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ULAM–HYERS–RASSIAS STABILITY OF A NONLINEAR STOCHASTIC ITO–VOLTERRA INTEGRAL EQUATION

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Abstract. In this paper, by using the classical Banach contraction principle, we investigate and establish the stability in the sense of Ulam-Hyers and in the sense of Ulam-Hyers-Rassias for the following stochastic integral equation

$$X_t = \xi_t + \int_0^t A(t,s,X_s)ds + \int_0^t B(t,s,X_s)dW_s,$$

where $\int_0^t B(t,s,X_s)dW_s$ is Ito integral.

1. Introduction

The study of stability problems for various functional equations originated from a famous talk given by Ulam in 1940. In the talk, Ulam discussed a problem concerning the stability of homomorphisms (see [24] and [25]). More precisely, he proposed the following problem:

Given a group G_1, a metric group (G_2,d) and a positive number ε, does there exist a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the following inequality

$$d(f(xy), f(x)f(y)) < \delta,$$

for all $x,y \in G_1$, then there exists a homomorphism $T : G_1 \longrightarrow G_2$ such that:

$$d(f(x), T(x)) < \varepsilon,$$

for all $x \in G_1$?

When this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable, or that the equation defining group homomorphisms is stable (in the sense of Ulam).

In 1941, D.H. Hyers (see [8]) gave a partial solution of Ulam's problem under the assumption that G_1 and G_2 are Banach spaces. In 1950, T. Aoki (see [2]) studied the stability problem for additive mappings by using unbounded Cauchy differences (see

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also [16]). In 1978, Th.M. Rassias (see [22]) studied a similar problem. The stability considered in [22] is often called the Ulam-Hyers-Rassias stability.

In [21], V. Radu introduced a simple and nice proof for the Hyers-Ulam stability of the Cauchy additive functional equation. Using the idea of V. Radu, S.M. Jung proved in [11] the Hyers-Ulam-Rassias stability of some Volterra integral equations defined on an finite interval. After that, in [5], L.P. Castro and D.A. Ramos investigated the stability of Volterra integral equation of second kind for not only the finite case but also the infinite case. A simple proof of Jung's problem was later given in [23] by using some Gronwall lemmas.

In the references, at the end of this paper, we have listed other papers dealing with the stability of functional equations.

For a large amount of information on the stability of functional equations, the reader is invited to consult the books [7], [9] and [12] (see also the papers [1], [4], [26], and others). Especially, in [4], the authors presented some recent developments in Ulam's type stability.

We point out that a fixed point method was used to investigate the stability of several functional equations. Works along these lines are achieved by L. Cădariu, V. Radu (see [6] and [21]), and H.A. Kenary et al ([14]). Fixed point methods were also used to study the stability of differential equations (see [13], [17], and other related papers).

Recently, N.P.N Ngoc ([18]) and X. Zhao ([27]) established the stability to stochastic differential equations on finite intervals. In this paper, we first introduce the notion of Hyers-Ulam-Rassias stability to the stochastic Ito-Volterra integral equation and then prove that kind of equations on not only finite but also infinite intervals has the Ulam-Hyers-Rassias stability.

2. Definitions and Preliminaries

Fix a probability space $(\Omega,\mathscr{F},\mathbf{P})$. Let $\|\cdot\|_2 = (E|\cdot|^2)^{\frac{1}{2}}$ be a norm of the space $L_2(\Omega,\mathbf{P})$. Let W_t be a Brownian motion defined in $(\Omega,\mathscr{F},\mathbf{P})$ and let $\{\mathscr{F}_t, t\in I\}$ be the natural filtration associated to W_t, where $I\subset\mathbb{R}$ ($I=[0,T]$ or $I=[0,\infty)$).

Denote by $L^2_{ad}(I,\Omega)$ the space of stochastic processes $f(t,\omega)$ such that each $f(t,\omega)$ is adapted to the filtration $\{\mathscr{F}_t\}$ and $E\left(\int_I |f(t)|^2 dt\right) < \infty$.

Let $A(t,s,x)$ and $B(t,s,x)$ be measurable functions of $s,t\in S$ and $x\in\mathbb{R}$, where $S=\{(s,t)\in I^2 : 0\leqslant s\leqslant t\}$. Consider the stochastic integral equation of Volterra second type:

$$X_t = \xi_t + \int_0^t A(t,s,X_s)ds + \int_0^t B(t,s,X_s)dWs, t\in I, \tag{1}$$

where ξ_t is a $\mathscr{F}_t$-adapted process.

About the existence and uniqueness of solution of Equation (1), we refer to [10] and [20] for more detail.

In the following definitions, we introduce the stability in the sense Ulam-Hyers and Ulam-Hyers-Rassias of the stochastic integral equation.

DEFINITION 1. Equation (1) is said to have the Ulam-Hyers stability with respect to ε if there exists a constant $M_\varepsilon > 0$ such that for each solution $X_t \in L^2_{ad}(I,\Omega)$ of the following inequation

$$\|X_t - \xi_t - \int_0^t A(t,s,X_s)ds - \int_0^t B(t,s,X_s)dW_s\|_2 \leqslant \varepsilon, \forall t \in I, \tag{2}$$

there exists a solution $U_t \in L^2_{ad}(I,\Omega)$ of Equation (1) such that:

$$\|X_t - U_t\|_2 \leqslant M_\varepsilon \varepsilon, \forall t \in I,$$

where M_ε is a constant that does not depend on X_t.

