A REMARK FOR SPATIAL ANALYTICITY AROUND STRAINING FLOWS

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Abstract. Time-local existence of unique smooth solutions to the Navier-Stokes equations in the whole space with linearly growing initial data has been established, via smoothing properties of Ornstein-Uhlenbeck semigroup. It has also been shown that the solution is real-analytic in spatial variables around rotating flows. This note is devoted to prove the spatial analyticity for cases of straining flows and shear flows. It is estimated the size of radius of convergence of Taylor series, due to estimates for higher order derivatives and Cauchy-Hadamard theorem.

1. Introduction

We consider the Navier-Stokes equations which describe incompressible, viscous fluid flows in the whole space \( \mathbb{R}^n \) for \( n \in \mathbb{N}, \geq 2 \):

\[
\begin{align*}
\frac{\partial}{\partial t}U - \Delta U + U \cdot \nabla U + \nabla P &= F \quad \text{in } \mathbb{R}^n \times (0,T), \\
\nabla \cdot U &= 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\
U \big|_{t=0} &= U_0 \quad \text{in } \mathbb{R}^n.
\end{align*}
\] (1.1)

Here, \( U = (U^1(x,t), \ldots, U^n(x,t)) \) and \( P = P(x,t) \) stand for the unknown velocity and the unknown pressure at \( x \in \mathbb{R}^n \) and \( t \in (0,T) \), respectively; \( U_0 = (U^1_0(x), \ldots, U^n_0(x)) \) is a given initial velocity, and \( F = (F^1(x,t), \ldots, F^n(x,t)) \) is a given external force. We have used the notation of differentiation; \( \partial_t := \partial / \partial t \), \( \Delta := \sum_{i=1}^n \partial_i^2 \), \( \partial_i := \partial / \partial x_i \) for \( i = 1, \ldots, n \), \( \nabla := (\partial_1, \ldots, \partial_n) \), \( \nabla \cdot U := \sum_{i=1}^n \partial_i U^i \). It is always imposed the compatibility condition, that is, \( \nabla \cdot U_0 = 0 \).

There is huge literature on time-local well-posedness (i.e., existence, uniqueness, smoothness and equi-continuity of solutions) to (1.1); see e.g. [5, 6, 9, 14, 16]. In their results, it is assumed that velocities decay at \( |x| \to \infty \). Dealing with nondecaying velocities, in [8] Giga and his collaborators have established time-local well-posedness in \( BUC \) as well as \( L^\infty \). In fact, one can construct time-local unique smooth mild solutions, when \( U_0 \) is bounded uniformly continuous; see also [2, 3, 15]. For the case of linearly growing velocities, in [12] Hieber and the second author of this note constructed time-local smooth solutions for

\[
U_0(x) = -Mx + u_0(x)
\]
at \( x \in \mathbb{R}^n \) with \( M = (m_{ij})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n} \) and \( u_0 \in L^q(\mathbb{R}^n) \) for \( q \in [n, \infty) \) satisfying the trace-free condition \( \text{tr}M = 0 \) and the divergence-free condition \( \nabla \cdot u_0 = 0 \). It might be supposed that \( U = -Mx \) is a stationary solution to (1.1) with some \( P \) and \( F \), thus we are required to investigate time-evolution of disturbance from the initial disturbance \( u_0 \). In the cases of more general situation, the reader can find the existence results in [11, 21].

In [12] the spatial real analyticity was also proved, provided \( M \) is skew-symmetric. The aim of this note is to show that \( U \) is real analytic in \( x \) for the cases of general \( M \), including straining flows and shear flows.

In \( n = 3 \), typical examples of \( M \) are as follows:

\[
R = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} -b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 2b \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

for \( a, b, c \in \mathbb{R} \), and these sum. Note that \( R \), \( J \) and \( S \) correspond to rotating, straining and shear flows, respectively.

