_1 \Theta\left(\xi \eta^{-1}\right) \xi^j \langle \xi \rangle^{-1} \chi_2(\xi t^\sigma) \right) \right|$$
$$\leqslant \frac{C \eta^j \langle \eta \rangle^{-1}}{1 + t\eta (\xi - \eta)^2}$$

in the domain $\frac{2}{3}t^{-\sigma} < \frac{\eta}{3} < \xi < 3\eta$. Therefore we obtain

$$|I_3| \leqslant Ct^{\frac{1}{2}}\eta^j \langle\eta\rangle^{-1} \int_{\frac{\eta}{3}}^{3\eta} \frac{\langle\xi\rangle |\phi(\xi) - \phi(\eta)|}{|\xi-\eta|} \frac{|\xi-\eta| \, d\xi}{1 + t\eta(\xi-\eta)^2}$$
$$+ Ct^{\frac{1}{2}}\eta^j \langle\eta\rangle^{-1} \int_{\frac{\eta}{3}}^{3\eta} \frac{|\xi-\eta| \, \partial_\xi(\langle\xi\rangle\phi(\xi)) \, d\xi}{1 + t\eta(\xi-\eta)^2}$$

for $\eta > 2t^{-\sigma}$. By the Hardy inequality $\int_{\frac{\eta}{3}}^{3\eta} \frac{\langle\xi\rangle^2 |\phi(\xi)-\phi(\eta)|^2}{|\xi-\eta|^2} d\xi \leqslant C\|\phi\|_{\mathbf{H}^{1,1}}$ and by the Cauchy-Schwarz inequality we find

$$\langle\eta\rangle^{\frac{7}{4}-j}\{\eta\}^{k-j}|I_3| \leqslant Ct^{\frac{1}{2}}\langle\eta\rangle^{\frac{3}{4}}\{\eta\}^k \|\phi\|_{\mathbf{H}^{1,1}} \left(\int_{\frac{\eta}{3}}^{3\eta} \frac{(\xi-\eta)^2 \, d\xi}{\left(1 + t\eta(\xi-\eta)^2\right)^2} \right)^{\frac{1}{2}}$$
$$\leqslant C\langle\eta\rangle^{\frac{3}{4}}\{\eta\}^k \left\langle t\eta^3 \right\rangle^{-\frac{1}{4}} \|\phi\|_{\mathbf{H}^{1,1}} \leqslant Ct^{-\delta_1(k)}\|\phi\|_{\mathbf{H}^{1,1}}$$

for $j,k \geqslant 0$, since $\eta > 2t^{-\sigma}$, where $\delta_1(0) = \frac{1-3\sigma}{4}$ and $\delta_1(k) = \frac{1}{4}$ if $k \geqslant 1$. To estimate the integral I_4 we integrate by parts via the identity

$$e^{-itS(\xi,\eta)} = H_2 \partial_\xi \left(\xi e^{-itS(\xi,\eta)} \right) \tag{3.2}$$

with $H_2 = \left(1 - it\xi\partial_\xi S(\xi,\eta)\right)^{-1}$, to get

$$I_4 = Ct^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \frac{\langle\xi\rangle\phi(\xi)}{\xi}$$
$$\times \xi^2 \partial_\xi \left(\left(\chi_1(\xi\eta^{-1}) - \Theta(\xi\eta^{-1})\right) H_2 \xi^j \langle\xi\rangle^{-1} \chi_2(\xi t^\sigma) \right) d\xi$$
$$+ Ct^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \left(\chi_1(\xi\eta^{-1}) - \Theta(\xi\eta^{-1})\right) H_2 \xi^{j+1} \langle\xi\rangle^{-1} \chi_2(\xi t^\sigma) \, \partial_\xi(\langle\xi\rangle\phi(\xi)) \, d\xi.$$

Using $\partial_\xi^k \Lambda(\xi) = O\left(\xi^{3-k}\right)$, $k = 0,1,2$, we find the estimates

$$\left| \left(\chi_1(\xi\eta^{-1}) - \Theta(\xi\eta^{-1})\right) H_2 \xi^{j+1} \langle\xi\rangle^{-1} \chi_2(\xi t^\sigma) \right|$$
$$+ \left| \xi^2 \partial_\xi \left(\left(\chi_1(\xi\eta^{-1}) - \Theta(\xi\eta^{-1})\right) H_2 \xi^j \langle\xi\rangle^{-1} \chi_2(\xi t^\sigma) \right) \right| \leqslant \frac{C\xi^{1+j}\langle\xi\rangle^{-1}}{1 + t\xi\eta^2}$$

in the domain $t^{-\sigma} \leqslant \xi \leqslant \frac{2}{3}\eta$ or $\frac{3}{2}\eta \leqslant \xi \leqslant 3\eta$. Then we obtain

$$|I_4| \leqslant Ct^{\frac{1}{2}} \int_{t^{-\sigma}}^{3\eta} \frac{\langle\xi\rangle |\phi(\xi)|}{\xi} \frac{\xi^{1+j}\langle\xi\rangle^{-1} \, d\xi}{1 + t\xi\eta^2}$$
$$+ Ct^{\frac{1}{2}} \int_{t^{-\sigma}}^{3\eta} \frac{\xi^{1+j}\langle\xi\rangle^{-1} \left|\partial_\xi(\langle\xi\rangle\phi(\xi))\right| d\xi}{1 + t\xi\eta^2}.$$

Hence by the Hardy inequality

$$|I_4| \leqslant Ct^{\frac{1}{2}} \|\phi\|_{\mathbf{H}^{1,1}} \left(\int_{t^{-\sigma}}^{3\eta} \frac{\xi^{2+2j} \langle \xi \rangle^{-2} d\xi}{(1+t\xi\eta^2)^2} \right)^{\frac{1}{2}}.$$

We obtain for $\frac{1}{2}t^{-\sigma} < \eta < 1$

$$\int_{t^{-\sigma}}^{3\eta} \frac{\xi^{2+2j} \langle \xi \rangle^{-2} d\xi}{(1+t\xi\eta^2)^2} \leqslant C\eta^{2j+3} \int_0^1 \frac{y^{2j+2} dy}{(1+t\eta^3 y)^2} \leqslant C\eta^{2j+3} \langle t\eta^3 \rangle^{-2}$$

and for $\eta \geqslant 1$

$$\int_{t^{-\sigma}}^{3\eta} \frac{\xi^{2+2j} \langle \xi \rangle^{-2} d\xi}{(1+t\xi\eta^2)^2} \leqslant Ct^{-2}\eta^{-4} \int_0^{3\eta} \langle \xi \rangle^{2j-2} d\xi \leqslant Ct^{-2}\eta^{2j-5}.$$

Thus we have

$$\langle \eta \rangle^{\frac{5}{2}-j} \{\eta\}^{k-j} |I_4| \leqslant Ct^{-\delta_2(k)} \|\phi\|_{\mathbf{H}^{1,1}}$$

for all $t \geqslant 1,\ \eta > 0$, where $\delta_2(0) = \frac{1-3\sigma}{2}$, $\delta_2(1) = \frac{1-\sigma}{2}$ and $\delta_2(k) = \frac{1}{2}$ if $k \geqslant 2$. We choose $\sigma = \frac{3}{11}$ if $k = 0$, and $\sigma = \frac{1}{4}$ if $k \geqslant 1$, then we get $\delta_1(0) = \frac{1}{22}$, $\delta_1(k) = \frac{1}{4}$ if $k \geqslant 1$. In the domain $\eta < 0$ we integrate by parts using the identity (3.2)

$$\begin{aligned}
\mathscr{V}_1 \xi^j \phi &= Ct^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \xi^{1+j} \chi_1 \left(\xi |\eta|^{-1} \right) \chi_1 (\xi t^\sigma) \phi(\xi) d\xi \\
&+ Ct^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \frac{\langle \xi \rangle \phi(\xi)}{\xi} \xi^2 \partial_\xi \left(H_2 \chi_1 \left(\xi |\eta|^{-1} \right) \xi^j \langle \xi \rangle^{-1} \chi_2 (\xi t^\sigma) \right) d\xi \\
&+ Ct^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \xi^{1+j} \langle \xi \rangle^{-1} \chi_2 (\xi t^\sigma) \chi_1 \left(\xi |\eta|^{-1} \right) H_2 \partial_\xi (\langle \xi \rangle \phi(\xi)) d\xi.
\end{aligned}$$

Then as above we obtain $\langle \eta \rangle^{\frac{5}{2}-j} \{\eta\}^{k-j} \left| \mathscr{V}_1 \xi^j \phi \right| \leqslant Ct^{-\delta_2(k)} \|\phi\|_{\mathbf{H}^{1,1}}$ for all $t \geqslant 1,\ \eta < 0$, where $\delta_2(0) = \frac{1-3\sigma}{2}$, $\delta_2(1) = \frac{1-\sigma}{2}$ and $\delta_2(k) = \frac{1}{2}$ if $k \geqslant 2$. We choose $\sigma = \frac{2}{7}$ if $k = 0,1$, and $\sigma = \frac{1}{4}$ if $k \geqslant 2$, then we get $\delta_2(0) = \frac{1}{14}$, $\delta_2(1) = \frac{5}{14}$ and $\delta_2(k) = \frac{1}{2}$ if $k \geqslant 2$. To estimate $\mathscr{V}_2 \xi^j \phi$ we integrate by parts via the identity (3.2) to get $\mathscr{V}_2 \xi^j \phi = \sum_{k=5}^7 I_k$, where

$$\begin{aligned}
I_5 &= Ct^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \xi^{1+j} \chi_1 (\xi t^\sigma) \chi_2 \left(\xi |\eta|^{-1} \right) \phi(\xi) d\xi, \\
I_6 &= Ct^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \frac{\langle \xi \rangle \phi(\xi)}{\xi} \xi^2 \partial_\xi \left(H_2 \chi_2 \left(\xi |\eta|^{-1} \right) \xi^j \langle \xi \rangle^{-1} \chi_2 (\xi t^\sigma) \right) d\xi, \\
I_7 &= Ct^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \xi^{1+j} \langle \xi \rangle^{-1} \chi_2 (\xi t^\sigma) \chi_2 \left(\xi |\eta|^{-1} \right) H_2 \partial_\xi (\langle \xi \rangle \phi(\xi)) d\xi.
\end{aligned}$$

As above we have $\langle \eta \rangle^{\frac{5}{2}-j} \{\eta\}^{k-j} |I_5| \leqslant Ct^{\frac{1}{2}-\sigma(2+k)} \left\| \xi^{-1} \phi \right\|_{\mathbf{L}^\infty(\mathbb{R}_+)}$. Using $\partial_\xi^k \Lambda(\xi) = O\left(\xi^{3-k}\right),\ k = 0,1,2,$ we find the estimates

$$\begin{aligned}
&\left| \xi^2 \partial_\xi \left(H_2 \chi_2 \left(\xi |\eta|^{-1} \right) \xi^j \langle \xi \rangle^{-1} \chi_2 (\xi t^\sigma) \right) \right| \\
&+ \left| \xi^{1+j} \langle \xi \rangle^{-1} \chi_2 \left(\xi |\eta|^{-1} \right) \chi_2 (\xi t^\sigma) H_2 \right| \leqslant \frac{C\xi^{1+j} \langle \xi \rangle^{-1}}{1+t\xi^3}
\end{aligned}$$

in the domain $\xi \geqslant \frac{3}{2}|\eta|$, $\xi > t^{-\sigma}$, we obtain

$$|I_6| \leqslant Ct^{\frac{1}{2}} \int_{\frac{3}{2}|\eta|}^{\infty} \frac{\langle\xi\rangle |\phi(\xi)|}{\xi} \frac{\xi^{1+j} \langle\xi\rangle^{-1} d\xi}{1+t\xi^3},$$
$$|I_7| \leqslant Ct^{\frac{1}{2}} \int_{\frac{3}{2}|\eta|}^{\infty} \frac{\xi^{1+j} \langle\xi\rangle^{-1} \left|\partial_\xi (\langle\xi\rangle \phi(\xi))\right| d\xi}{1+t\xi^3}.$$