DEFINITION 2. Equation (1) is said to have the Ulam-Hyers-Rassias stability with respect to $u(t)$ if there exists a constant $M_u > 0$ such that for each solution $X_t \in L^2_{ad}(I,\Omega)$ of the following inequation

$$\|X_t - \xi_t - \int_0^t A(t,s,X_s)ds - \int_0^t B(t,s,X_s)dW_s\|_2 \leqslant u(t), \forall t \in I, \tag{3}$$

there exists a solution $U_t \in L^2_{ad}(I,\Omega)$ of the equation (1) such that:

$$\|X_t - U_t\|_2 \leqslant M_u u(t), \forall t \in I,$$

where M_u is a constant that does not depend on X_t.

In order to show that Equation (1) is stable in the sense of Ulam-Hyers and Ulam-Hyers-Rassias, we shall need some definitions and remarks in [20].

DEFINITION 3. ([20]) Let C_u denote the space of all processes in $L^2_{ad}(I,\Omega)$ that satisfy the following condition

$$\|x(t)\|_2 \leqslant Ku(t), \forall t \in I,$$

where $u(t) > 0$ is a given continuous function and K is some positive constant.

REMARK 1. It is well known that C_u is a Banach space when a norm $\|\cdot\|_{C_u}$ is defined by

$$\|x\|_{C_u} = \sup_{t\in I}\left\{\frac{\|x(t,\omega\|_2}{u(t)}\right\}.$$

DEFINITION 4. ([20]) If $u(t) = 1, \forall t \in I$ in Definition 3, we shall denote the corresponding C_u by C_b.

DEFINITION 5. ([20]) Let $C_{1,u}$ denote the space of all processes $x(t,s;\omega)$ in $C_1 = \{x(t,s;\omega) : \|x\| := \sup_{(s,t)\in S}\|x(t,s;\omega)\|_2 < \infty\}$ such that

$$\|x(t,s)\|_2 \leqslant Ku(t)u(s), \forall (s,t) \in S,$$

for some constant $K > 0$ and bounded positive continuous function $u(t)$.

DEFINITION 6. ([20]) If $u(t) = 1, \forall t \in I$ in Definition 5, we shall denote the corresponding $C_{1,u}$ by $C_{1,b}$.

REMARK 2. It is known that $C_{1,u}$ is a Banach space with the norm $\|\cdot\|_{C_{1,u}}$ defined by

$$\|x\|_{C_{1,u}} = \sup_{(s,t)\in S} \left\{ \frac{\|x(t,s)\|_2}{u(t)u(s)} \right\}.$$

We define the integral operators Λ_1, Λ_2 as follows:

$$(\Lambda_1 x)(t,\omega) = \int_0^t x(t,s;\omega)ds,$$

$$(\Lambda_2 x)(t,\omega) = \int_0^t x(t,s;\omega)dW_s.$$

REMARK 3. According to [20], with suitable conditions, $(C_{1,u}, C_u)$ is admissible with respect to both Λ_1 and Λ_2. It means that $\Lambda_1(C_{1,u}) \subset C_u$ and $\Lambda_2(C_{1,u}) \subset C_u$. In this case, there are constants K_1 and K_2 such that:

$$\begin{cases} \|\Lambda_1 x\|_{C_u} \leqslant K_1 \|x\|_{C_{1,u}}, \\ \|\Lambda_2 x\|_{C_u} \leqslant K_2 \|x\|_{C_{1,u}}. \end{cases}$$

We now introduce Banach's fixed point theory. This theorem will play an important role in proving our main theorems.

THEOREM 1. ([3]) *(Banach's fixed point theorem) Suppose (X,d) is a complete metric space and $T : X \to X$ is a contraction (for some $\lambda \in [0,1)$), $d(T(x),T(y) \leqslant \lambda d(x,y)$ for all $x,y \in X$. Also suppose that $u \in X, \delta > 0$, and $d(u,T(u)) \leqslant \delta$. Then there exists a unique $p \in X$ such that $p = T(p)$. Moreover,*

$$d(u,p) \leqslant \frac{\delta}{1-\lambda}. \tag{4}$$

In the rest of the paper, we shall use the following operator

$$\Lambda(X_t) = \xi_t + \int_0^t A(t,s,X_s)ds + \int_0^t B(t,s,X_s)dW_s.$$

3. Ulam-Hyers-Rassias stability on a finite interval

In this section, we show that Equation (1) on the finite interval $I = [0,T]$, under some conditions given in [10], has Ulam-Hyers-Rassias property. Furthermore, this equation also has a unique solution.