Substituting \( u := U + Mx \), (1.1) are rewritten as

\[
\begin{align*}
\partial_t u - \Delta u + u \cdot \nabla u - Mx \cdot \nabla u - Mu + \nabla p &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T), \\
\nabla \cdot u &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T), \\
\partial_t u|_{t=0} &= u_0 \quad \text{in} \quad \mathbb{R}^n.
\end{align*}
\]

(1.2)

Here, \( p \) is a scalar function satisfying \( \nabla p = \nabla P - F + M^2x \). It is rather easy to prove the existence of weak solutions to (1.2); see e.g. [1, 4]. However, for solving (1.1) by the converse transformation \( U = -Mx + u \), we are forced to construct classical solutions to (1.2). For this purpose, we select a semigroup approach. Hence, (1.2) is formally equivalent to the integral equation

\[
(\text{INT}) \quad u(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A}\mathbb{P}\{u(s) \cdot \nabla u(s) - 2Mu(s)\}ds
\]

with \( u_0 \in L^q(\mathbb{R}^n) \) for \( q \in [n, \infty) \), since \( -A \) generates the Ornstein-Uhlenbeck semigroup \( \{e^{tA}\}_{t \geq 0} \) in \( L^q_\sigma \). Here, we have used the Helmholtz projection \( \mathbb{P} = (\delta_{ij} + R_i R_j)_{1 \leq i,j \leq n} \) onto the solenoidal subspace \( L^q_\sigma \) of the Lebesgue space \( L^q \) for \( q \in (1, \infty) \) associated with Kronecker’s delta \( \delta_{ij} \) and the Riesz transform \( R_i \) defined as \( R_i := \partial_i(-\Delta)^{-1/2} \) for \( i = 1, \ldots, n \) as well as \( Av := \Delta v + Mx \cdot \nabla v - Mv \) with domain \( D(A) := \{v \in W^{2,q} \cap L^q_\sigma; Mx \cdot \nabla v \in L^q\} \). Note that \( A \) and \( \mathbb{P} \) commute, since \( \nabla \cdot Av = 0 \) provided \( \nabla \cdot v = 0 \). The solution \( u \in C([0, T]; L^q_\sigma(\mathbb{R}^n)) \) to the integral equation (INT) is often called a mild solution, so we use the terminology.

The aim of this note is to show the real analyticity of a mild solution \( u \) with respect to spatial variables \( x \), whence it exists. We will establish the \( L^\infty \)-norm estimates of higher order derivatives, and appeal to the Cauchy-Hadamard theorem for estimating the size of radius of convergence of Taylor series. Besides, in [17, 18] Masuda discussed the real analyticity of solutions to (1.1) in \( t \) and \( x \) from a different approach; his proof is based on the implicit function theory. The reader can find recent improvement of his
method in e.g. [7] and references therein. It however looks hard to apply his method into our situation, at least directly. The difficulty comes from the fact that the semigroup \( \{e^{tA}\}_{t \geq 0} \) is not analytic, this means that arguments of the maximal regularity do not work, so it is not clear how to obtain certain a priori estimates.

This note is organized as follows. In section 2 we state the main results. Section 3 is to recall smoothing properties of the Ornstein-Uhlenbeck semigroup and Kahane’s lemma for bilinear estimates. We will give a complete proof of our main results in section 4.

Throughout this note, we denote positive constants by \( C \) the value of which may differ from one occasion to another.

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2. Main Results

This section is devoted to state the main results of this note. We first recall the time-local existence and uniqueness results for mild solutions.

**Theorem 1.** ([12]) Let \( n \geq 2, k \in \mathbb{N}, q \in [n, \infty) \). If \( M \in \mathbb{R}^{n \times n}, \text{tr}M = 0 \) and \( u_0 \in L^q_\sigma(\mathbb{R}^n) \), then there exist \( T_k > 0 \) and a unique mild solution \( u \in C([0, T_k]; L^q_\sigma(\mathbb{R}^n)) \) such that

\[
\frac{1}{t^{k/2}+(q-1/r)n/2} \nabla^k u \in C([0, T_k]; L^r(\mathbb{R}^n)) \quad \text{for} \quad r \in [q, \infty].
\]