Hence by the Hardy inequality

$$|I_6| + |I_7| \leqslant Ct^{\frac{1}{2}} \|\phi\|_{\mathbf{H}^{1,1}} \left(\int_{\max\left(\frac{3}{2}|\eta|, t^{-\sigma}\right)}^{\infty} \frac{\xi^{2+2j} \langle\xi\rangle^{-2} d\xi}{(1+t\xi^3)^2} \right)^{\frac{1}{2}}.$$

We have for $|\eta| < 1$

$$\int_{\max\left(\frac{3}{2}|\eta|, t^{-\sigma}\right)}^{\infty} \frac{\xi^{2+2j} \langle\xi\rangle^{-2} d\xi}{(1+t\xi^3)^2} \leqslant C \int_{t^{-\sigma}}^{\frac{3}{2}} \frac{\xi^{2+2j} d\xi}{(1+t\xi^3)^2} + Ct^{-2} \int_{\frac{3}{2}}^{\infty} \xi^{2j-6} d\xi$$
$$\leqslant Ct^{-\frac{2j}{3}-1} \int_{t^{\frac{1}{3}-\sigma}}^{\frac{3}{2}t^{\frac{1}{3}}} \xi^{2j+2} \langle\xi\rangle^{-6} d\xi + Ct^{-2}$$

if $0 \leqslant j \leqslant 2$, and for $|\eta| \geqslant 1$

$$\int_{\max\left(\frac{3}{2}|\eta|, t^{-\sigma}\right)}^{\infty} \frac{\xi^{2+2j} \langle\xi\rangle^{-2} d\xi}{(1+t\xi^3)^2} \leqslant Ct^{-2} \int_{\frac{3}{2}|\eta|}^{\infty} \xi^{2j-6} d\xi \leqslant Ct^{-2} \langle\eta\rangle^{2j-5}.$$

Thus as above we have $\langle\eta\rangle^{\frac{5}{2}-j} \{\eta\}^{k-j} (|I_6| + |I_7|) \leqslant Ct^{-\delta_2(k)} \|\phi\|_{\mathbf{H}^{1,1}}$ for all $t \geqslant 1$ if $0 \leqslant j \leqslant 2$. Lemma 3.1 is proved.

3.3. Asymptotics for the operator $\mathscr{V}^*$

We next consider the operator $\mathscr{V}^*$. Since $\|\mathscr{V}^*\phi\|_{\mathbf{L}^\infty(\mathbb{R}_+)} \leqslant C|t|^{\frac{1}{2}} \|\eta\phi\|_{\mathbf{L}^1(\mathbb{R})}$ and $\|\mathscr{V}^*\phi\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C \left\| |\eta|^{\frac{1}{2}} \phi \right\|_{\mathbf{L}^2(\mathbb{R})}$, then by the Riesz interpolation theorem (see [18], p. 52) we have

$$\|\mathscr{V}^*\phi\|_{\mathbf{L}^p(\mathbb{R}_+)} \leqslant C|t|^{\frac{1}{2}-\frac{1}{p}} \left\| |\eta|^{1-\frac{1}{p}} \phi \right\|_{\mathbf{L}^{\frac{p}{p-1}}} \tag{3.3}$$

for $2 \leqslant p \leqslant \infty$. In the next lemma we find the asymptotics of $\mathscr{V}^*$.

LEMMA 3.2. *Let $0 \leqslant \alpha < \frac{5}{2}$. Then the following estimate is valid for all $t \geqslant 1$*

$$\|\mathscr{V}^*\phi - A_\alpha^* \xi^\alpha \phi\|_{\mathbf{L}^\infty(\mathbb{R}_+)} \leqslant C \max\left(t^{-\frac{1}{4}}, t^{\frac{\alpha-1}{3}}\right) \|\partial_\eta (|\eta|^\alpha \phi)\|_{\mathbf{L}^2(\mathbb{R})}.$$

Proof. We write $\mathscr{V}^*\phi - A_\alpha^*\xi^\alpha\phi = \sum_{k=1}^2 I_k$, where

$$I_1 = \sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{itS(\xi,\eta)}\left(\eta^\alpha\phi(\eta) - \xi^\alpha\phi(\xi)\right)\left|\Lambda''(\eta)\right|\Theta\left(\eta\xi^{-1}\right)\eta^{-\alpha}d\eta,$$

$$I_2 = \sqrt{\frac{|t|}{2\pi}}\int_{-\infty}^\infty e^{itS(\xi,\eta)}|\eta|^\alpha\phi(\eta)\left|\Lambda''(\eta)\right|\left(1-\Theta\left(\eta\xi^{-1}\right)\right)|\eta|^{-\alpha}d\eta$$

for $\xi > 0$. In the integral I_1 we use the identity $e^{itS(\xi,\eta)} = H_3\partial_\eta\left((\eta-\xi)e^{itS(\xi,\eta)}\right)$ with $H_3 = (1+it(\eta-\xi)\partial_\eta S(\xi,\eta))^{-1}$, $\partial_\eta S(\xi,\eta) = -|\Lambda''(\eta)|(\xi-\eta)$, and integrate by parts

$$I_1 = Ct^{\frac{1}{2}}\int_0^\infty e^{itS(\xi,\eta)}\frac{\eta^\alpha\phi(\eta) - \xi^\alpha\phi(\xi)}{\eta-\xi}$$
$$\times(\eta-\xi)^2\partial_\eta\left(H_3\left|\Lambda''(\eta)\right|\eta^{-\alpha}\Theta\left(\eta\xi^{-1}\right)\right)d\eta$$
$$+Ct^{\frac{1}{2}}\int_0^\infty e^{itS(\xi,\eta)}(\eta-\xi)H_3\left|\Lambda''(\eta)\right|\eta^{-\alpha}\Theta\left(\eta\xi^{-1}\right)\partial_\eta\left(\eta^\alpha\phi(\eta)\right)d\eta.$$

Then applying the estimates $\Lambda''(\eta) = O(\eta)$,

$$\left|(\eta-\xi)H_3\left|\Lambda''(\eta)\right|\eta^{-\alpha}\Theta\left(\eta\xi^{-1}\right)\right|$$
$$+\left|(\eta-\xi)^2\partial_\eta\left(H_3\left|\Lambda''(\eta)\right|\eta^{-\alpha}\Theta\left(\eta\xi^{-1}\right)\right)\right| \leqslant \frac{C\xi^{1-\alpha}|\eta-\xi|}{1+t\xi(\eta-\xi)^2}$$

in the domain $0 < \frac{1}{3}\xi \leqslant \eta \leqslant 3\xi$, we find

$$|I_1| \leqslant Ct^{\frac{1}{2}}\xi^{1-\alpha}\left\|\partial_\eta\left(\eta^\alpha\phi\right)\right\|_{\mathbf{L}^2(\mathbb{R}_+)}\left(\int_{\frac{1}{3}\xi}^{3\xi}\frac{(\eta-\xi)^2d\eta}{\left(1+t\xi(\eta-\xi)^2\right)^2}\right)^{\frac{1}{2}}.$$

Changing $\eta = y\xi$, we have

$$\int_{\frac{1}{3}\xi}^{3\xi}\frac{(\eta-\xi)^2d\eta}{\left(1+t\xi(\eta-\xi)^2\right)^2} \leqslant C\int_{\frac{1}{3}}^{3}\frac{\xi^3(1-y)^2dy}{\left(1+t\xi^3(1-y)^2\right)^2} \leqslant \frac{C\xi^3}{\langle t\xi^3\rangle^{\frac{3}{2}}}.$$

Hence $|I_1| \leqslant C\max\left(t^{-\frac{1}{4}}, t^{-\frac{1-\alpha}{3}}\right)\left\|\partial_\eta\left(|\eta|^\alpha\phi\right)\right\|_{\mathbf{L}^2(\mathbb{R}_+)}$ if $0 \leqslant \alpha \leqslant \frac{5}{2}$. In the integral I_2 using the identity $e^{itS(\xi,\eta)} = H_4\partial_\eta\left(\eta e^{itS(\xi,\eta)}\right)$ with $H_4 = (1+it\eta\partial_\eta S(\xi,\eta))^{-1}$, $\partial_\eta S(\xi,\eta) = -\Lambda''(\eta)(\xi-\eta)$, we integrate by parts

$$I_2 = Ct^{\frac{1}{2}}\int_{-\infty}^\infty e^{itS(\xi,\eta)}\frac{|\eta|^\alpha\phi(\eta)}{\eta}\eta^2\partial_\eta\left(H_4\left(1-\Theta\left(\eta\xi^{-1}\right)\right)\left|\Lambda''(\eta)\right||\eta|^{-\alpha}\right)d\eta$$
$$+Ct^{\frac{1}{2}}\int_{-\infty}^\infty e^{itS(\xi,\eta)}\eta H_4\left(1-\Theta\left(\eta\xi^{-1}\right)\right)\left|\Lambda''(\eta)\right||\eta|^{-\alpha}\partial_\eta\left(|\eta|^\alpha\phi(\eta)\right)d\eta.$$

Then using the estimates $\partial_\eta S(\xi,\eta) = O(\eta(\xi+|\eta|))$ in the domains $\eta < \frac{1}{3}\xi$ and $\eta \geqslant 3\xi > 0$, we get

$$\left|\eta^2 \partial_\eta \left(H_4 \left(1-\Theta\left(\eta\xi^{-1}\right)\right)\left|\Lambda''(\eta)\right||\eta|^{-\alpha}\right)\right|$$
$$+\left|\eta H_4 \left(1-\Theta\left(\eta\xi^{-1}\right)\right)\left|\Lambda''(\eta)\right||\eta|^{-\alpha}\right| \leqslant \frac{C|\eta|^{2-\alpha}\langle\eta\rangle}{1+t\eta^2(\xi+|\eta|)}.$$

Therefore by the Hardy inequality

$$|I_2| \leqslant Ct^{\frac{1}{2}} \left\|\partial_\eta \left(|\eta|^\alpha \phi\right)\right\|_{\mathbf{L}^2(\mathbb{R})} \left(\int_{-\infty}^{\infty} \frac{|\eta|^{4-2\alpha} d\eta}{\left(1+t\eta^2(\xi+|\eta|)\right)^2}\right)^{\frac{1}{2}}.$$

We have

$$\int_{-\infty}^{\infty} \frac{|\eta|^{4-2\alpha} d\eta}{\left(1+t\eta^2(\xi+|\eta|)\right)^2} \leqslant C\int_0^1 \frac{|\eta|^{4-2\alpha} d\eta}{\left(1+t\eta^2(\xi+|\eta|)\right)^2}$$
$$+Ct^{-2}\int_1^{\infty} (\xi+|\eta|)^{-2} d\eta \leqslant C\max\left(t^{-2}, t^{-\frac{5-2\alpha}{3}}\right)$$

if $0 \leqslant \alpha < \frac{5}{2}$. Therefore we get $|I_2| \leqslant C\max\left(t^{-\frac{1}{2}}, t^{-\frac{1-\alpha}{3}}\right) \left\|\partial_\eta \left(|\eta|^\alpha \phi\right)\right\|_{\mathbf{L}^2(\mathbb{R})}$. Lemma 3.2 is proved.

4. Commutators with $\mathscr{V}$

First we estimate the Fourier type integral

$$\mathscr{W}\phi = t^{\frac{1}{2}} \int_{\mathbb{R}_+} e^{-itS(\xi,\eta)} q(\xi,\eta)\phi(\xi)\, d\xi$$

in the $\mathbf{L}^2$ - norm. In the particular factorized case $q(\xi,\eta) = q_1(\xi) q_2(\eta)$, with estimate $|q_2(\mu(x))| \leqslant |\Lambda''(\mu(x))|^{\frac{1}{2}}$, we find

$$\|\mathscr{W}\phi\|_{\mathbf{L}^2} \leqslant t^{\frac{1}{2}} \left\| q_2(\mu(x)) \left|\Lambda''(\mu(x))\right|^{-\frac{1}{2}} \int_{\mathbb{R}_+} e^{itx\xi} e^{-it\Lambda(\xi)} q_1(\xi)\phi(\xi)\, d\xi \right\|_{\mathbf{L}^2}$$
$$\leqslant C\left\|e^{-it\Lambda} q_1\phi\right\|_{\mathbf{L}^2} = C\|q_1\phi\|_{\mathbf{L}^2}.$$

Next we obtain a more general result.