THEOREM 2. ***(Ulam-Hyers stability)*** *We suppose that the following assumptions are satisfied:*

a) $\xi_t \in C_b$;

b) $\begin{cases} |A(t,s,X_s)| \leqslant K(1+|X_s|), \forall 0 \leqslant s \leqslant t \leqslant T, a.s; \\ |B(t,s,X_s)| \leqslant K(1+|X_s|), \forall 0 \leqslant s \leqslant t \leqslant T, a.s; \end{cases}$

c) $\begin{cases} |A(t,s,X_s)-A(t,s,Y_s)| \leqslant \alpha_1 |X_s - Y_s|, \forall 0 \leqslant s \leqslant t \leqslant T, a.s; \\ |B(t,s,X_s)-B(t,s,Y_s)| \leqslant \alpha_2 |X_s - Y_s|, \forall 0 \leqslant s \leqslant t \leqslant T, a.s; \end{cases}$

d) $(\alpha_1 T + \alpha_2 \sqrt{T}) < 1$.

Then:

i) Equation (1) *has a unique solution belonging to the space* C_b.

ii) Equation (1) *has the Ulam-Hyers stability.*

Proof. For all $X_t \in C_b$, using the triangle inequality, the estimation $\| \int_0^t \cdot ds\|_2 \leqslant \int_0^t \| \cdot \|_2 ds$ and Ito isometry, we get

$$\begin{aligned} \|\Lambda(X_t)\|_2 &\leqslant \|\xi_t\|_2 + \| \int_0^t A(t,s,X_s)ds\|_2 + \| \int_0^t B(t,s,X_s)dW_s\|_2 \\ &\leqslant \|\xi_t\|_2 + \int_0^t \|A(t,s,X_s)\|_2 ds + \sqrt{\int_0^t \|B(t,s,X_s)\|_2^2 ds} \\ &\leqslant \|\xi_t\|_2 + \int_0^t K(1+\|X_s\|_2)ds + \sqrt{\int_0^t K^2(1+\|X_s\|_2)^2 ds} \\ &\leqslant \|\xi_t\|_{C_b} + K(T+\sqrt{T})(1+\|X_s\|_{C_b}), \end{aligned}$$

which implies that $\|\Lambda(X_t)\|_{C_b} \leqslant \|\xi_t\|_{C_b} + K(T+\sqrt{T})(1+\|X_s\|_{C_b})$. Hence, $\Lambda(C_b) \subset C_b$.

Furthermore, we have

$$\begin{aligned} &\|\Lambda(X_t) - \Lambda(Y_t)\|_2 \leqslant \\ &\leqslant \| \int_0^t A(t,s,X_s) - A(t,s,Y_s)ds\|_2 + \| \int_0^t B(t,s,X_s) - B(t,s,Y_s)dW_s\|_2 \\ &\leqslant \int_0^t \|A(t,s,X_s) - A(t,s,Y_s)\|_2 ds + \sqrt{\int_0^t \|B(t,s,X_s) - B(t,s,Y_s)\|_2^2 ds} \\ &\leqslant \int_0^t \alpha_1 \|X_s - Y_s\|_2 ds + \sqrt{\int_0^t \alpha_2^2 \|X_s - Y_s\|_2^2 ds} \\ &\leqslant (\alpha_1 T + \alpha_2 \sqrt{T}) \|X_s - Y_s\|_{C_b}, \end{aligned}$$

which implies that $\|\Lambda(X_t) - \Lambda(X_t)\|_{C_b} \leqslant (\alpha_1 T + \alpha_2\sqrt{T})\|X_s - Y_s\|_{C_b}$. By assumption d), the mapping Λ is strictly contractive. Thus, by the Banach's fixed point principle, Equation (1) has a unique solution $U_t \in C_b$.

Let $X_t \in C_b$ be a solution of Inequation (2). It means that $\|X_t - \Lambda(X_t)\|_2 \leqslant \varepsilon, \forall t \in [0,T]$, from which we get $\|X_t - \Lambda(X_t)\|_{C_b} \leqslant \varepsilon$. By the estimate (4) in Theorem 1, we obtain

$$\|X_t - U_t\|_{C_b} \leqslant \frac{\varepsilon}{1-M_1}, \tag{5}$$

where $M_1 = \alpha_1 T + \alpha_2\sqrt{T}$. On the other hand, we have

$$\|X_t - U_t\|_2 \leqslant \|X_t - U_t\|_{C_b}, \forall t \in [0,T]. \tag{6}$$

Thus, $\|X_t - U_t\|_2 \leqslant \dfrac{\varepsilon}{1-M_1}$, which implies that Equation (1) is stable in the sense Ulam-Hyers and completes the proof.

THEOREM 3. *(**Ulam-Hyers-Rassias stability**) We suppose that the following assumptions are satisfied:*

a) $\xi_t \in L^2_{ad}([0,T],\Omega)$;

b) $\begin{cases} |A(t,s,X_s)| \leqslant K(1+|X_s|), \forall 0 \leqslant s \leqslant t \leqslant T, a.s; \\ |B(t,s,X_s)| \leqslant K(1+|X_s|), \forall 0 \leqslant s \leqslant t \leqslant T, a.s; \end{cases}$

c) $\begin{cases} |A(t,s,X_s) - A(t,s,Y_s)| \leqslant \alpha_1 |X_s - Y_s|, \forall 0 \leqslant s \leqslant t \leqslant T, a.s; \\ |B(t,s,X_s) - B(t,s,Y_s)| \leqslant \alpha_2 |X_s - Y_s|, \forall 0 \leqslant s \leqslant t \leqslant T, a.s; \end{cases}$

d) The function $u(t)$ is positive and there exists a constant $N_u > 0$ such that

$$\int_0^t u^2(s)ds \leqslant N_u u^2(t), \forall t \in [0,T];$$

e) $\sqrt{2(T\alpha_1^2 + \alpha_2^2)N_u} < 1$.