To prove this theorem, one may argue by successive approximation as in [9, 14]. When \( n = 2 \), it is easy to obtain a time-global unique mild solution. It seems to be hard to gain a time-global solution even for small initial \( u_0 \in L^q_\sigma(\mathbb{R}^n) \). For the case when \( q = \infty \), because there is a lack of boundedness of \( \mathbb{P} \) in \( L^\infty \), we need some restriction, for example, \( u_0 \in B^0_{\infty, 1} \subset L^\infty \) for dealing with nondecaying data; see [21]. Remark that \( T_k \) must be chosen small for large \( k \in \mathbb{N} \). Indeed, when \( q > n \), by the iteration scheme, we deduce \( T_k \geq C k^{-k} ||u_0||_q^{2-2n/q} \) with some \( C \) depending only on \( n, q \) and \( M \). Nevertheless, the mild solution \( u \) is unique as long as it exists, one can extend the existence time of the mild solution up to \( T_1 \) having bounds for \( k \)-th derivatives. We hence confirm that \( u(t) \in C^k(\mathbb{R}^n) \) for all \( k \in \mathbb{N} \) and \( t \in (0, T_1] \), which means that \( u(t) \in C^\infty(\mathbb{R}^n) \), whence the mild solution exists.

Because the semigroup \( \{e^{tA}\}_{t \geq 0} \) is not analytic, it is impossible to control \( L^r \)-norm of \( \partial_t u \), at least directly. In fact, \( u \notin C^1(0, T_1; L^q) \), that is, \( u \) can not be a strong solution. However, it might be shown that \( u \) is smooth in \( t \), using the notion of weak solutions; see [21]. We finally reach to \( u \in C^\infty(\mathbb{R}^n \times (0, T_1]) \). Therefore, \( u \) is a classical solution to (1.2) associated with

\[
p = \sum_{i,j=1}^n R_i R_j u^i u^j - 2(-\Delta)^{-1} \partial_t m_{ij} u^j.
\]
We thus construct a time-local classical solution to (1.1) by $U = -Mx + u$ and suitable choice of $P$.

In [12], the real analyticity of mild solutions in $x$ is also shown for skew-symmetric $M$. In this note, we relax the condition on $M$ to obtain the spatial analyticity. For making short of this note, we deal with the case when $u_0 \in L^n$, and estimate for $\|\nabla^k u(t)\|_{\infty}$, only. Note that the same assertion holds when $u_0 \in L^q$ for $q \in (n, \infty)$ or $u_0 \in B^0_{\infty,1}$.

**Theorem 2.** Let $n \geq 2$, $M \in \mathbb{R}^{n \times n}$, $\text{tr}M = 0$, $u_0 \in L^q_\alpha(\mathbb{R}^n)$, $T > 0$, and let $u$ be a mild solution in $[0, T]$. Assume further that there exist constants $L_1$ and $L_2$ such that

$$\sup_{0 \leq t \leq T} \|u(t)\|_n \leq L_1 \quad \text{and} \quad \sup_{0 < t \leq T} t^{1/2} \|u(t)\|_{\infty} \leq L_2.$$  

Then there exist constants $K_1$ and $K_2$ depending only on $n$, $M$, $T$, $L_1$ and $L_2$ such that

$$\|\nabla^k u(t)\|_{\infty} \leq K_1(K_2 k)^k t^{-k/2 - 1/2} \quad \text{for} \quad t \in (0, T], \quad k \in \mathbb{N}. \tag{2.2}$$

When $M = 0$, the same assertion was proved in [10] with $u_0 \in L^q_\alpha(\mathbb{R}^n)$ for $q \in [n, \infty]$; with $u_0 \in BMO^{-1}$ see [20]. One may take $L_1$ and $L_2$ in (2.1) as finite quantities, whence the mild solution exists up to $T$. Consequently, it follows from (2.2) that the mild solution $u(t)$ is real analytic in $x$ as long as it exists. More precisely, the size of radius of convergence of Taylor series ($=: \rho$) is estimated from below by

$$\rho = \rho(t) = \liminf_{k \to \infty} \left( \frac{\|\nabla^k u(t)\|_{\infty}}{k!} \right)^{-1/k} \geq \frac{e}{K_2} \sqrt{t}, \quad t \in (0, T].$$

Here, we use Stirling’s formula (3.3) in section 3 below and the Cauchy-Hadamard theorem. This assertion implies that the propagation speed is infinite as well as the heat equation, that is, even if the support of initial data is compact, the support of solutions coincides the whole space, instantaneously. It is open whether $u$ is real analytic in $t$.