LEMMA 4.1. *Suppose that* $\left|(\eta\partial_\eta)^k q(\xi,\eta)\right| \leqslant C\langle\eta\rangle^{-\nu_1}\langle\xi\rangle^{-\nu_2}\{\eta\}^{\nu_3}$ *for all* $\xi \in \mathbb{R}_+$, $\eta \in \mathbb{R}$, $k = 0,1,2$, *where* $\nu_1 > 0$, $\nu_2 \geqslant 0$, $\nu_3 > -1$, $\nu_2+\nu_3 > 0$. *Then the estimate*

$$\left\|\left|\Lambda''\right|^{\frac{1}{2}} \mathscr{W}\phi\right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant C\|\phi\|_{\mathbf{L}^2(\mathbb{R}_+)} \begin{cases} 1 \text{ if } \nu_3 > 0, \\ \log\langle t\rangle \text{ if } \nu_3 = 0, \\ |t|^{-\frac{1}{2}\nu_3} \text{ if } -1 < \nu_3 < 0 \end{cases}$$

is true for all $t \geqslant 1$.

Proof. We write

$$\begin{aligned}
&\left\| \left|\Lambda''\right|^{\frac{1}{2}} \mathscr{W} \phi \right\|^2_{\mathbf{L}^2(\mathbb{R})} \\
&= Ct \int_{\mathbb{R}} d\eta \left|\Lambda''\right| \int_{\mathbb{R}_+} d\xi \int_{\mathbb{R}_+} d\zeta e^{it(S(\zeta,\eta)-S(\xi,\eta))} q(\xi,\eta)\phi(\xi)\overline{q(\zeta,\eta)\phi(\zeta)} \\
&= Ct \int_{\mathbb{R}_+} d\xi e^{-it\Lambda(\xi)}\phi(\xi) \int_{\mathbb{R}_+} d\zeta e^{it\Lambda(\zeta)}\overline{\phi(\zeta)} K(\xi,\zeta),
\end{aligned}$$

where the kernel

$$K(\xi,\zeta) = \int_{\mathbb{R}} d\eta \left|\Lambda''\right| e^{it\frac{\eta}{|\eta|}\Lambda'(\eta)(\xi-\zeta)} q(\xi,\eta)\overline{q(\zeta,\eta)}.$$

Integrating two times by parts via the identity

$$e^{it\frac{\eta}{|\eta|}\Lambda'(\eta)(\xi-\zeta)} = H_5\partial_\eta\left(\eta e^{it\frac{\eta}{|\eta|}\Lambda'(\eta)(\xi-\zeta)}\right) \text{ with } H_5 = \left(1+it\eta\left|\Lambda''(\eta)\right|(\xi-\zeta)\right)^{-1}$$

we get

$$K(\xi,\zeta) = \int_{\mathbb{R}} e^{it\frac{\eta}{|\eta|}\Lambda'(\eta)(\xi-\zeta)} \eta\partial_\eta\left(H_5\eta\partial_\eta\left(H_5\left|\Lambda''\right| q(\xi,\eta)\overline{q(\zeta,\eta)}\right)\right)d\eta.$$

Since

$$\left|\eta\partial_\eta\left(H_5\eta\partial_\eta\left(H_5\left|\Lambda''\right| q(\xi,\eta)\overline{q(\zeta,\eta)}\right)\right)\right| \leqslant \frac{C\left|\Lambda''\right|\{\eta\}^{2\nu}\langle\eta\rangle^{-2\nu_1}\langle\xi\rangle^{-\nu_2}\langle\zeta\rangle^{-\nu_2}}{\left(1+t\eta^2|\xi-\zeta|\right)^2},$$

we get

$$\begin{aligned}
&|K(\xi,\zeta)| \leqslant C\langle\xi\rangle^{-\nu_2}\langle\zeta\rangle^{-\nu_2}\int_0^1 \frac{\eta^{1+2\nu_3}d\eta}{\left(1+t\eta^2|\xi-\zeta|\right)^2} + C\int_1^\infty \frac{\eta^{-2\nu_1}d\eta}{\left(1+t\eta^2|\xi-\zeta|\right)^2} \\
&\leqslant C\langle\xi\rangle^{-\nu_2}\langle\zeta\rangle^{-\nu_2}(t|\xi-\zeta|)^{-1-\nu_3}\int_0^{(t|\xi-\zeta|)^{\frac{1}{2}}} \eta^{1+2\nu_3}\langle\eta\rangle^{-4}d\eta \\
&+C(|\xi-\zeta|t)^{2\nu_3-1}\int_{t|\xi-\zeta|}^\infty \eta^{-2\nu_1}\langle\eta\rangle^{-4}d\eta \\
&\leqslant C\langle\xi-\zeta\rangle^{-\nu_2}\langle(\xi-\zeta)t\rangle^{-1-\nu_3} + C(|\xi-\zeta|t)^{2\nu_1-1}\langle(\xi-\zeta)t\rangle^{-3-2\nu_1}
\end{aligned}$$

if $v_3 > -1$. Then by the Cauchy-Schwarz inequality and Young inequality we obtain

$$\begin{aligned}
&\left\| \left|\Lambda''\right|^{\frac{1}{2}} \mathscr{W}\phi \right\|^2_{\mathbf{L}^2(\mathbb{R})} \\
&\leqslant Ct\|\phi\|_{\mathbf{L}^2(\mathbb{R}_+)} \left\| \int_{\mathbb{R}_+} \langle(\xi-\zeta)t\rangle^{-1-v_3} \langle \xi-\zeta\rangle^{-v_2} |\phi(\zeta)| d\zeta \right\|_{\mathbf{L}^2(\mathbb{R}_+)} \\
&+ Ct\|\phi\|_{\mathbf{L}^2(\mathbb{R}_+)} \left\| \int_{\mathbb{R}_+} (|\xi-\zeta|t)^{2v_1-1} \langle(\xi-\zeta)t\rangle^{-3-2v_1} |\phi(\zeta)| d\zeta \right\|_{\mathbf{L}^2(\mathbb{R}_+)} \\
&\leqslant C|t| \left(\left\| \langle \xi t\rangle^{-1-v_3} \langle\xi\rangle^{-v_2} \right\|_{\mathbf{L}^1(\mathbb{R})} + \left\| |\xi t|^{2v_1-1} \langle \xi t\rangle^{-3-2v_1} \right\|_{\mathbf{L}^1(\mathbb{R})} \right) \|\phi\|^2_{\mathbf{L}^2(\mathbb{R}_+)} \\
&\leqslant C\|\phi\|^2_{\mathbf{L}^2(\mathbb{R}_+)} \left(|t| \int_0^{|t|^{-1}} d\xi + |t|^{-v_3} \int_{|t|^{-1}}^1 \xi^{-1-v_3} d\xi \right. \\
&\left. + |t|^{-v_3} \int_1^\infty \xi^{-1-v_3-v_2} d\xi + \int_0^\infty \xi^{2v_1-1} \langle\xi\rangle^{-3-2v_1} d\xi \right) \\
&\leqslant C\|\phi\|^2_{\mathbf{L}^2(\mathbb{R}_+)} \begin{cases} 1 \text{ if } v_3 > 0, \\ \log\langle t\rangle \text{ if } v_3 = 0, \\ |t|^{-v_3} \text{ if } -1 < v_3 < 0 \end{cases}
\end{aligned}$$

if $v_1 > 0,\ v_3 > -1,\ v_2 > -v_3$. Lemma 4.1 is proved.

Consider the commutator $[h, \mathscr{V}_1]$.

LEMMA 4.2. *Let $\phi(0) = 0$. Suppose that $h \in \mathbf{C}^4(\mathbb{R}_+)$ and $\partial_\xi^k h(\xi) = O(\xi^{\rho-k})$ for all $\xi \in \mathbb{R}_+,\ 0 \leqslant k \leqslant 4$. Then the following estimate is true*

$$\left\| \left|\Lambda''\right|^{\frac{1}{2}} \{\eta\}^\alpha \langle\eta\rangle^\beta t[h,\mathscr{V}_1]\xi^j\phi \right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant C\|\phi\|_{\mathbf{H}^{1,1}(\mathbb{R}_+)}$$

for all $t \geqslant 1$, if $\alpha > 2-j-\rho,\ \beta < 2-\rho-\max(j-1,0),\ j \geqslant 0$.

Proof. Integration by parts yields $\{\eta\}^\alpha\langle\eta\rangle^\beta t[h,\mathscr{V}_1]\xi^j\phi = I_1 + I_2$, where

$$I_1 = \sqrt{\frac{|t|}{2\pi}} \int_0^\infty e^{-itS(\xi,\eta)} q_1(\xi,\eta) \partial_\xi(\langle\xi\rangle\phi(\xi)) d\xi$$

$$I_2 = \sqrt{\frac{|t|}{2\pi}} \int_0^\infty e^{-itS(\xi,\eta)} q_2(\xi,\eta) \frac{\langle\xi\rangle\phi(\xi)}{\xi} d\xi,$$

with $q_1(\xi,\eta) = \frac{h(\eta)-h(\xi)}{i\partial_\xi S(\xi,\eta)} \{\eta\}^\alpha\langle\eta\rangle^\beta\{\xi\}^j\langle\xi\rangle^{j-1}\chi_1\left(\xi|\eta|^{-1}\right)$ and $q_2(\xi,\eta) = \xi\partial_\xi q_1(\xi,\eta)$. Since $\partial_\xi S(\xi,\eta) = \Lambda'(\xi) - \frac{\eta}{|\eta|}\Lambda'(\eta),\ \partial_\xi^k\Lambda(\xi) = O(\xi^{3-k})$, we estimate

$$\begin{aligned}
q_1(\xi,\eta) &= O\left(\frac{|\eta|^{\rho-1}}{\xi+|\eta|} \{\eta\}^\alpha\langle\eta\rangle^\beta\{\xi\}^j\langle\xi\rangle^{j-1}\chi_1\left(\xi|\eta|^{-1}\right) \right) \\
&= O\left(\{\eta\}^{\rho+\alpha+j-2} \langle\eta\rangle^{\rho+\beta-2+\max(j-1,0)} \right),
\end{aligned}$$

since $\xi \leqslant 3|\eta|$. In the same manner $q_2(\xi,\eta) = O\left(\{\eta\}^{\rho+\alpha+j-2}\langle\eta\rangle^{\rho+\beta-2+\max(j-1,0)}\right)$. Hence taking $\nu = \min(\rho+\alpha+j-2, 2-\max(j-1,0)-\rho-\beta) > 0$, we get $\left|(\eta\partial_\eta)^k q_l(\xi,\eta)\right| \leqslant C\{\eta\}^\nu \langle\eta\rangle^{-\nu}$ for all $\xi \in \mathbb{R}_+$, $\eta \in \mathbb{R}$, $k = 0,1,2$. Then by the Hardy inequality and by Lemma 4.1 we have $\left\||\Lambda''|^{\frac{1}{2}} I_k\right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant C\|\phi\|_{\mathbf{H}^{1,1}(\mathbb{R}_+)}$ for $k = 1,2$. Lemma 4.2 is proved.

In the next lemma we estimate the operator $\mathscr{V}_2$.

LEMMA 4.3. *Let $\phi(0) = 0$. Then the following estimate is true*

$$\left\||\Lambda''|^{\frac{1}{2}}\{\eta\}^\alpha\langle\eta\rangle^\beta t\mathscr{V}_2\xi^j\phi\right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant C\|\phi\|_{\mathbf{H}^{1,1}(\mathbb{R}_+)}\begin{cases} 1 \text{ if } \alpha > 2-j, \\ \log\langle t\rangle \text{ if } \alpha = 2-j, \\ |t|^{\frac{2-j-\alpha}{2}} \text{ if } 1-j < \alpha < 2-j \end{cases}$$

for all $t \geqslant 1$, if $0 \leqslant j \leqslant 3$, $\alpha \geqslant 0$, $\alpha > 1-j$, $\beta < \min(3-j, 1+\alpha)$.