Then:

i) Equation (1) has a unique solution belonging to the space $L^2_{ad}([0,T],\Omega)$.

ii) Equation (1) has the Ulam-Hyers-Rassias stability with respect to $u(t)$.

Proof.

For all $X_t, Y_t \in L^2_{ad}([0,T],\Omega)$, we set

$$d_u(X_t,Y_t) = \sup_{t\in[0,T]} \frac{\|X_t - Y_t\|_2}{u(t)} < \infty.$$

Notice that $\Lambda(L^2_{ad}([0,T],\Omega)) \subset L^2_{ad}([0,T],\Omega)$ and $(L^2_{ad}([0,T],\Omega), d_u)$ is a complete metric space.

We assert that Λ is strictly contractive on $L^2_{ad}([0,T],\Omega)$. Given any $X_t, Y_t \in L^2_{ad}([0,T],\Omega)$, let $M_{X_t,Y_t} \in [0,\infty)$ be an arbitrary constant such that $d_u(X_t,Y_t) \leqslant M_{X_t,Y_t}$, from which we deduce that

$$\|X_t - Y_t\|_2 \leqslant M_{X_t,Y_t} u(t), \forall t \in [0,T]. \tag{7}$$

Using the inequality $\|x+y\|_2^2 \leqslant 2(\|x\|_2^2+\|y\|_2^2)$, Schwarz inequality and Ito isometry, we have the following estimates:

$$\begin{aligned}
&\|\Lambda(X_t)-\Lambda(Y_t)\|_2^2 \leqslant \\
&\leqslant 2\left(\|\int_0^t A(t,s,X_s)-A(t,s,Y_s)ds\|_2^2+\|\int_0^t B(t,s,X_s)-B(t,s,Y_s)dW_s\|_2^2\right)\\
&\leqslant 2\left(T\int_0^t \|A(t,s,X_s)-A(t,s,Y_s)\|_2^2 ds+\int_0^t \|B(t,s,X_s)-B(t,s,Y_s)\|_2^2 ds\right)\\
&\leqslant 2\left(T\int_0^t \alpha_1^2\|X_s-Y_s\|_2^2 ds+\int_0^t \alpha_2^2\|X_s-Y_s\|_2^2 ds\right)\\
&\leqslant 2(T\alpha_1^2+\alpha_2^2)\int_0^t \|X_s-Y_s\|_2^2 ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\Lambda(X_t)-\Lambda(Y_t)\|_2^2 &\leqslant 2(T\alpha_1^2+\alpha_2^2)\int_0^t M_{X_t,Y_t}^2 u^2(s)ds\\
&\leqslant 2(T\alpha_1^2+\alpha_2^2)M_{X_t,Y_t}^2 N_u u^2(t).
\end{aligned}$$

Hence,

$$\|\Lambda(X_t)-\Lambda(Y_t)\|_2 \leqslant M_2 M_{X_t,Y_t} u(t), \tag{8}$$

where $M_2=\sqrt{2(T\alpha_1^2+\alpha_2^2)N_u}$. It implies that $d_u(\Lambda(X_t),\Lambda(Y_t)) \leqslant M_2 M_{X_t,Y_t}$. We may conclude that $d_u(\Lambda(X_t),\Lambda(Y_t)) \leqslant M_2 d_u(X_t,Y_t)$ for any $X_t,Y_t \in L_{ad}^2([0,T],\Omega)$. By assumption e), the mapping Λ is strictly contractive on the metric space $(L_{ad}^2([0,T],\Omega),d_u)$. Thus, by the Banach's fixed point principle, Equation (1) has a unique solution.

Let X_t be a solution of Inequation (3) and let U_t be the solution of Equation (1). From $\|X_t-\Lambda(X_t)\|_2 \leqslant u(t), \forall t\in[0,T]$, we get $d_u(X_t,\Lambda(X_t)) \leqslant 1$. By the triangle inequality, we have

$$\begin{aligned}
d_u(X_t,U_t) &\leqslant d_u(X_t,\Lambda(X_t))+d_u(\Lambda(X_t),U_t)\\
&\leqslant d_u(X_t,\Lambda(X_t))+d_u(\Lambda(X_t),\Lambda(U_t))\\
&\leqslant 1+M_2 d_u(X_t,U_t),
\end{aligned}$$

which implies that

$$d_u(X_t,U_t) \leqslant \frac{1}{1-M_2}. \tag{9}$$

Hence,

$$\|X_t-U_t\|_2 \leqslant M_u u(t), \tag{10}$$

where $M_u=\dfrac{1}{1-M_2}$. It means that Equation (1) has the Ulam-Hyers-Rassias stability. The proof of the theorem thus is complete.

4. Ulam-Hyers-Rassias stability on an infinite interval

In this section, we investigate the stability of Equation (1) on the infinite interval $I=[0,\infty)$ making use of some results given in the paper [20]. In the first two theorems, we use the triangle inequality, the estimation $\|\int_0^t \cdot ds\|_2 \leqslant \int_0^t \|\cdot\|_2 ds$ and Ito isometry in order to evaluate the L_2-norm of $\Lambda(X_t)-\Lambda(Y_t)$. In the last two theorems, by using Remark 3, we quickly obtain estimations for $\|\Lambda(X_t)-\Lambda(Y_t)\|_{C_b}$ and $\|\Lambda(X_t)-\Lambda(Y_t)\|_{C_u}$.