By (2.1), it is easy to see that

$$\sup_{0 < t \leq T} t^{1/2} \|\nabla u(t)\|_n \leq L_3 \tag{2.3}$$

with some constant $L_3$ depending only on $n$, $M$, $L_1$ and $L_2$.

Similarly, $\sup_{0 < t \leq T} t^{k/2} \|\nabla^k u(t)\|_n$ is bounded for each finite $k \in \mathbb{N}$.

The assumption on $L_2$ may be relaxed slightly. In fact, instead of assuming on $L_2$, we are allowed to suppose

$$\sup_{0 < t \leq T} t^{(1/q - 1/s) n/2} \|u(t)\|_s \leq L_4 \quad \text{for some} \quad s \in (q, \infty],$$

since bounds of $L_2$ and $L_3$ are ensured by those of $L_1$ and $L_4$. Notice that the uniform bound of $L_2$ (or $L_3$, $L_4$) in $t$, up to $T = \infty$ in particular, is still open for $n \geq 3$, even if $M = 0$ and $\|u_0\|_n$ is small.
3. Ornstein-Uhlenbeck Semigroup

Let \( n \in \mathbb{N} \). For \( q \in [1, \infty] \), \( L^q = L^q(\mathbb{R}^n) \) denote by the usual Lebesgue spaces in \( \mathbb{R}^n \) with norm \( \| f \|_q := (\int_{\mathbb{R}^n} |f(x)|^q \, dx)^{1/q} \) for \( q < \infty \), and \( \| f \|_\infty := \text{ess sup}_{x \in \mathbb{R}^n} |f(x)| \). We often omit the notation \((\mathbb{R}^n)\), if no confusion occurs likely; we sometimes do not distinguish the vector valued function and scalar as well as function spaces. The Sobolev space stands for \( W^{m,q} \) for \( q \in [1, \infty] \) and \( m \in \mathbb{N}_0 \), where \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). The solenoidal subspace of \( L^q \) denotes by \( L^q_\sigma \) for \( q \in (1, \infty) \).

Secondly, we recall properties of the Ornstein-Uhlenbeck semigroup.

**Proposition 1.** ([12, 19]) (a) Let \( n \in \mathbb{N} \), \( n \geq 2 \), \( q \in (1, \infty) \), \( M \in \mathbb{R}^{n \times n} \), \( \text{tr}M = 0 \). Put \( A := \Delta + Mx \cdot \nabla - M \) with domain \( D(A) := \{ v \in W^{2,q} \cap L^2_\sigma; Mx \cdot \nabla v \in L^q \} \). \(-A\) generates a non-analytic \((C_0)\)-semigroup \( \{e^{tA}\}_{t \geq 0} \) on \( L^q_\sigma \). Further, it has the following explicit formula

\[
e^{tA}v(x) := \frac{e^{-tM}}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} v(e^{tM}x - y)e^{-Q_t^{-1}y^2/4} \, dy
\]

for \( x \in \mathbb{R}^n \), \( t > 0 \) and \( v \in L^q_\sigma \), where \( Q_t := \int_0^t e^{sM}e^{sMT} \, ds \).

(b) Let \( T > 0 \), \( r \in [q, \infty] \). Thus, there exist constants \( C > 0 \) depending only on \( n \), \( q \), \( r \), \( M \) and \( \omega \geq 0 \) depending only on \( M \) such that

\[
\| \nabla^k e^{tA}v \|_r \leq Ce^{\omega kt}t^{-(1/q-1/r)n/2}\| \nabla^k v \|_q
\]

(3.1)

for \( t > 0 \), \( k \in \mathbb{N}_0 \) and \( v \in W^{k,q} \) as well as

\[
\| \nabla^k e^{tA}v \|_r \leq C(Ck)^{1/2}e^{\omega kt}t^{-k/2-(1/q-1/r)n/2}\| v \|_q
\]

(3.2)

for \( t > 0 \), \( k \in \mathbb{N}_0 \) and \( v \in L^q \).