Proof. Integration by parts yields $\{\eta\}^\alpha\langle\eta\rangle^\beta t\mathscr{V}_2\xi^j\phi = I_1 + I_2$, where

$$I_1 = \sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\widetilde{q_1}(\xi,\eta)\partial_\xi(\langle\xi\rangle\phi(\xi))\,d\xi,$$
$$I_2 = \sqrt{\frac{|t|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\widetilde{q_2}(\xi,\eta)\frac{\langle\xi\rangle\phi(\xi)}{\xi}d\xi,$$

where $q_1(\xi,\eta) = \frac{\{\eta\}^\alpha\langle\eta\rangle^\beta\xi^j\langle\xi\rangle^{-1}}{i\partial_\xi S(\xi,\eta)}\chi_2\left(\xi|\eta|^{-1}\right)$, $q_2(\xi,\eta) = \xi\partial_\xi q_1(\xi,\eta)$. Next we have $q_1(\xi,\eta) = O\left(\{\eta\}^{\alpha+j-2}\langle\eta\rangle^\beta(\langle\eta\rangle+\langle\xi\rangle)^{j-3}\right)$ and $q_2(\xi,\eta) = O\left(\{\eta\}^{\alpha+j-2}\langle\eta\rangle^\beta(\langle\eta\rangle+\langle\xi\rangle)^{j-3}\right)$. Hence we have $\left|(\eta\partial_\eta)^k q_l(\xi,\eta)\right| \leqslant C\langle\eta\rangle^{-\nu_1}\langle\xi\rangle^{-\nu_2}\{\eta\}^{\nu_3}$, for all $\xi \in \mathbb{R}_+$, $\eta \in \mathbb{R}$, $k = 0,1,2$, $l = 1,2$, with $0 < \nu_1 < 1+\alpha-\beta$, $\nu_1 \leqslant 3-j-\beta$, $\nu_2 = 3-j-\beta-\nu_1 \geqslant 0$, $\nu_3 = \alpha+j-2 > -1$, so that $\nu_2+\nu_3 > 0$. Then by the Hardy inequality and by Lemma 4.1 we get

$$\left\||\Lambda''|^{\frac{1}{2}} I_k\right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant C\|\phi\|_{\mathbf{H}^{1,1}(\mathbb{R}_+)}\begin{cases} 1 \text{ if } \nu_3 > 0, \\ \log\langle t\rangle \text{ if } \nu_3 = 0, \\ |t|^{-\frac{1}{2}\nu_3} \text{ if } -1 < \nu_3 < 0, \end{cases}$$

$k = 1,2$. Lemma 4.3 is proved.

In the next lemma, we estimate the derivatives $\partial_\eta\mathscr{V}_k$.

LEMMA 4.4. *Let $\phi(0) = 0$. Then the following estimates are true for all $t \geqslant 1$*

$$\left\||\Lambda''|^{\frac{1}{2}}\{\eta\}^\alpha\langle\eta\rangle^\beta\partial_\eta\mathscr{V}_1\xi^j\phi\right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant C\|\phi\|_{\mathbf{H}^{1,1}(\mathbb{R}_+)},$$

if $\alpha > -j,\ \beta < \min(1-j,0),\ j \geqslant 0$, *and*

$$\left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\alpha} \langle\eta\rangle^{\beta} \partial_\eta \mathscr{V}_2 \xi^j \phi \right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant C \|\phi\|_{\mathbf{H}^{1,1}(\mathbb{R}_+)},$$

if $\alpha \geqslant -1,\ \alpha > -j,\ \beta < \min(1-j, 1+\alpha),\ 0 \leqslant j \leqslant 2.$

Proof. Since $\mathscr{A}_1 \mathscr{V} = \mathscr{V} i\xi$ with $\mathscr{A}_1 = \mathscr{A}_0 + i\eta,\ \mathscr{A}_0 = \frac{1}{t|\Lambda''(\eta)|}\partial_\eta$, then we obtain the commutator $\partial_\eta \mathscr{V} = t|\Lambda''(\eta)|[i\eta, \mathscr{V}]$. By Lemma 4.2 we find

$$\begin{aligned} \left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\alpha} \langle\eta\rangle^{\beta} \partial_\eta \mathscr{V}_1 \xi^j \phi \right\|_{\mathbf{L}^2(\mathbb{R})} &\leqslant Ct \left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\alpha+1} \langle\eta\rangle^{\beta+1} [\eta, \mathscr{V}_1] \xi^j \phi \right\|_{\mathbf{L}^2(\mathbb{R})} \\ &\leqslant C \|\phi\|_{\mathbf{H}^{1,1}(\mathbb{R}_+)} \end{aligned}$$

for all $t \geqslant 1$, if $\alpha > -j,\ \beta < \min(1-j,0),\ j \geqslant 0$. Also by Lemma 4.3 we get

$$\begin{aligned} &\left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\alpha} \langle\eta\rangle^{\beta} \partial_\eta \mathscr{V}_2 \xi^j \phi \right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant Ct \left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\alpha+2} \langle\eta\rangle^{\beta+2} \mathscr{V}_2 \xi^j \phi \right\|_{\mathbf{L}^2(\mathbb{R})} \\ &+Ct \left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\alpha+1} \langle\eta\rangle^{\beta+1} \mathscr{V}_2 \xi^{j+1} \phi \right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant C \|\phi\|_{\mathbf{H}^{1,1}(\mathbb{R}_+)} \end{aligned}$$

for all $t \geqslant 1$, if $\alpha \geqslant -1,\ \alpha > -j,\ \beta < \min(1-j, 1+\alpha),\ 0 \leqslant j \leqslant 2$. Lemma 4.4 is proved.

5. Commutators with $\mathscr{V}^*$

First we estimate the Fourier type integral in the $\mathbf{L}^2$ - norm

$$I = t^{\frac{1}{2}} \int_{\mathbb{R}_+} e^{itS(\xi,\eta)} q(\xi,\eta) |\Lambda''(\eta)|^{\frac{1}{2}} \phi(\eta) d\eta.$$

Consider a particular factorized case $q(\xi,\eta) = q_1(\xi) q_2(\eta)$, with estimate $|q_1(\xi)| \leqslant C$. Then we find

$$\begin{aligned} \|I\|_{\mathbf{L}^2(\mathbb{R}_+)} &\leqslant t^{\frac{1}{2}} \left\| \int_{\mathbb{R}_+} e^{-itx\xi} M q_2(\mu(x)) |\Lambda''(\mu(x))|^{-\frac{1}{2}} \phi(\mu(x)) dx \right\|_{\mathbf{L}^2_\xi(\mathbb{R}_+)} \\ &\leqslant C \left\| q_2(\mu(x)) |\Lambda''(\mu(x))|^{-\frac{1}{2}} \phi(\mu(x)) \right\|_{\mathbf{L}^2_x(\mathbb{R}_+)} = C \|q_2(\eta)\phi(\eta)\|_{\mathbf{L}^2_\eta}. \end{aligned}$$

In particular taking $q_1(\xi) = 1,\ q_2(\eta) = |\Lambda''(\eta)|^{\frac{1}{2}}$, we obtain $\|\mathscr{V}^* \theta(\eta)\phi\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C \left\| |\Lambda''|^{\frac{1}{2}} \phi \right\|_{\mathbf{L}^2(\mathbb{R}_+)}$.

In the next lemma we generalize this result.

LEMMA 5.1. *Suppose that* $\left|\left(\xi\partial_{\xi}\right)^{k} q(\xi,\eta)\right| \leqslant C\{\xi\}^{\nu}\langle\xi\rangle^{-\nu}$ *for all* $\xi,\ \eta \in \mathbb{R}_{+}$, $k=0,1,2,$ *where* $\nu>0$. *Then the estimate* $\|I\|_{\mathbf{L}^{2}(\mathbb{R}_{+})} \leqslant C\|\phi\|_{\mathbf{L}^{2}(\mathbb{R}_{+})}$ *is true for all* $t \geqslant 1$.

Proof. Changing $\eta=\mu(x)$, we find

$$\begin{aligned}
&\|I\|_{\mathbf{L}^{2}(\mathbb{R}_{+})}^{2} = Ct\int_{0}^{\infty} d\xi \int_{\mathbb{R}_{+}} e^{itS(\xi,\mu(x))} q(\xi,\mu(x))\,\phi(\mu(x))\left|\Lambda''(\mu(x))\right|^{-\frac{1}{2}} dx \\
&\times \int_{\mathbb{R}_{+}} e^{-itS(\xi,\mu(y))} \overline{q(\xi,\mu(y))\phi(\mu(y))}\left|\Lambda''(\mu(y))\right|^{-\frac{1}{2}} dy \\
&= Ct\int_{\mathbb{R}_{+}} M\phi(\mu(x))\left|\Lambda''(\mu(x))\right|^{-\frac{1}{2}} dx \\
&\times \int_{\mathbb{R}_{+}} \overline{M\phi(\mu(y))}\left|\Lambda''(\mu(y))\right|^{-\frac{1}{2}} dy K(t,x,y),
\end{aligned}$$

where

$$K(t,x,y) = \int_{0}^{\infty} e^{it\xi(y-x)} q(\xi,\mu(x))\,\overline{q(\xi,\mu(y))}d\xi.$$

Integrating two times by parts via the identity $e^{it\xi(y-x)} = H_{6}\partial_{\xi}\left(\xi e^{it\xi(y-x)}\right)$ with $H_{6} = (1+it\xi(y-x))^{-1}$ we get

$$K(t,x,y) = \int_{0}^{\infty} e^{it\xi(y-x)}\xi\partial_{\xi}\left(\xi H_{6}\partial_{\xi}\left(H_{6}q(\xi,\mu(x))\overline{q(\xi,\mu(y))}\right)\right)d\xi.$$

Since

$$\left|\xi\partial_{\xi}\left(\xi H_{6}\partial_{\xi}\left(H_{6}q(\xi,\mu(x))\overline{q(\xi,\mu(y))}\right)\right)\right| \leqslant \frac{C\{\xi\}^{2\nu}\langle\xi\rangle^{-2\nu}}{(1+t\xi|y-x|)^{2}}$$

we find

$$\begin{aligned}
|K(t,x,y)| &\leqslant C\int_{0}^{1}\frac{\xi^{2\nu}d\xi}{(1+t\xi|y-x|)^{2}} + C\int_{1}^{\infty}\frac{\xi^{-2\nu}d\xi}{(1+t\xi|y-x|)^{2}} \\
&\leqslant C\langle(y-x)t\rangle^{-1-2\nu} + C|(y-x)t|^{2\nu-1}\langle(y-x)t\rangle^{-1-2\nu}
\end{aligned}$$

if $\nu>0$. Then by the Hölder and the Young inequalities we obtain

$$\begin{aligned}
&\|I\|_{\mathbf{L}^{2}(\mathbb{R}_{+})}^{2} \leqslant Ct\left\|\left|\Lambda''(\mu(x))\right|^{-\frac{1}{2}}\phi(\mu(x))\right\|_{\mathbf{L}_{x}^{2}(\mathbb{R}_{+})} \\
&\times\left(\left\|\int_{0}^{\infty}\langle(y-x)t\rangle^{-1-2\nu}\left|\Lambda''(\mu(y))\right|^{-\frac{1}{2}}|\phi(\mu(y))|dy\right\|_{\mathbf{L}_{x}^{2}(\mathbb{R}_{+})}\right. \\
&\left.+\left\|\int_{0}^{\infty}|(y-x)t|^{2\nu-1}\langle(y-x)t\rangle^{-1-2\nu}\left|\Lambda''(\mu(y))\right|^{-\frac{1}{2}}|\phi(\mu(y))|dy\right\|_{\mathbf{L}_{x}^{2}(\mathbb{R}_{+})}\right) \\
&\leqslant Ct\left(\left\|\langle xt\rangle^{-1-2\nu}\right\|_{\mathbf{L}_{x}^{1}(\mathbb{R})} + \left\||xt|^{2\nu-1}\langle xt\rangle^{-1-2\nu}\right\|_{\mathbf{L}_{x}^{1}(\mathbb{R})}\right)\|\phi\|_{\mathbf{L}_{\eta}^{2}(\mathbb{R}_{+})}^{2} \\
&\leqslant C\|\phi\|_{\mathbf{L}_{\eta}^{2}(\mathbb{R}_{+})}^{2},
\end{aligned}$$

since $\nu > 0$. Lemma 5.1 is proved.