THEOREM 4. ***(Ulam-Hyers stability)*** *We suppose that the following assumptions are satisfied:*

a) $\sup_{t\geqslant 0}\int_0^t(\gamma(t,s)+\gamma^2(t,s))ds<\infty$;

b) $\xi_t\in C_b$;

c) $\begin{cases} |A(t,s,X_s)| \leqslant \gamma(t,s)|X_s|, \forall 0\leqslant s\leqslant t, a.s; \\ |B(t,s,X_s)| \leqslant \gamma(t,s)|X_s|, \forall 0\leqslant s\leqslant t, a.s; \end{cases}$

d) $\begin{cases} |A(t,s,X_s)-A(t,s,Y_s)| \leqslant \alpha_1\gamma(t,s)|X_s-Y_s|, \forall 0\leqslant s\leqslant t, a.s; \\ |B(t,s,X_s)-B(t,s,Y_s)| \leqslant \alpha_2\gamma(t,s)|X_s-Y_s|, \forall 0\leqslant s\leqslant t, a.s; \end{cases}$

e) $\sup_{t\geqslant 0}\left(\alpha_1\int_0^t\gamma(t,s)ds+\alpha_2\sqrt{\int_0^t\gamma^2(t,s)ds}\right)<1$.

Then:

i) Equation (1) *has a unique solution belonging to the space* C_b.

ii) Equation (1) *has the Ulam-Hyers stability.*

Proof.

For all $X_t\in C_b$, we have

$$\begin{aligned}\|\Lambda(X_t)\|_2 &\leqslant \|\xi_t\|_2+\|\int_0^t A(t,s,X_s)ds\|_2+\|\int_0^t B(t,s,X_s)dW_s\|_2\\ &\leqslant \|\xi_t\|_2+\int_0^t\|A(t,s,X_s)\|_2ds+\sqrt{\int_0^t\|B(t,s,X_s)\|_2^2ds}\\ &\leqslant \|\xi_t\|_2+\int_0^t\gamma(t,s)\|X_s\|_2ds+\sqrt{\int_0^t\gamma^2(t,s)\|X_s\|_2^2ds}\\ &\leqslant \|\xi_t\|_{C_b}+\|X_s\|_{C_b}\sup_{t\geqslant 0}\left(\int_0^t\gamma(t,s)ds+\sqrt{\int_0^t\gamma^2(t,s)ds}\right).\end{aligned}$$

Hence,

$$\|\Lambda(X_t)\|_{C_b}\leqslant\|\xi_t\|_{C_b}+\|X_s\|_{C_b}\sup_{t\geqslant 0}\left(\int_0^t\gamma(t,s)ds+\sqrt{\int_0^t\gamma^2(t,s)ds}\right), \tag{11}$$

which implies that $\Lambda(C_b)\subset C_b$.

As in Theorem 2, we have

$$\begin{aligned}
&\|\Lambda(X_t)-\Lambda(Y_t)\|_2 \leqslant \\
&\leqslant \int_0^t \|A(t,s,X_s)-A(t,s,Y_s)\|_2 ds + \sqrt{\int_0^t \|B(t,s,X_s)-B(t,s,Y_s)\|_2^2 ds} \\
&\leqslant \int_0^t \alpha_1\gamma(t,s)\|X_s-Y_s\|_2 ds + \sqrt{\int_0^t \alpha_2^2\gamma^2(t,s)\|X_s-Y_s\|_2^2 ds} \\
&\leqslant \sup_{t\geqslant 0}\left(\alpha_1\int_0^t \gamma(t,s)ds + \alpha_2\sqrt{\int_0^t \gamma^2(t,s)ds}\right)\|X_s-Y_s\|_{C_b}.
\end{aligned}$$

Hence,

$$\|\Lambda(X_t)-\Lambda(Y_t)\|_{C_b} \leqslant \sup_{t\geqslant 0}\left(\alpha_1\int_0^t \gamma(t,s)ds + \alpha_2\sqrt{\int_0^t \gamma^2(t,s)ds}\right)\|X_s-Y_s\|_{C_b}.$$

By assumption e), Λ is a contraction. Therefore, there exists unique solution $U(t)\in C_b$ of Equation (1) such that $\Lambda(U_t)=U_t, t\geqslant 0$.

We assume that X_t is a solution of Inequation (2). We have $\|X_t-\Lambda(X_t)\|_2 \leqslant \varepsilon, \forall t\geqslant 0$, which implies that $\|X_t-\Lambda(X_t)\|_{C_b}\leqslant\varepsilon$. By the estimate (4) in Theorem 1, we obtain

$$\|X_t-U_t\|_{C_b} \leqslant \frac{\varepsilon}{1-M_3}, \tag{12}$$

where $M_3=\sup_{t\geqslant 0}\left(\alpha_1\int_0^t\gamma(t,s)ds+\alpha_2\sqrt{\int_0^t\gamma^2(t,s)ds}\right)$. Hence, $\|X_t-U_t\|_2\leqslant\dfrac{\varepsilon}{1-M_3}$ for all $t\geqslant 0$, which shows that the stochastic integral equation (1) is stable in the sense of Ulam-Hyers and completes the proof.