This proposition is based on the results by Metafune and his collaborators in [19]. The proofs of above estimates are precisely shown in [12], so we omit them in this note. Remark that the semigroup \( \{e^{tA}\}_{t \geq 0} \) is neither analytic nor commutative to \( \nabla \). Indeed, it holds that

\[
\nabla e^{tA}v = e^{tM}e^{tA}\nabla v \quad \text{for} \quad v \in (W^{1,q})^n.
\]

From above, it is clear that \( \omega \) essentially depends only on the maximum of absolute value of real part of eigenvalues of \( M \). If \( M \) is skew-symmetric, then \( \omega = 0 \), since \( e^{tM} \) is unitary and \( Q_t = tI \).

We recall Kahane’s lemma for control the bilinear terms.

**Lemma 1.** ([13]) Let \( n \in \mathbb{N} \). Put \( |\alpha| := \sum_{i=1}^n \alpha_i \) and \( \alpha! := \prod_{i=1}^n (\alpha_i !) \) for \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \). Denote \( \beta \leq \alpha \) by \( \beta_i \leq \alpha_i \) for all \( i \), and \( (\alpha/\beta)! := \frac{\alpha!}{\beta!(\alpha-\beta)!} \). Assume that multi-sequences \( \{S_\alpha\} \), \( \{T_\alpha\} \) satisfy

\[
|S_0| \leq \sigma, \quad |T_0| \leq \theta, \quad |S_\alpha| \leq \sigma |\alpha|^{\alpha-\delta} \quad \text{and} \quad |T_\alpha| \leq \theta |\alpha|^{\alpha-\delta'}
\]
for $\alpha \in \mathbb{N}_0^\mathbb{N}, \neq 0$, where $\delta, \delta' \in \mathbb{R}$, $\sigma, \theta \geq 0$ are constants. If $\delta, \delta' > 1/2$, then there exists $\gamma > 0$ depending only on $n, \delta$ and $\delta'$ such that

$$\left| \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) S_\beta T_{\alpha-\beta} \right| \leq \gamma \sigma \theta |\alpha|^{-\min\{\delta, \delta'\}} \quad \text{for} \quad \alpha \in \mathbb{N}_0^\mathbb{N}, \neq 0.$$ 

This lemma follows from Stirling’s formula

$$k! \sim \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \quad \text{for} \quad k \in \mathbb{N}. \quad (3.3)$$

### 4. Proof

We give the proof of Theorem 2. For $M = 0$, it was obtained by [10] that mild solution is real analytic in $x$. We modify their proof. It suffices to show the following assertion essentially equivalent to (2.2).

**Proposition 2.** Suppose that the assumptions of Theorem 2 are satisfied. Let $\delta \in (1/2, 1]$. Then there exist positive constants $K_1$ and $K_2$ depending only on $n, M, L_1, L_2, T$ and $\delta$ such that

$$\|\nabla^k u(t)\|_{\infty} \leq K_1 (K_2 k)^{k-\delta} t^{-k/2-1/2}, \quad t \in (0, T], \quad k \in \mathbb{N}. \quad (4.1)$$

**Proof.** We may assume that $u$ is smooth, i.e., $u \in C^\infty(\mathbb{R}^n \times (0, T])$ by construction. It is used an induction argument with respect to $k \in \mathbb{N}$. For $k = 0$ and $1$, we see that $t^{k/2+1/2-n/2} \nabla^k u(t)$ is uniformly bounded in $[0, T]$ with valued in $L^r(\mathbb{R}^n)$ for $r \in [n, \infty]$, using $L_1$ and $L_2$ as (2.3). This means that (4.1) holds for $k = 1$, taking $K_1 \geq C_1'$ with some large $C_1 > 0$ and $K_2 = 1$. Similarly, assuming that for $k_* \geq 2$ determined later, (4.1) holds for $k \leq k_*$ with some large $K_1 \geq C_1$ and $K_2 = 1$.