In the next lemma we estimate the commutator $[h, \mathscr{V}^*]$. We denote $q_1(\xi,\eta) = \frac{h(\xi)-h(\eta)}{\xi-\eta}$, $q_2(\xi,\eta) = \eta\partial_\eta q_1(\xi,\eta)$ and $\widetilde{q_l}(\xi,\eta) = q_l(\xi,\eta) - \langle\xi\rangle^{-2} q_l(0,\eta)$.

LEMMA 5.2. *Assume that* $\left|(\xi\partial_\xi)^k \widetilde{q_l}(\xi,\eta)\right| \leqslant C\{\xi\}\langle\xi\rangle^{-1}$ *for all* $\xi,\eta > 0$, $k = 0,1,2$, $l = 1,2$. *Also suppose that* $|q_l(0,\eta)| \leqslant C$. *Then the estimate*

$$\|[h,\mathscr{V}^*]\,\theta(\eta)\phi(\eta)\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C\left\||\Lambda''|^{\frac{1}{2}}\mathscr{A}_0\phi\right\|_{\mathbf{L}^2(\mathbb{R}_+)} + Ct^{-1}\left\||\eta|^{-\frac{3}{2}}\phi(\eta)\right\|_{\mathbf{L}^2(\mathbb{R}_+)}$$

is true for all $t \geqslant 1$.

Proof. Since $\partial_\eta S(\xi,\eta) = -|\Lambda''(\eta)|(\xi-\eta)$, integrating by parts we obtain $it[h,\mathscr{V}_1^*]\phi = \sum_{k=1}^4 I_k$, where

$$I_1 = \langle\xi\rangle^{-2}\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}_+} e^{itS(\xi,\eta)} q_1(0,\eta)\,\partial_\eta(\theta(\eta)\phi(\eta))\,d\eta,$$

$$I_2 = \langle\xi\rangle^{-2}\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}_+} e^{itS(\xi,\eta)} q_2(0,\eta)\frac{\phi(\eta)}{\eta}d\eta,$$

$$I_3 = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}_+} e^{itS(\xi,\eta)}\widetilde{q_1}(\xi,\eta)\,\partial_\eta(\theta(\eta)\phi(\eta))\,d\eta,$$

$$I_4 = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}_+} e^{itS(\xi,\eta)}\widetilde{q_2}(\xi,\eta)\frac{\phi(\eta)}{\eta}d\eta.$$

Using inequality $\|\mathscr{V}^*\theta(\eta)\phi\|_{\mathbf{L}^2(\mathbb{R}_+)} = \left\||\Lambda''|^{\frac{1}{2}}\phi\right\|_{\mathbf{L}^2(\mathbb{R}_+)}$, we get

$$\|I_1\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C\left\||\Lambda''|^{\frac{1}{2}}t\mathscr{A}_0\phi\right\|_{\mathbf{L}^2(\mathbb{R}_+)} \text{ and } \|I_2\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C\left\||\eta|^{-\frac{3}{2}}\phi(\eta)\right\|_{\mathbf{L}^2(\mathbb{R}_+)}.$$

Next using the assumption of the lemma, we apply Lemma 5.1 with $\nu = 1$ to get $\|I_3\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C\left\||\Lambda''|^{\frac{1}{2}}t\mathscr{A}_0\phi\right\|_{\mathbf{L}^2(\mathbb{R}_+)}$ and $\|I_4\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C\left\||\eta|^{-\frac{3}{2}}\phi(\eta)\right\|_{\mathbf{L}^2(\mathbb{R}_+)}$. Lemma 5.2 is proved.

6. A priori estimates

6.1. Estimates for the nonlinearity

Define the norm

$$\|\phi\|_{\mathbf{Y}} = \left\|\xi^{-1}\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty(\mathbb{R}_+)} + \langle t\rangle^{-\gamma}\|\langle\xi\rangle\phi_\xi\|_{\mathbf{L}^2(\mathbb{R}_+)} + \langle t\rangle^{-\gamma}\|\langle\xi\rangle\phi\|_{\mathbf{L}^2(\mathbb{R}_+)}.$$

LEMMA 6.1. *Assume that* $\|\widehat{\varphi}\|_{\mathbf{Y}} \leqslant \varepsilon$. *Then the asymptotics*

$$t\mathscr{F}\mathscr{U}(-t)u^3 = e^{it\Omega(\xi)}\mathscr{D}_3\frac{1}{i\Lambda''(\xi)}\widehat{\varphi}^3 + \frac{3}{\Lambda''(\xi)}|\widehat{\varphi}|^2\widehat{\varphi} + O\left(\varepsilon^3 t^{\gamma-\frac{1}{22}}\right)$$

is true for all $t \geqslant 1$, $\xi > 0$, *where* $\widehat{\varphi}(t) = \mathscr{F}\mathscr{U}(-t)u(t)$, $\Omega(\xi) = \Lambda(\xi) - 3\Lambda\left(\frac{\xi}{3}\right)$.

Proof. In view of the factorization formula (2.1) we find

$$t\mathscr{F}\mathscr{U}(-t)u^3 = e^{it\Omega}\mathscr{D}_3\mathscr{V}^*(3t)\psi^3 + 3\mathscr{V}^*(t)|\psi|^2\psi + 3\mathscr{D}_{-1}\mathscr{V}^*(-t)|\psi|^2\overline{\psi} + e^{it\Omega}\mathscr{D}_{-3}\mathscr{V}^*(-3t)\overline{\psi}^3,$$

where $\psi = \mathscr{V}\widehat{\varphi}$. By Lemma 3.2 with $\alpha = \frac{3}{4}$, we get

$$\mathscr{V}^*(3t)\psi^3 = A^*_{\frac{3}{4}}(3t)\xi^{\frac{3}{4}}\psi^3 + O\left(t^{-\frac{1}{12}}\left\|\partial_\eta\left(|\eta|^{\frac{3}{4}}\psi^3\right)\right\|_{\mathbf{L}^2(\mathbb{R})}\right).$$

By Lemma 3.1 with $j = 0$, we have

$$\langle\eta\rangle|\psi| \leqslant |\langle\eta\rangle\mathscr{V}_1\widehat{\varphi}| + |\langle\eta\rangle\mathscr{V}_2\widehat{\varphi}| \leqslant C\left\|\xi^{-\frac{1}{2}}\langle\xi\rangle\widehat{\varphi}\right\|_{\mathbf{L}^\infty(\mathbb{R}_+)} + Ct^{\gamma-\frac{1}{22}}\|\widehat{\varphi}\|_{\mathbf{Y}} \leqslant C\varepsilon.$$

for $\eta\in\mathbb{R}$. Next applying Lemma 4.4 with $\alpha = \frac{1}{4},\ \beta = -1,\ j = 0$ we get $\left\||\Lambda''|^{\frac{1}{2}}\{\eta\}^{\frac{1}{4}}\langle\eta\rangle^{-1}\partial_\eta\psi\right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant C\varepsilon t^{\gamma}$. Therefore $\mathscr{V}^*(3t)\psi^3 = A^*_{\frac{3}{4}}(3t)\xi^{\frac{3}{4}}\psi^3 + O\left(\varepsilon^3 t^{\gamma-\frac{1}{12}}\right)$. Again by Lemma 3.1 with $j = 0$, we have $\psi = A_0\widehat{\varphi} + O\left(\varepsilon t^{\gamma-\frac{1}{22}}\right)$. Then using the asymptotics $A_0(t,\xi) = \frac{1}{\sqrt{i\Lambda''(\xi)}} + O\left(t^{\frac{1}{2}}\xi\langle t\xi^3\rangle^{-1}\right)$ and $A^*_{\frac{3}{4}}(3t,\xi)\xi^{\frac{3}{4}} = \sqrt{i\Lambda''(\xi)} + O\left(\xi^2 t^{\frac{1}{2}}\langle t\xi^3\rangle^{-1}\right)$, we get

$$\mathscr{V}^*(3t)\psi^3 = A^*_{\frac{3}{4}}(3t)\xi^{\frac{3}{4}}(A_0\widehat{\varphi})^3 + O\left(\varepsilon^3 t^{\gamma-\frac{1}{22}}\right) = \frac{1}{i\Lambda''(\xi)}\widehat{\varphi}^3 + O\left(\varepsilon^3 t^{\gamma-\frac{1}{22}}\right).$$

In the same manner we find for the second summand

$$\mathscr{V}^*(t)|\psi|^2\psi = A^*_{\frac{3}{4}}(t)\xi^{\frac{3}{4}}|A_0\widehat{\varphi}|^2 A_0\widehat{\varphi} + O\left(\varepsilon^3 t^{\gamma-\frac{1}{22}}\right) = \frac{1}{\Lambda''(\xi)}|\widehat{\varphi}|^2\widehat{\varphi} + O\left(\varepsilon^3 t^{\gamma-\frac{1}{22}}\right).$$

The third and fourth terms are remainders $\mathscr{D}_{-1}\mathscr{V}^*(-t)|\psi|^2\overline{\psi} = O\left(\varepsilon^3 t^{\gamma-\frac{1}{22}}\right)$ and $\mathscr{D}_{-3}\mathscr{V}^*(-3t)\overline{\psi}^3 = O\left(\varepsilon^3 t^{\gamma-\frac{1}{22}}\right)$. Lemma 6.1 is proved.

6.2. Uniform norm

We prove a priori estimate for the uniform norm $\|\widehat{\varphi}\|_{\mathbf{S}_T} = \sup_{t\in[1,T]}\left\|\xi^{-1}\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty}$ uniformly in time $T > 1$. Also we define the norm $\|u\|_{\mathbf{Q}_T} = \sup_{t\in[1,T]} t^{-\gamma}\left(\|\mathscr{J}u(t)\|_{\mathbf{H}^1} + \|u(t)\|_{\mathbf{H}^1}\right)$ with some small $\gamma > 0$.

LEMMA 6.2. *Assume that* $\|u\|_{\mathbf{Q}_T} \leqslant C\varepsilon$ *for any* $T > 1$ *and for sufficiently small* ε. *Then the estimate* $\|\widehat{\varphi}\|_{\mathbf{S}_T} < C\varepsilon$ *is true.*

Proof. By the contradiction we can find the first time $T_1 \in (1,T]$ such that $\|\widehat{\varphi}\|_{\mathbf{S}_{T_1}} = C\varepsilon$. We use equation (2.1) for $\widehat{\varphi} = \mathscr{F}\mathscr{U}(-t)u(t)$. In view of Lemma 6.1, we get

$$\partial_t y = e^{it\Omega}\mathscr{D}_3 \frac{1}{it\Lambda''(\xi)}\widehat{\varphi}^3 + \frac{3i\xi}{t\Lambda''(\xi)\langle\xi\rangle^2}|\widehat{\varphi}|^2 y + O\left(\varepsilon^3 t^{\gamma-\frac{1}{22}-1}\right).$$

where $y(t,\xi) = \xi^{-1}\langle\xi\rangle^2\widehat{\varphi}(t,\xi)$. Choosing $\Psi(t,\xi) = \exp\left(-\frac{3i\xi}{|\Lambda''(\xi)|\langle\xi\rangle^2}\int_1^t |\widehat{\varphi}(\tau,\xi)|^2 \frac{d\tau}{\tau}\right)$, we get

$$\partial_t(y(t,\xi)\Psi(t,\xi)) = \Psi(t,\xi)e^{it\Omega}\mathscr{D}_3\frac{1}{it\Lambda''(\xi)}\widehat{\varphi}^3 + O\left(\varepsilon^3 t^{\gamma-\frac{1}{22}-1}\right).$$

Integration in time yields

$$|y(t,\xi)\Psi(t,\xi)| \leqslant |y(0,\xi)| + C\left|\int_0^t \Psi(\tau,\xi)e^{i\tau\Omega}\mathscr{D}_3\frac{1}{\Lambda''(\xi)}\widehat{\varphi}^3\frac{d\tau}{\tau}\right| + O(\varepsilon^3).$$

Integrating by parts we get $|y(t,\xi)| \leqslant \varepsilon + O(\varepsilon^3) < C\varepsilon$ for $\xi > 0$. Since the solution u is real, we have $\overline{\widehat{\varphi}(t,\xi)} = \widehat{\varphi}(t,-\xi)$. Therefore $\|\widehat{\varphi}\|_{\mathbf{S}_{T_1}} < C\varepsilon$. This is the desired contradiction. Lemma 6.2 is proved.