THEOREM 5. ***(Ulam-Hyers-Rassias stability)*** *We suppose that the following assumptions are satisfied:*

a) $u(t)>0$ *is a continuous function and* $\sup_{t\geqslant 0}\int_0^t(u(s)+u^2(s))ds<\infty$*;*

b) $\xi_t\in C_u$*;*

c) $\begin{cases}|A(t,s,X_s)|\leqslant u(t)\left[z(t,\omega)+\gamma(t,s)|X_s|\right], \forall 0\leqslant s\leqslant t<\infty, a.s;\\ |B(t,s,X_s)|\leqslant u(t)\left[z(t,\omega)+\gamma(t,s)|X_s|\right], \forall 0\leqslant s\leqslant t<\infty, a.s;\end{cases}$

for $0\leqslant s\leqslant t<\infty$*, where* $z(s,\omega)$ *is a second order stochastic process in* C_u *and* $\gamma(t,s)$ *is a bounded continuous function defined for* $0\leqslant s\leqslant t$.

d) $\begin{cases}|A(t,s,X_s)-A(t,s,Y_s)|\leqslant\alpha_1 u(t)|X_s-Y_s|, \forall 0\leqslant s\leqslant t<\infty, a.s;\\ |B(t,s,X_s)-B(t,s,Y_s)|\leqslant\alpha_2 u(t)|X_s-Y_s|, \forall 0\leqslant s\leqslant t<\infty, a.s;\end{cases}$

e) $\sup_{t\geqslant 0}\left(\alpha_1\int_0^t u(s)ds+\alpha_2\sqrt{\int_0^t u^2(s)ds}\right)<1$.

Then:

i) Equation (1) *has a unique solution belonging to the space* C_u.

ii) Equation (1) *has the Ulam-Hyers-Rassias stability with respect to* $u(t)$.

Proof. According to [20], $(C_{1,u}, C_u)$ is admissible with respect to both the operators Λ_1 and Λ_2. Condition c) implies that $A(t,s,X_s)$ and $B(t,s,X_s)$ are in $C_{1,u}$ whenever $X_t \in C_u$. Therefore, $\Lambda(C_u) \subset C_u$.

We show that if $X_t, Y_t \in C_u$ then $(A(t,s,X_s) - A(t,s,Y_s))$ and $(B(t,s,X_s) - B(t,s,Y_s))$ belong to $C_{1,u}$.

From the condition d), we get

$$\begin{aligned}\frac{\|A(t,s,X_s) - A(t,s,Y_s)\|_2}{u(t)u(s)} &\leqslant \frac{\alpha_1 u(t)\|X_s - Y_s\|_2}{u(t)u(s)} = \alpha_1 \frac{\|X_s - Y_s\|_2}{u(s)} \\ &\leqslant \alpha_1 \|X_s - Y_s\|_{C_u}.\end{aligned}$$

Thus, $A(t,s,X_s) - A(t,s,Y_s) \in C_{1,u}$.

Similarly, we have $B(t,s,X_s) - B(t,s,Y_s) \in C_{1,u}$.

Hence, $\begin{cases} \int_0^t A(t,s,X_s) - A(t,s,Y_s)ds \in C_u; \\ \int_0^t B(t,s,X_s) - B(t,s,Y_s)dW_s \in C_u; \end{cases}$

As in Theorem 4, we have the following estimates:

$$\begin{aligned}&\|\Lambda(X_t) - \Lambda(Y_t)\|_2 \leqslant \\ &\leqslant \int_0^t \|A(t,s,X_s) - A(t,s,Y_s)\|_2 ds + \sqrt{\int_0^t \|B(t,s,X_s) - B(t,s,Y_s)\|_2^2 ds} \\ &\leqslant \int_0^t \alpha_1 u(t)\|X_s - Y_s\|_2 ds + \sqrt{\int_0^t \alpha_2^2 u^2(t)\|X_s - Y_s\|_2^2 ds} \\ &\leqslant \alpha_1 u(t) \int_0^t \|X_s - Y_s\|_2 ds + \alpha_2 u(t)\sqrt{\int_0^t \|X_s - Y_s\|_2^2 ds}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\|\Lambda(X_t) - \Lambda(Y_t)\|_2}{u(t)} \leqslant \\ &\leqslant \alpha_1 \int_0^t \|X_s - Y_s\|_2 ds + \alpha_2 \sqrt{\int_0^t \|X_s - Y_s\|_2^2 ds} \\ &\leqslant \alpha_1 \int_0^t \frac{\|X_s - Y_s\|_2}{u(s)} u(s) ds + \alpha_2 \sqrt{\int_0^t \frac{\|X_s - Y_s\|_2^2}{u^2(s)} u^2(s) ds} \\ &\leqslant \sup_{t \geqslant 0} \left(\alpha_1 \int_0^t u(s)ds + \alpha_2 \sqrt{\int_0^t u^2(s)ds} \right) \|X_s - Y_s\|_{C_u},\end{aligned}$$

from which we deduce that

$$\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_u} \leqslant \sup_{t \geqslant 0} \left(\alpha_1 \int_0^t u(s)ds + \alpha_2 \sqrt{\int_0^t u^2(s)ds} \right) \|X_s - Y_s\|_{C_u}.$$

By assumption e), the mapping Λ is strictly contractive. Thus, by the Banach's fixed point principle, there exits a unique solution (say) U_t in C_u of Equation (1).