Let $k \geq k_* + 1$. Suppose that (4.1) hold from 1 to $k - 1$. We now claim that (4.1) holds for $k$ with suitable $K_1$ and $K_2$. Put $\varepsilon \in (0, 1)$, we divide the integral in $(0, t)$ of (INT) into two parts to have

$$\|\nabla^k u(t)\|_{\infty} \leq \|\nabla^k e^{tA} u_0\|_{\infty}$$

$$+ \left( \int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t \right) \|\nabla^k e^{(t-s)A} \nabla u(s)\|_{\infty} ds$$

$$+ \left( \int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t \right) \|\nabla^k e^{(t-s)A} \nabla 2M u(s)\|_{\infty} ds$$

$$=: I_1 + I_2 + I_3 + I_4 + I_5.$$

We shall estimate $I_1, \ldots, I_5$ above, separately.

In what follows, for the sake of simplicity, let $T \leq 1$, and then $t \leq 1$. For estimating $I_1$, by the smoothing estimate (3.2) it holds that

$$I_1 \leq C (C_\phi k)^{k/2} e^{|\alpha| t^{k/2-1/2}} \|u_0\|_n \quad \text{for} \quad t \in (0, T]$$
with some $C\triangle$ and $C\vee$ independent of $k$. By Stirling’s formula (3.3),

$$I_1 \leq C_2(C_3k)^{k-\delta}t^{-k/2-1/2}$$

for all $k \geq 2$ with constants chosen as $C_3' := C\vee e^{3\alpha}$ and $C_2' := 2C\triangle L_1 C_3'$. It is also easy to derive the estimate to $I_2$. Indeed, for $t \in (0, T]$

$$I_2 \leq \int_0^{(1-\varepsilon)t} C(Ck)^{k/2}e^{k\alpha(t-s)}(t-s)^{-k/2-1/2} \|u(s)\|_n ds \leq C(Ck)^{k/2}e^{k\alpha} \int_0^{(1-\varepsilon)t} (t-s)^{-k/2-1} \|u(s)\|_n \|\nabla u(s)\|_n ds \leq C(Ck)^{k/2}e^{k\alpha} e^{-k/2-1/2}(1-\varepsilon)^{1/2}t^{-k/2-1/2}.$$ 

Here, we have used the Hölder inequality and (2.3). Similarly as $I_2$,

$$I_4 \leq \int_0^{(1-\varepsilon)t} C(Ck)^{k/2}e^{k\alpha(t-s)}(t-s)^{-k/2-1} \|u(s)\|_n ds \leq C(Ck)^{k/2}e^{k\alpha} e^{-k/2-1/2}(1-\varepsilon)^{1/2}t^{-k/2+1/2}$$

hold. Hereafter, we choose $\varepsilon := \eta/k$ for $\eta \geq 1$ and $k \geq \eta + 1$. So,

$$I_2 + I_4 \leq C\triangle C_\triangledown^{k/2}k^{k-1}e^{k\alpha} \eta^{-k/2-1}t^{-k/2-1/2} \leq C_2(C_3k)^{k-\delta}t^{-k/2-1/2}$$

are satisfied with constants $C\triangle$ and $C_\triangledown$; we have chosen $C_2' := C\triangle$, $C_3' := \max\{1, C_\triangledown\}$ and $\eta \geq e^{2\omega+2}$, since $k/2 \leq k-\delta$ for $k \geq 2$ and $\delta \leq 1$ as well as $k^2 \leq e^k$. Hence, one can see

$$I_1 + I_2 + I_4 \leq C_2(C_3k)^{k-\delta}t^{-k/2-1/2}$$

for $k \geq \eta + 1$ with $\eta := e^{2\omega+2}$, $C_2 := C_2' + C_2''$ and $C_3 := \max\{C_3', C_3''\}$. The estimate for $I_5$ is as follows. By (3.1) we shift $\nabla^k$ to $u$, then

$$I_5 \leq \int_0^t C e^{k\alpha(t-s)} \|\nabla^k u(s)\|_\infty ds \leq Ce^{k\alpha t} \int_0^t \|\nabla^k u(s)\|_\infty ds.$$ 

This term remains for applying the Gronwall inequality, later.