6.3. $\mathbf{L}^2$ - norm

We consider a-priori estimates in $\mathbf{L}^2$ - norm uniformly in time.

LEMMA 6.3. *Suppose that* $\|\widehat{\varphi}\|_{\mathbf{S}_T} \leqslant C\varepsilon$ *for any* $T > 1$ *and for sufficiently small* ε. *Then the estimate* $\|u\|_{\mathbf{Q}_T} < C\varepsilon$ *is true.*

Proof. Arguing by contradiction, we can find the first time $T_1 \in (1,T]$ such that $\|u\|_{\mathbf{Q}_{T_1}} = C\varepsilon$. By the energy method we get $\frac{d}{dt}\|u\|_{\mathbf{H}^1} \leqslant C\|uu_x\|_{\mathbf{L}^\infty}\|u\|_{\mathbf{H}^1}$. Via Lemma 3.1 we have $|\psi_k| \leqslant C\varepsilon$, therefore $\left\|\partial_x^k u\right\|_{\mathbf{L}^\infty} = \|\mathscr{D}_t M\mathscr{B}\psi_k\|_{\mathbf{L}^\infty}$ $\leqslant Ct^{-\frac{1}{2}}\|\psi_k\|_{\mathbf{L}^\infty} \leqslant C\varepsilon t^{-\frac{1}{2}}$, $k = 0,1$, from which it follows that $\frac{d}{dt}\|u\|_{\mathbf{H}^3} \leqslant C\varepsilon^2 t^{-1}\|u\|_{\mathbf{H}^1}$. Integration in time yields $\|u\|_{\mathbf{H}^1} \leqslant \varepsilon + C\varepsilon^3 t^\gamma$. Note that $\|\mathscr{J}u\|_{\mathbf{H}^1} = \left\|\langle\xi\rangle\widehat{\varphi_\xi}\right\|_{\mathbf{L}^2(\mathbb{R}_+)}$. To estimate the norm $\left\|\langle\xi\rangle\widehat{\varphi_\xi}\right\|_{\mathbf{L}^2(\mathbb{R}_+)}$ we differentiate equation (2.1) to get $\partial_t \langle\xi\rangle \widehat{\varphi_\xi} = \sum_{k=1}^{10} I_k$, where

$$I_1 = \frac{1}{t\langle\xi\rangle} e^{it\Omega} \mathscr{D}_3 i\xi \partial_\xi \mathscr{V}^*(3t)\psi^3,\ I_2 = \frac{3}{t\langle\xi\rangle} i\xi\partial_\xi \mathscr{V}^*(t)|\psi|^2\psi,$$
$$I_3 = \frac{3}{t\langle\xi\rangle}\mathscr{D}_{-1} i\xi\partial_\xi \mathscr{V}^*(-t)|\psi|^2\overline{\psi},\ I_4 = \frac{1}{t\langle\xi\rangle}e^{it\Omega}\mathscr{D}_{-3} i\xi\partial_\xi\mathscr{V}^*(-3t)\overline{\psi}^3,$$
$$I_5 = \left(\langle\xi\rangle\partial_\xi \frac{i\xi}{\langle\xi\rangle^2}\right)\mathscr{F}\mathscr{U}(-t)u^3,$$
$$I_6 = -\frac{\xi\Omega'}{\langle\xi\rangle}e^{it\Omega}\left(\mathscr{D}_3\mathscr{V}^*(3t)\psi^3 + \mathscr{D}_{-3}\mathscr{V}^*(-3t)\overline{\psi}^3\right).$$

Consider the first term $I_1 = \frac{1}{t\langle\xi\rangle} e^{it\Omega}\mathscr{D}_3 i\xi\partial_\xi\mathscr{V}^*(3t)\psi^3$. Denote $\omega_j = \mathscr{V}_1 (i\xi)^j \widehat{\varphi}$ and $r_j = \mathscr{V}_2 (i\xi)^j \widehat{\varphi}$, then we substitute $\psi = \omega + r$. Hence $\psi^3 = \left(r^2 + 3\omega^2 + 3r\omega\right) r + \omega^3$. Then

$$I_1 = \frac{1}{t\langle\xi\rangle}e^{it\Omega}\mathscr{D}_3 i\xi\partial_\xi\mathscr{V}^*(3t)\left(r^2+3\omega^2+3r\omega\right)r$$
$$+\frac{1}{t\langle\xi\rangle}e^{it\Omega}\mathscr{D}_3 i\xi\partial_\xi\mathscr{V}^*(3t)\omega^3 = I_{11}+I_{12}.$$

In view of the identity $i\xi\mathscr{V}^*\phi = \mathscr{V}^*\mathscr{A}_1\phi$ we find

$$t^{-1} i\xi\partial_\xi\mathscr{V}^* = -\xi\Lambda'(\xi)\mathscr{V}^* + \xi\mathscr{V}^*\frac{\eta}{|\eta|}\Lambda'(\eta)$$
$$= \frac{\Lambda'(\xi)}{(i\xi)^2}\mathscr{V}^*\mathscr{A}_1^3 - \mathscr{V}^*\left[\mathscr{A}_1, \frac{\eta}{|\eta|}\Lambda'(\eta)\right] - \mathscr{V}^*\frac{\eta}{|\eta|}\Lambda'(\eta)\mathscr{A}_1.$$

Hence $I_{11} = I_{13} + I_{14} + I_{15}$, where

$$I_{13} = \frac{i}{\langle\xi\rangle}e^{it\Omega}\mathscr{D}_3\frac{\Lambda'(\xi)}{(i\xi)^2}\mathscr{V}^*(3t)\mathscr{A}_1^3(3t)\left(r^2+3\omega^2+3r\omega\right)r$$
$$I_{14} = -\frac{i}{\langle\xi\rangle}e^{it\Omega}\mathscr{D}_3\mathscr{V}^*(3t)\left[\mathscr{A}_1(3t), \frac{\eta}{|\eta|}\Lambda'(\eta)\right]\left(r^2+3\omega^2+3r\omega\right)r$$
$$I_{15} = -\frac{i}{\langle\xi\rangle}e^{it\Omega}\mathscr{D}_3\mathscr{V}^*(3t)\frac{\eta}{|\eta|}\Lambda'(\eta)\mathscr{A}_1(3t)\left(r^2+3\omega^2+3r\omega\right)r.$$

We have

$$I_{13} = \frac{i}{\langle\xi\rangle}e^{it\Omega}\mathscr{D}_3\frac{\Lambda'(\xi)}{(i\xi)^2}\mathscr{V}^*(3t)(3((r+2\omega)\omega_3 + 6\omega_1\omega_2)r$$
$$+18\left(\omega_1^2+\psi\omega_2\right)r_1 + 18\psi\omega_1 r_2 + 3\psi^2 r_3 + 18\omega_1 r_1^2 + 18(r+\omega)r_1 r_2 + 6r_1^3)$$
$$= \frac{i}{\langle\xi\rangle}e^{it\Omega}\mathscr{D}_3\frac{\Lambda'(\xi)}{(i\xi)^2}(I_{16}+I_{17}+I_{18}),$$

with $I_{16} = 3\sum_{j=0}^{3} \mathscr{V}^*(3t)\left(B_{3-j}r_j\right)$, $I_{17} = 18\mathscr{V}^*(3t)\left(\omega_1 r_1^2 + \psi r_1 r_2\right)$,

$I_{18} = 6\mathscr{V}^*(3t)\, r_1^3$, and $B_0 = \psi^2$, $B_1 = 6\psi\omega_1$, $B_2 = 6\left(\omega_1^2 + \psi\omega_2\right)$ and $B_3 = (\psi + \omega)\,\omega_3 + 6\omega_1\omega_2$. Also we represent

$$I_{14} = -\frac{i}{t\langle\xi\rangle} e^{it\Omega} \mathscr{D}_3 \mathscr{V}^*(3t)\left(r^2 + 3\omega^2 + 3r\omega\right) r$$

and

$$I_{15} = -\frac{3i}{\langle\xi\rangle} e^{it\Omega} \mathscr{D}_3 \mathscr{V}^*(3t)\, \frac{\eta}{|\eta|} \Lambda'(\eta)\left(\psi^2 r_1 + (\omega + \psi)\,\omega_1 r\right).$$

We have $\|I_{16}\|_{\mathbf{L}^2} \leqslant C\sum_{j=0}^{3} \left\| |\Lambda''|^{\frac{1}{2}} \left|B_{3-j}\right| r_j \right\|_{\mathbf{L}^2}$. Next using Lemma 3.1 we find

$\sum_{j=0}^{3} \left\| \langle\eta\rangle^{\frac{7}{2}} \eta^{-j} B_j \right\|_{\mathbf{L}^\infty(\mathbb{R}_+)} \leqslant C\|\widehat{\varphi}\|_{\mathbf{Y}}^2$, $j = 0,1,2,3$, then by Lemma 4.3 with $\alpha = 3 - j$, $\beta = -j$, $0 \leqslant j \leqslant 3$, we get

$$\|I_{16}\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C\|\widehat{\varphi}\|_{\mathbf{Y}}^2 \sum_{j=0}^{3} \left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{3-j} \langle\eta\rangle^{-j} r_j \right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant Ct^{\gamma-1}\|\widehat{\varphi}\|_{\mathbf{Y}}^3.$$

Next by Lemma 3.1 we estimate $\|\langle\eta\rangle r_1\|_{\mathbf{L}^\infty(\mathbb{R})} \leqslant Ct^{\gamma-\frac{5}{14}}\|\widehat{\varphi}\|_{\mathbf{Y}}$, $\left\|\{\eta\}^{-1}\omega_1\right\|_{\mathbf{L}^\infty(\mathbb{R})} \leqslant C\|\widehat{\varphi}\|_{\mathbf{Y}}$, $\|\langle\eta\rangle\psi\|_{\mathbf{L}^\infty(\mathbb{R})} \leqslant C\|\widehat{\varphi}\|_{\mathbf{Y}}$, then by Lemma 4.3 with $\alpha = 2 - j$, $\beta = -j$, $0 \leqslant j \leqslant 2$, we find

$$\begin{aligned}
&\|I_{17}\|_{\mathbf{L}^2(\mathbb{R}_+)} = C\left\|\mathscr{V}^*(3t)\left(\omega_1 r_1^2 + \psi r_1 r_2\right)\right\|_{\mathbf{L}^2(\mathbb{R}_+)} \\
&\leqslant C\|\langle\eta\rangle r_1\|_{\mathbf{L}^\infty(\mathbb{R})}\left(\left\|\langle\eta\rangle\{\eta\}^{-1}\omega_1\right\|_{\mathbf{L}^\infty(\mathbb{R})} + \|\langle\eta\rangle\psi\|_{\mathbf{L}^\infty(\mathbb{R})}\right) \\
&\times \sum_{j=1}^{2}\left\||\Lambda''|^{\frac{1}{2}}\{\eta\}^{2-j}\langle\eta\rangle^{-j} r_j\right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant Ct^{2\gamma-\frac{5}{14}-1}\log\langle t\rangle\|\widehat{\varphi}\|_{\mathbf{Y}}^3 \leqslant Ct^{-1}\|\widehat{\varphi}\|_{\mathbf{Y}}^3.
\end{aligned}$$