Let $X_t \in C_u$ be a solution of Inequation (3). We have

$$\|X_t - \Lambda(X_t)\|_2 \leqslant u(t), \tag{13}$$

from which, we deduce the following inequality $\|X_t - \Lambda(X_t)\|_{C_u} \leqslant 1$.

By the triangle inequality, we get:

$$\begin{aligned}\|X_t - U_t\|_{C_u} &\leqslant \|X_t - \Lambda(X_t)\|_{C_u} + \|\Lambda(X_t) - \Lambda(U_t)\|_{C_u} \leqslant \\ &\leqslant 1 + M_4\|X_t - Y_t\|_{C_u},\end{aligned}$$

where $M_4 = \sup_{t\geqslant 0}\left(\alpha_1 \int_0^t u(s)ds + \alpha_2\sqrt{\int_0^t u^2(s)ds}\right)$. Therefore,

$$\|X_t - U_t\|_{C_u} \leqslant \frac{1}{1-M_4}. \tag{14}$$

Thus, $\|X_t - U_t\|_2 \leqslant \dfrac{1}{1-M_4}u(t), \forall t \geqslant 0$, which implies that Equation (1) has the Ulam-Hyers-Rasiass stability with respect to $u(t)$. This ends the proof.

REMARK 4. Theorem 2 is a consequence of Theorem 5.

In the next two theorems, we keep the assumptions in Theorem 4 and Theorem 5. Remark 3 will be used to evaluate $\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_b}$ and $\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_u}$.

THEOREM 6. ***(Ulam-Hyers stability)*** *Suppose that the assumptions a), b), c) and d) in Theorem 4 together with the following assumption are satisfied:*

e) $(K_1\alpha_1 + K_2\alpha_2)\sup_{0\leqslant s\leqslant t<\infty}\gamma(t,s) < 1$, *where* K_1, K_2 *are the constants in Remark 3.*

Then:

i) Equation (1) has a unique solution belonging to the space C_b.

ii) Equation (1) has the Ulam-Hyers stability.

Proof.

With $X_t \in C_b$, we get the following estimates:

$$\begin{aligned}\|\int_0^t A(t,s,X_s)ds\|_2 &\leqslant \int_0^t \|A(t,s,X_s)\|_2 ds \\ &\leqslant \int_0^t \gamma(t,s)\|X_s\|_2 ds \\ &\leqslant \|X_s\|_{C_b}\sup_{t\geqslant 0}\int_0^t \gamma(t,s)ds < \infty,\end{aligned}$$

and

$$\begin{aligned}\|\int_0^t B(t,s,X_s)dW_s\|_2^2 &= \int_0^t \|B(t,s,X_s)\|_2^2 ds\\ &\leqslant \int_0^t \gamma^2(t,s)\|X_s\|_2^2 ds\\ &\leqslant \|X_s\|_{C_b}^2 \sup_{t\geqslant 0}\int_0^t \gamma^2(t,s)ds < \infty.\end{aligned}$$

Hence, $\int_0^t A(t,s,X_s)ds \in C_b, \int_0^t B(t,s,X_s)dW_s \in C_b$.

As in Theorem 4, we have $\Lambda(C_b) \in C_b$.

From

$$\begin{aligned}&|\Lambda(X_t)-\Lambda(Y_t)| \leqslant |\int_0^t A(t,s,X_s)-A(t,s,Y_s)ds|\\ &+|\int_0^t B(t,s,X_s)-B(t,s,Y_s)dW_s|,\end{aligned}$$

we get that

$$\begin{aligned}&\|\Lambda(X_t)-\Lambda(Y_t)\|_{C_b} \leqslant\\ &\leqslant \|\int_0^t A(t,s,X_s)-A(t,s,Y_s)ds\|_{C_b} + \|\int_0^t B(t,s,X_s)-B(t,s,Y_s)dW_s\|_{C_b}\\ &\leqslant K_1\|A(t,s,X_s)-A(t,s,Y_s)\|_{C_{1,b}} + K_2\|B(t,s,X_s)-B(t,s,Y_s)\|_{C_{1,b}}.\end{aligned}$$

We also have

$$|A(t,s,X_s)-A(t,s,Y_s)| \leqslant \alpha_1\gamma(t,s)|X_s-Y_s|, \forall 0 \leqslant s \leqslant t,$$

then

$$\|A(t,s,X_s)-A(t,s,Y_s)\|_{C_{1,b}} \leqslant \alpha_1 \sup_{0\leqslant s\leqslant t<\infty}\gamma(t,s)\|X_s-Y_s\|_{C_b}.$$

Similarly, we obtain

$$\|B(t,s,X_s)-B(t,s,Y_s)\|_{C_{1,b}} \leqslant \alpha_2 \sup_{0\leqslant s\leqslant t<\infty}\gamma(t,s)\|X_s-Y_s\|_{C_b}.$$