For $I_3$, we shift $\nabla^{k-1}$ to the bilinear terms. It leads us to

$$I_3 \leq \int_0^t C e^{(k-1\omega)(t-s)}(t-s)^{-1/2} \left\| \sum_{j=0}^{k} \binom{k}{j} \{ \nabla^j u(s) \} \nabla^{k-j} u(s) \right\|_\infty ds \leq C e^{k\omega t} \int_0^t (t-s)^{-1/2} \|u(s)\|_\infty \|\nabla^k u(s)\|_\infty ds$$

$$+ C e^{k\omega t} \int_0^t (t-s)^{-1/2} \sum_{j=1}^{k-1} \binom{k}{j} \|\nabla^j u(s)\|_\infty \|\nabla^{k-j} u(s)\|_\infty ds \leq I_3'' + I_3'''.$$
For \( t \leq 1 \) and \( \varepsilon := \eta/k \), combining \( I_3^* \) and \( I_5 \), we derive
\[
I_3^* + I_5 \leq Ce^{on} \int_{(1-\eta/k)t}^t (t-s)^{-1/2}s^{-k/2-1/2}\|\nabla^k u(s)\|_{\infty}ds.
\]

Put \( \psi(t) := \sup_{0 \leq \tau \leq t} t^{k/2+1/2}\|\nabla^k u(\tau)\|_{\infty} \). We thus see that
\[
I_3^* + I_5 \leq Ce^{on} \int_{(1-\eta/k)t}^t (t-s)^{-1/2}s^{-k/2-1/2}\psi(s)ds
\]
\[
\leq Ce^{on} \{(1-\eta/k)t\}^{-k/2-1/2}\psi(t) \int_{(1-\eta/k)t}^t (t-s)^{-1/2}s^{-1/2}ds
\]
\[
\leq t^{-k/2-1/2}\psi(t)/2 \quad \text{for } k \geq k_*,
\]
with some large \( k_* \geq \eta + 1 \). For the last inequality, the definition of Napier’s constant leads us to take a constant \( C_* \) independently of \( k \):
\[
(1-\eta/k)^{-k/2-1/2} \leq 2e^{\eta/2}(1-\eta/k)^{-1/2} \leq C_*.
\]

By assumption of induction and Lemma 1, the last term \( I_3^* \) is estimated as follows:
\[
I_3^* \leq Ce^{k_0\varepsilon} \int_{(1-\varepsilon)t}^t (t-s)^{-1/2} \sum_{j=1}^{k-1} \binom{k}{j} K_1(K_2 j)^{j-\delta} s^{-j/2-1/2}
\]
\[
\cdot K_1(K_2(k-j))^{k-j-\delta} s^{-k/2+j/2-1/2} ds
\]
\[
\leq Ce^{k_0\varepsilon} K_1^2 K_2^{k-2\delta} \int_{(1-\varepsilon)t}^t (t-s)^{-1/2}s^{-k/2-1} ds
\]
\[
\cdot \sum_{j=0}^{k} \binom{k}{j} j^{j-\delta} (k-j)^{k-j-\delta}
\]
\[
\leq Ce^{k_0\varepsilon} K_1^2 K_2^{k-2\delta} k^{-\delta} (1-\eta/k)^{-k/2-1/2} t^{1/2} t^{k/2-1/2}
\]
\[
\leq C_4 K_1^2 K_2^{k-2\delta} t^{-k/2-1/2}
\]
with some \( C_4 \) independent of \( k \), since \( \eta \) is fixed. Therefore, we have
\[
t^{k/2+1/2}\|\nabla^k u(t)\|_{\infty} \leq \psi(t) \leq 2C_2(C_3k)^{k-\delta} + 2C_4K_1^2 K_2^{-\delta}(K_2 k)^{k-\delta}
\]
for \( k \geq k_* \). Finally, we take
\[
K_1 := \max\{C_1, 4C_2\} \quad \text{and} \quad K_2 := \max\{C_3, (4C_4K_1)^{1/\delta}\}
\]
to get (4.1) with \( k \). This completes the proof of Proposition 2.

REFERENCES


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