Also by estimate (3.3) with $p = 2 + \gamma$, then by Lemma 4.3 with $\alpha = \frac{p-2}{2p} > 0$, $\beta = 0$, $j = 1$, we obtain

$$\begin{aligned}
&\left\|\langle\xi\rangle^{-1} I_{18}\right\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C\left\|\mathscr{V}^*(3t)\, r_1^3\right\|_{\mathbf{L}^p(\mathbb{R}_+)} \leqslant C|t|^{\frac{1}{2}-\frac{1}{p}}\left\||\eta|^{1-\frac{1}{p}} r_1^3\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R})} \\
&\leqslant C|t|^{\frac{p-2}{2p}}\|\langle\eta\rangle r_1\|_{\mathbf{L}^\infty(\mathbb{R})}^2\left\|\langle\eta\rangle^{-1}\right\|_{\mathbf{L}^{\frac{2p}{p-2}}(\mathbb{R})}\left\||\Lambda''|^{\frac{1}{2}}\{\eta\}^{\frac{p-2}{2p}} r_1\right\|_{\mathbf{L}^2(\mathbb{R})} \\
&\leqslant Ct^{-\frac{5}{7}-\frac{1}{2}+4\gamma}\|\widehat{\varphi}\|_{\mathbf{Y}}^3 \leqslant Ct^{-1}\|\widehat{\varphi}\|_{\mathbf{Y}}^3.
\end{aligned}$$

Next we estimate

$$\|I_{14}\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant Ct^{-1}\left(\|\omega\|_{\mathbf{L}^\infty(\mathbb{R})}^2 + \|r\|_{\mathbf{L}^\infty(\mathbb{R})}^2\right)\left\||\Lambda''|^{\frac{1}{2}} r\right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant Ct^{-1}\|\widehat{\varphi}\|_{\mathbf{Y}}^3$$

and

$$\begin{aligned}\|I_{15}\|_{\mathbf{L}^2(\mathbb{R}_+)} &\leqslant C\left\|\mathscr{V}^*(3t)\frac{\eta}{|\eta|}\Lambda'(\eta)\left(\psi^2 r_1+(\omega+\psi)\omega_1 r\right)\right\|_{\mathbf{L}^2(\mathbb{R}_+)}\\ &\leqslant C\left(\|\langle\eta\rangle\omega\|^2_{\mathbf{L}^\infty(\mathbb{R})}+\|\langle\eta\rangle r\|^2_{\mathbf{L}^\infty(\mathbb{R})}\right)\sum_{j=0}^{1}\left\||\Lambda''|^{\frac{1}{2}}\{\eta\}^{3-j}\langle\eta\rangle^{2-j}r_j\right\|_{\mathbf{L}^2(\mathbb{R})}\\ &\leqslant Ct^{\gamma-1}\|\widehat{\varphi}\|^3_{\mathbf{Y}}.\end{aligned}$$

Thus we get $\|I_{11}\|_{\mathbf{L}^2}\leqslant Ct^{\gamma-1}\|\widehat{\varphi}\|^3_{\mathbf{Y}}$. Next we consider $I_{12}=\frac{3}{t\langle\xi\rangle}e^{it\Omega}\mathscr{D}_3\partial_\xi\mathscr{V}^*(3t)\omega^2\omega_1+\frac{i}{t\langle\xi\rangle}e^{it\Omega}\mathscr{D}_3\mathscr{V}^*(3t)\omega^3$. We have $\partial_\xi\mathscr{V}^*=\left[\frac{\xi}{|\xi|}\Lambda',\mathscr{V}^*\right]$. Also we have the identity $\mathscr{A}_1^l=\sum_{j=0}^{l-1}(i\eta)^j\mathscr{A}_0\mathscr{A}_1^{l-1-j}+(i\eta)^l$, therefore

$$\begin{aligned}\left[(i\xi)^l h,\mathscr{V}^*\right]&=h(\xi)\mathscr{V}^*\mathscr{A}_1^l-\mathscr{V}^*h(\eta)(i\eta)^l\\ &=\sum_{j=0}^{l-1}h(\xi)\mathscr{V}^*\theta(\eta)(i\eta)^j\mathscr{A}_0\mathscr{A}_1^{l-1-j}+[h,\mathscr{V}^*]\theta(\eta)(i\eta)^l\\ &\quad+h(\xi)\mathscr{V}^*(1-\theta(\eta))\mathscr{A}_1^l-\mathscr{V}^*(1-\theta(\eta))h(\eta)(i\eta)^l.\end{aligned}\tag{6.1}$$

We take $h(\xi)=\frac{\Lambda'(\xi)}{(i\xi)^2},\ l=2,$ then

$$\begin{aligned}\partial_\xi\mathscr{V}^*&=\sum_{j=0}^{1}h(\xi)\mathscr{V}^*\theta(\eta)(i\eta)^j\mathscr{A}_0\mathscr{A}_1^{1-j}+[h,\mathscr{V}^*]\theta(\eta)(i\eta)^2\\ &\quad+h(\xi)\mathscr{V}^*(1-\theta(\eta))\mathscr{A}_1^2-\mathscr{V}^*(1-\theta(\eta))h(\eta)(i\eta)^2.\end{aligned}$$

Applying Lemma 5.2 with $h(\xi)=\frac{\Lambda'(\xi)}{\xi^2}$ we get the estimate

$$\begin{aligned}\left\|\partial_\xi\mathscr{V}^*(3t)\omega^2\omega_1\right\|_{\mathbf{L}^2(\mathbb{R}_+)}&\leqslant Ct^{-1}\left\||\Lambda''|^{\frac{1}{2}}\{\eta\}^{-1}\langle\eta\rangle^{-1}\partial_\eta\left(\omega^2\omega_2+2\omega\omega_1^2\right)\right\|_{\mathbf{L}^2(\mathbb{R}_+)}\\ &+Ct^{-1}\left\||\Lambda''|^{\frac{1}{2}}\langle\eta\rangle\partial_\eta\omega^2\omega_1\right\|_{\mathbf{L}^2(\mathbb{R}_+)}+Ct^{-1}\left\|\eta^{\frac{1}{2}}\omega^2\omega_1\right\|_{\mathbf{L}^2(\mathbb{R}_+)}\\ &+C\left\||\Lambda''|^{\frac{1}{2}}\left(\omega^2\omega_3+\omega\omega_1\omega_2+\omega_1^3+\eta^2\omega^2\omega_1\right)\right\|_{\mathbf{L}^2(\mathbb{R}_-)}.\end{aligned}$$

Next applying Lemma 4.4 with $\alpha=1-j,\ \beta=-j,\ j\geqslant 0$ we get $\left\||\Lambda''|^{\frac{1}{2}}\{\eta\}^{1-j}\langle\eta\rangle^{-j}\partial_\eta\omega_j\right\|_{\mathbf{L}^2(\mathbb{R}_+)}\leqslant Ct^\gamma\|\widehat{\varphi}\|_{\mathbf{Y}}$ and by Lemma 3.1 with $k=1$ we find $\left\|\langle\eta\rangle^{\frac{5}{2}-j}\{\eta\}^{1-j}\omega_j\right\|_{\mathbf{L}^\infty(\mathbb{R}_-)}\leqslant Ct^{\gamma-\frac{5}{14}}\|\widehat{\varphi}\|_{\mathbf{Y}}$ for $j\geqslant 0$. Hence $\|I_{12}\|_{\mathbf{L}^2}\leqslant Ct^{\gamma-1}\|\widehat{\varphi}\|^3_{\mathbf{Y}}$ and therefore $\|I_1\|_{\mathbf{L}^2}\leqslant Ct^{\gamma-1}\|\widehat{\varphi}\|^3_{\mathbf{Y}}$. In the same manner we estimate $I_2,\ I_3,\ I_4$ and I_5.

Consider now the term I_6. Again we substitute $\psi^3 = \left(r^2+3\omega^2+3r\omega\right)r+\omega^3$, then $I_6 = I_{19}+I_{20}$, where

$$I_{19} = -\frac{27i\Omega'}{\xi^2\langle\xi\rangle}e^{it\Omega}\mathscr{D}_3(i\xi)^3\mathscr{V}^*(3t)\left(r^2+3\omega^2+3r\omega\right)r \\ +\frac{27i\Omega'}{\xi^2\langle\xi\rangle}e^{it\Omega}\mathscr{D}_{-3}(i\xi)^3\mathscr{V}^*(-3t)\overline{(r^2+3\omega^2+3r\omega)r},$$

and $I_{20} = -\frac{\xi\Omega'}{\langle\xi\rangle}e^{it\Omega}\Phi$, where $\Phi = \mathscr{D}_3\mathscr{V}^*(3t)\omega^3+\mathscr{D}_{-3}\mathscr{V}^*(-3t)\overline{\omega}^3$. As above in the estimate of I_{13}, we obtain $\|I_{19}\|_{\mathbf{L}^2} \leqslant Ct^{\gamma-1}\|\widehat{\varphi}\|_{\mathbf{Y}}^3$. Next we represent

$$I_{20} = -\partial_t\left(e^{it\Omega}\frac{\xi\Omega'(\xi)}{\langle\xi\rangle i\Omega(\xi)}\Phi\right)+e^{it\Omega}\frac{\xi\Omega'(\xi)}{\langle\xi\rangle i\Omega(\xi)}\partial_t\Phi.$$

To estimate the derivative $\partial_t\mathscr{V}^*\omega^3$, we apply the identity

$$\partial_t\mathscr{V}^* = \frac{1}{2t}\mathscr{V}^*+\left[\frac{i\xi}{|\xi|}\Lambda,\mathscr{V}^*\right]+[i\xi,\mathscr{V}^*]\frac{\eta}{|\eta|}\Lambda'(\eta).$$

We take $h(\xi) = \frac{\Lambda(\xi)}{(i\xi)^3}$ then by (6.1) take $h(\xi) = \frac{\Lambda(\xi)}{(i\xi)^3}$ and $l = 3$

$$\left[(i\xi)^3h,\mathscr{V}^*\right] = \sum_{j=0}^{2}h(\xi)\mathscr{V}^*\theta(\eta)(i\eta)^j\mathscr{A}_0\mathscr{A}_1^{2-j}+[h(\xi),\mathscr{V}^*]\theta(\eta)(i\eta)^3 \\ +h(\xi)\mathscr{V}^*(1-\theta(\eta))\mathscr{A}_1^3-\mathscr{V}^*(1-\theta(\eta))h(\eta)(i\eta)^3.$$

Also we have

$$[i\xi,\mathscr{V}^*]\frac{\eta}{|\eta|}\Lambda'(\eta) = \mathscr{V}^*\theta(\eta)\mathscr{A}_0\frac{\eta}{|\eta|}\Lambda'(\eta) \\ +\mathscr{V}^*(1-\theta(\eta))\mathscr{A}_1\frac{\eta}{|\eta|}\Lambda'(\eta)-\mathscr{V}^*(1-\theta(\eta))i\eta\frac{\eta}{|\eta|}\Lambda'(\eta).$$

Then applying Lemma 5.2 with $h(\xi) = \frac{\Lambda(\xi)}{(i\xi)^3}$ we get the estimate

$$\left\|\partial_t\mathscr{V}^*(3t)\omega^3\right\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant Ct^{-1}\left\|\left|\Lambda''\right|^{\frac12}\langle\eta\rangle\omega^3\right\|_{\mathbf{L}^2(\mathbb{R})} \\ +C\sum_{j=0}^{2}\left\|\left|\Lambda''\right|^{\frac12}\eta^j\langle\eta\rangle\mathscr{A}_0\mathscr{A}_1^{2-j}\omega^3\right\|_{\mathbf{L}^2(\mathbb{R}_+)}+C\sum_{j=0,3}\left\|\left|\Lambda''\right|^{\frac12}\eta^{3-j}\mathscr{A}_1^j\omega^3\right\|_{\mathbf{L}^2(\mathbb{R}_-)}.$$