Therefore,

$$\|\Lambda(X_t)-\Lambda(Y_t)\|_{C_b} \leqslant (K_1\alpha_1+K_2\alpha_2)\sup_{0\leqslant s\leqslant t<\infty}\gamma(t,s)\|X_s-Y_s\|_{C_b}. \tag{15}$$

According to Theorem 1, with U_t is the solution of Equation (1) and X_t is a solution of Inequation (3), we have the following estimate $\|X_t-U_t\|_2 \leqslant \dfrac{\varepsilon}{1-M_5}$, where $M_5 = (K_1\alpha_1+K_2\alpha_2)\sup_{0\leqslant s\leqslant t<\infty}\gamma(t,s)$, which implies that Equation (1) has the Ulam-Hyers stability. This completes the proof.

THEOREM 7. ***(Ulam-Hyers-Rassias stability)*** *Suppose that the assumptions a), b), c) and d) in Theorem 5 together with the following assumption are satisfied:*

e) $K_1\alpha_1 + K_2\alpha_2 < 1$*, where* K_1, K_2 *are the constants in Remark 3.*

Then:

i) Equation (1) *has a unique solution belonging to the space* C_u*.*

ii) Equation (1) *has the Ulam-Hyers-Rassias stability with respect to* $u(t)$*.*

Proof. We have

$$\begin{aligned}
&\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_u} \leqslant \\
&\leqslant \|\int_0^t A(t,s,X_s) - A(t,s,Y_s)ds\|_{C_u} + \|\int_0^t B(t,s,X_s) - B(t,s,Y_s)dW_s\|_{C_u}. \\
&\leqslant K_1\|A(t,s,X_s) - A(t,s,Y_s)\|_{C_{1,u}} + K_2\|B(t,s,X_s) - B(t,s,Y_s)\|_{C_{1,u}}.
\end{aligned}$$

Thus,

$$\frac{\|A(t,s,X_s) - A(t,s,Y_s)\|_2}{u(t)u(s)} \leqslant \alpha_1 \frac{\|X_s - Y_s\|_2}{u(s)} \leqslant \alpha_1\|X_s - Y_s\|_{C_u}.$$

Therefore, $\|A(t,s,X_s) - A(t,s,Y_s)\|_{C_{1,u}} \leqslant \alpha_1\|X_t - Y_t\|_{C_u}$.

Similarly, we have $\|B(t,s,X_s) - B(t,s,Y_s)\|_{C_{1,u}} \leqslant \alpha_2\|X_t - Y_t\|_{C_u}$.

We get the following estimate

$$\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_u} \leqslant (K_1\alpha_1 + K_2\alpha_2)\|X_t - Y_t\|_{C_u}. \tag{16}$$

By assumption e), the mapping Λ is strictly contractive. Thus, according to the Banach's fixed point principle, Equation (1) has a unique solution $U_t \in C_u$.

Using the estimate $\|X_t - \Lambda(X_t)\|_{C_u} \leqslant 1$ and the triangle inequality, we get that

$$\|X_t - U_t\|_{C_u} \leqslant \frac{1}{1 - M_6}, \tag{17}$$

where X_t is a solution of Inequation (3) and $M_6 = K_1\alpha_1 + K_2\alpha_2$.

Thus, $\|X_t - U_t\|_2 \leqslant \dfrac{1}{1 - M_6}u(t)$, which implies that Equation (1) has the Ulam-Hyers-Rasiass stability with respect to $u(t)$.

5. Examples

In this section, we consider Section 3 with the case $T = 1$. Remark that $u(t) = t$, $t \in [0,1]$, is a function satisfying the condition $d)$ in Theorem 3 with $N_u = \frac{1}{3}$.

Consider the following stochastic integral equation

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dW_s, \tag{18}$$

where μ and σ are constants. Here, ξ and the functions A, B are given by

$$\xi = X_0, \;\; A(t,s,x) = \mu x, \;\; B(t,s,x) = \sigma x,$$

The functions A and B satisfy Lipschitz condition in x with Lipschitz constants μ and σ, respectively. In the case $\mu+\sigma<1$, all the hypotheses of Theorem 2 are satisfied. Hence, Equation (18) has Ulam-Hyer stability and its solution is a geometry Brownian motion given by

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

We continue considering the Langevin equation (see Example 10.1.1. in [15])

$$X_t = X_0 - \int_0^t \alpha X_s ds + \int_0^t \beta dW_s, \tag{19}$$

where α, β are constants.

In the case $T=1$ and $u(t)=t$, the condition $e)$ in Theorem 3 is equivalent to $\alpha_1^2+\alpha_2^2<\frac{3}{2}$. It is evident that the functions $A=-\alpha x$ and $B=\beta$ satisfy Lipschitz condition in x with Lipschitz constant $|\alpha|$. Hence, with $|\alpha|<\frac{\sqrt{3}}{2}$, all the assumptions of Theorem 3 are satisfied. Thus, Equation (19) has Ulam-Hyer-Rassias stability with respect to $u(t)=t$ and its solution is an Ornstein-Uhlenbeck process given by

$$X_t = e^{-\alpha t}x_0 + \beta \int_0^t e^{-\alpha(t-s)} dW_s.$$

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