Then as in the estimate of I_{12} applying Lemma 4.4 with $\alpha = 1-j$, $\beta = -j$, $j \geqslant 0$ and Lemma 3.1 with $k = 1$ we get $\left\|(\partial_t\mathscr{V}^*)(3t)\omega^3\right\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant Ct^{\gamma-1}\|\widehat{\varphi}\|_{\mathbf{Y}}^3$. Also by equation (2.1) we find $\|\partial_t\widehat{\varphi}\|_{\mathbf{L}^2} \leqslant C\varepsilon^3t^{\gamma-1}$ and by the estimate of Lemma 4.2 with

$\rho = 3,\ j = 0,\ \alpha = 0,\ \beta = -2$ we obtain

$$\begin{aligned}\left\|\mathscr{V}^*\partial_t\omega^3\right\|_{\mathbf{L}^2(\mathbb{R}_+)} &\leqslant C\|\widehat{\varphi}\|_{\mathbf{Y}}^2\left(\left\||\Lambda''|^{\frac{1}{2}}\langle\eta\rangle^{-2}\partial_t\mathscr{V}_1\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbb{R})}+\left\||\Lambda''|^{\frac{1}{2}}\mathscr{V}_1\partial_t\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbb{R})}\right)\\ &\leqslant Ct^{\gamma-1}\|\widehat{\varphi}\|_{\mathbf{Y}}^3.\end{aligned}$$

Hence $\left\|\partial_t\left(\mathscr{V}^*(3t)\,\omega^3\right)\right\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C\varepsilon^3 t^{\gamma-1}$. Therefore we get $\partial_t\left\|\langle\xi\rangle\widehat{\varphi_\xi}-e^{it\Omega}\Phi\right\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C\varepsilon^3 t^{\gamma-1}$. We have $\|\Phi\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant C\left\||\Lambda''|^{\frac{1}{2}}\omega^3\right\|_{\mathbf{L}^2(\mathbb{R})} \leqslant C\varepsilon^3 t^{\gamma}$. Then the integration in time of the above equation yields $\left\|\langle\xi\rangle\widehat{\varphi_\xi}-e^{it\Omega}\Phi\right\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant \varepsilon + C\varepsilon^3 t^{\gamma}$. Hence $\left\|\langle\xi\rangle\widehat{\varphi_\xi}\right\|_{\mathbf{L}^2(\mathbb{R}_+)} \leqslant \varepsilon + C\varepsilon^3 t^{\gamma}$. Consequently $\|\mathscr{J}u\|_{\mathbf{H}^1} < C\varepsilon t^{\gamma}$. This is the desired contradiction. Lemma 6.3 is proved.

7. Proof of Theorem 1.1

By Lemma 6.2 and Lemma 6.3, we see that the a priori estimate $\|u\|_{\mathbf{X}_T} \leqslant C\varepsilon$ is true for all $T > 0$. Therefore the global existence of solutions of the Cauchy problem (1.1) follows by a standard continuation argument.

Now we turn to the proof of the asymptotic formula (1.2) for the solutions u of the Cauchy problem (1.1). We need to compute the asymptotics of the function $\widehat{\varphi}(t,\xi)$. As in the proof of Lemma 6.2 we get

$$\partial_t\left(y(t,\xi)\Psi(t,\xi)\right) = \Psi(t,\xi)e^{it\Omega}\langle\xi\rangle^3\mathscr{D}_3\frac{1}{it\Lambda''(\xi)}\widehat{\varphi}^3 + O\left(\varepsilon^3 t^{\gamma-\frac{23}{22}}\right).$$

Integrating by parts implies $|z(t,\xi)-z(s,\xi)| \leqslant C\varepsilon^3 s^{-\frac{1}{22}+\gamma}$ for all $t > s > 0$, where $z(t,\xi) = y(t,\xi)\Psi(t,\xi)$. Therefore there exists a unique final state $z_+ \in \mathbf{L}^\infty$ such that $\|z(t)-z_+\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{-\frac{1}{22}+\gamma}$ for all $t > 0$. We write

$$\int_1^t|\widehat{\varphi}(\tau,\xi)|^2\frac{d\tau}{\tau} = \xi^2\langle\xi\rangle^{-4}\int_1^t|z(\tau,\xi)|^2\frac{d\tau}{\tau} = \xi^2\langle\xi\rangle^{-4}|z_+|^2\log t + \Phi_1(t).$$

We have

$$\begin{aligned}&\|\Phi_1(t)-\Phi_1(s)\|_{\mathbf{L}^\infty} = \xi^2\langle\xi\rangle^{-4}\int_s^t\left(|z(\tau)|^2-|z(t)|^2\right)\frac{d\tau}{\tau}\\ &+\xi^2\langle\xi\rangle^{-4}\left(|z(t)|^2-|z_+|^2\right)\log\frac{t}{s} \leqslant C\varepsilon^3 s^{-\delta}\end{aligned}$$

for all $t > s > 0$. Hence there exists a unique real-valued function $\Phi_+ \in \mathbf{L}^\infty$ such that $\|\Phi(t)-\Phi_+\|_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 t^{-\frac{1}{22}+\gamma}$ for all $t > 0$. Hence

$$\left\|\Psi(t,\xi)-\exp\left(\frac{3i\xi^3|z_+|^2}{|\Lambda''(\xi)|\langle\xi\rangle^4}\log t+\frac{3i\xi}{|\Lambda''(\xi)|}\Phi_+\right)\right\|_{\mathbf{L}^\infty} \leqslant Ct^{-\frac{1}{22}+\gamma}$$

for all $t > 0$. Thus we get the large time asymptotics

$$\left\|\widehat{\varphi}(t,\xi)-\xi\langle\xi\rangle^{-2}z_{+}\Psi(t,\xi)\right\|_{\mathbf{L}^{\infty}}=\|z(t)-z_{+}\|_{\mathbf{L}^{\infty}}\leqslant Ct^{-\frac{1}{22}+\gamma} \text{ and}$$

$$\left\|z_{+}\Psi(t,\xi)-z_{+}\exp\left(\frac{3i\xi^{3}|z_{+}|^{2}}{|\Lambda''(\xi)|\langle\xi\rangle^{4}}\log t+\frac{3i\xi}{|\Lambda''(\xi)|}\Phi_{+}\right)\right\|_{\mathbf{L}^{\infty}}\leqslant Ct^{-\frac{1}{22}+\gamma}.$$

Therefore we obtain the estimate

$$\left\|\widehat{\varphi}(t,\xi)-W_{+}\exp\left(\frac{3i\xi}{|\Lambda''(\xi)|}|W_{+}|^{2}\log t\right)\right\|_{\mathbf{L}^{\infty}}\leqslant Ct^{-\frac{1}{22}+\gamma}$$

with $W_{+}=\xi\langle\xi\rangle^{-2}z_{+}\exp\left(\frac{3i\xi}{|\Lambda''(\xi)|}\Phi_{+}\right)$. Using the factorization $u(t)=2\mathrm{Re}\mathscr{D}_{t}M\mathscr{B}\mathscr{V}\widehat{\varphi}(t)$, by Lemma 3.1 we find

$$\begin{aligned}u(t)&=2\mathrm{Re}\mathscr{D}_{t}M\mathscr{B}\mathscr{V}\widehat{\varphi}(t)=2\mathrm{Re}\mathscr{D}_{t}M\mathscr{B}A\widehat{\varphi}(t)+O\left(t^{-\frac{1}{22}+\gamma}\right)\\&=2\mathrm{Re}\mathscr{D}_{t}M\mathscr{B}\frac{\theta(\xi)}{\sqrt{i\Lambda''(\xi)}}W_{+}\exp\left(\frac{3i\xi}{|\Lambda''(\xi)|}|W_{+}|^{2}\log t\right)+O\left(t^{-\frac{1}{22}+\gamma}\right).\end{aligned}$$

This completes the proof of asymptotics (1.2). Theorem 1.1 is proved.

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REFERENCES

[1] L. ABDELOUHAB, *Nonlocal dispersive equations in weighted Sobolev spaces.* Differential Integral Equations **5**, 2 (1992), 307–338.

[2] L. ABDELOUHAB, J. BONA, M. FELLAND AND J. SAUT, *Nonlocal Models for Nonlinear Dispersive Waves,* Phys. D, **40** (1989), 360–392.

[3] J. P. ALBERT, J. BONA AND D. B. HENRY, *Sufficient Conditions for Stability of Solitary-Wave solutions of model Equation for Long-Waves,* Phys. **240** (1987), 343, 366.

[4] A. B. AL'SHIN, M. O. KORPUSOV AND A. G. SVESHNIKOV, *Blow-up in nonlinear Sobolev type equations.* De Gruyter Series in Nonlinear Analysis and Applications, **15**. Walter de Gruyter & Co., Berlin, 2011. xii+648 pp.

[5] T. B. BENJAMIN, J. L. BONA, AND MAHONY, *Model equations for long waves in nonlinear dispersive systems,* Philos. Trans. Royal Soc. London, Ser. A, **272** (1972), pp. 47–78.

[6] M. V. FEDORYUK, *Asymptotic Methods in Analysis*, in: Analysis. I. Integral representations and asymptotic methods. Encyclopaedia of Mathematical Sciences, **13**. Springer-Verlag, Berlin, 1989. vi+238 pp.

[7] N. HAYASHI, *Global existence of small solutions to quadratic nonlinear Schrödinger equations,* Commun. P.D.E., **18** (1993), pp. 1109–1124.

[8] N. HAYASHI AND P. I. NAUMKIN, *The initial value problem for the cubic nonlinear Klein-Gordon equation,* Zeitschrift fur Angewandte Mathematik und Physik, **59**, 6 (2008), 1002–1028.

[9] N. HAYASHI AND P. NAUMKIN, *Factorization technique for the fourth-order nonlinear Schrödinger equation.* Z. Angew. Math. Phys. **66**, 5 (2015), 2343–2377.

[10] N. HAYASHI AND P. NAUMKIN, *On the inhomogeneous fourth-order nonlinear Schrödinger equation.* J. Math. Phys. **56**, 9 (2015), 093502, 25 pp.

[11] N. Hayashi and P. I. Naumkin, *Factorization technique for the modified Korteweg–de Vries equation.* SUT J. Math. **52**, 1 (2016), 49–95.

[12] N. Hayashi and T. Ozawa, *Scattering theory in the weighted $L^2(R^n)$ spaces for some Schrödinger equations,* Ann. I.H.P. (Phys. Théor.), **48** (1988), 17–37.

[13] B. Juarez-Campos, E. Kaikina, and H. F. Ruiz-Paredes, *Nonlinear model of quasi-stationary process in crystalline semiconductor.* Differ. Equ. Appl. **9**, 1 (2017), 37–55.

[14] M. O. Korpusov, and E. V. Yushkov, *Local solvability and blow-up for Benjamin-Bona-Mahony-Burgers, Rosenau-Burgers and Korteweg-de Vries-Benjamin-Bona-Mahony equations*, Electron. J. Differential Equations **2014**, No. 69, 16 pp.

[15] I. P. Naumkin, *Sharp asymptotic behavior of solutions for cubic nonlinear Schrödinger equations with a potential,* Journal of Mathematical Physics **57**, 051501 (2016); http://doi.org/10.1063/1.4948743.

[16] I. P. Naumkin, *Initial-boundary value problem for the one dimensional Thirring model,* Journal of Differential equations, **261**, 8 (2016), 4486–4523.

[17] I.P. Naumkin, *Nonlinear Schrödinger equations with exceptional potentials.* J. Differential Equations, **265**, 9 (2018), 4575–4631.

[18] E. M. Stein and R. Shakarchi, *Functional analysis. Introduction to further topics in analysis.* Princeton Lectures in Analysis, 4. Princeton University Press, Princeton, NJ, 2011. xviii+423 pp.

[19] A. G. Sveshnikov, A. B. Al'shin, M. O. Korpusov, and Yu. D. Pletner, *Linear and nonlinear equations of Sobolev type*, (FIZMATLIT, Moscow, 2007), pp. 734.

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