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PLANT–PEST–NATURAL ENEMY MODEL WITH IMPULSIVE BIOLOGICAL AND CHEMICAL CONTROL

VANDANA KUMARI, SUDIPA CHAUHAN, SUMIT KAUR BHATIA AND JOYDIP DHAR

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Abstract. In this paper, a plant pest mathematical model is presented with integrated pest management through impulse. Two control measures: Biological(Natural Enemies) and Chemical pesticides are taken in consideration in the model through impulse. Boundedness and the sufficient conditions of existence of the positive periodic solutions is established. Further, the local stability of the pest extinction equilibrium point is studied using Floquet's theory. It is proved that the pest extinction equilibrium point is globally stable at $T < T_{\max}$ and the system is permanent for $T > T_{\max}$. Numerical data per week are taken to illustrate the theoretical results using MATLAB software.

1. Introduction

For years, discussion on the rivalry over plants and pests has gained the major cause of attention. Farmer used all possible methods to tackle with the pests for the conservation of plants, humans, revenue and the capitalism of the the country. As the time passed on, new technologies stepped in to take control over deadly pests. Pests not only affects the plants but also affects the ecological balance of the biosphere [1, 2, 3, 4]. Our only focus was on to suppress the pests for better results and for the conservation of the biosphere.

To overcome the deadly pests destroying the plants, natural enemies were only the way agriculturists came through. They released the natural enemies impulsively and they could see the benefits over the method they applied. Example: Mirid bugs, plays the key role in suppressing the pests [6]. Mirid bugs prey on spider mites, aphids, leafhoppers, scale insects harming the growth of the plants. But these natural enemies has to cross life stages which transform them from immature to mature stage as generally mature natural enemies can only feed the pest.

With the advancement in technologies, farmers found it difficulty on dealing with the pests by releasing of the natural enemy due to their decreasing rate. Again, at this phase farmers had to face the situation of how to overcome the rate of pests without the help of natural enemies. Then, they introduced the new technology of controlling the pests by releasing of the man made chemical pesticides. Chemical pesticides

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kills the pests destroying plants which let the farmers tension free. But the farmers came up with the drawbacks of using the chemical pesticides which led the farmers to ponder over such situation. Chemical pesticides such as Organophosphate pesticides, Organochlorine insecticides, Carbamate pesticides etc were few of the pesticides used by the farmers which not only destroys the pests but also has an adverse effect on the environment. So such chemical pesticides with time had to be removed and were replaced by the pesticides which were harmful only to the pests and not the plants or the biosphere. Bio-pesticides were the advantageous method practised by the agriculturists which benefitted both ecologically and biologically to the ecosystem. Thus the farmers thought of a new technique called integrated pest management which involves both biological and chemical control measures impulsively to prevent the plant population. Many mathematical models have been studied by the authors [5, 7, 8, 9, 10, 11, 12] for biological and chemical control of pest population.

Keeping in view of the above discussions we have developed a mathematical model to show the hybrid approach of pest control by impulsive release of natural enemies and chemical pesticides which are not hazardous to the ecosystem. The interaction between the plant and pest population is by Holling Type-2 functional response as it incorporate the time taken by the pest to process the food and searching of prey which differentiates it from Holling Type-1 functional response and given better result. The sequence of paper is as follows: In section 2, mathematical model is formulated. A preliminary section of definition is discussed in section 3. In section 4, boundedness of the system is studied. The local stability of pest extinction equilibrium point is studied in section 5. In section 6, global attractivity of the system is analysed. Permanence of the system is studied in the section 7. In the last section of this paper numerical simulation with the hypothetical parameters is analysed.

2. Formulation of mathematical model

In this section,the model is formulated with the following assumptions:

1. The plant population $x(t)$ is growing logistically with growth rate r and carrying capacity k. The per capita rate at which pest population $y(t)$ captures plant population $x(t)$ is represented by the term $bx(t)/(\beta+x(t))$. Thus, the evolution equation is:
$$\frac{dx(t)}{dt} = rx(t)\left(1-\frac{x(t)}{k}\right) - b\frac{x(t)y(t)}{\beta+x(t)} \tag{1}$$

2. $y(t)$ is the density of pest population in that region of consideration at time t. It is also assume that only mature natural enemy can harvest the pest population. Further, d is the natural death rate of pest population. Thus, the evolution equation is:
$$\frac{dy(t)}{dt} = b\frac{x(t)y(t)}{\beta+x(t)} - dy(t) - \alpha y(t)z_2(t) \tag{2}$$

3. The growth rate of immature natural enemy population $z_1(t)$ is depend on mature natural enemy $z_2(t)$. The death rate and maturity rate of natural enemy is d_3 and

m. Thus, the evolution model is:

$$\frac{dz_1(t)}{dt} = \alpha y(t)z_2(t) - (d_3 + m)z_1(t) \tag{3}$$

4. The mature enemy population $z_2(t)$ are growing due to immature enemy population $z_1(t)$ at a rate m. Further, d_3 is the death rate of mature enemy population. Thus, the evolution model is:

$$\frac{dz_2(t)}{dt} = mz_1(t) - d_3 z_2(t) \tag{4}$$

We will extend this model by releasing amount of immature and mature natural enemies μ_1 and μ_2 at a fixed moment and a impulsive harvesting rate of pests δ. T is the impulsive period.

Hence, the proposed model is:

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= rx(t)\left(1 - \frac{x(t)}{k}\right) - b\frac{x(t)y(t)}{\beta + x(t)} \\ \frac{dy(t)}{dt} &= b\frac{x(t)y(t)}{\beta + x(t)} - dy(t) - \alpha y(t)z_2(t) \\ \frac{dz_1(t)}{dt} &= \alpha y(t)z_2(t) - (d_3 + m)z_1(t) \\ \frac{dz_2(t)}{dt} &= mz_1(t) - d_3 z_2(t) \end{aligned}\right\} t \neq nT \tag{A}$$

$$\left.\begin{aligned} x(t^+) &= x(t) \\ y(t^+) &= (1 - \delta)y(t) \\ z_1(t^+) &= z_1(t) + \mu_1 \\ z_2(t^+) &= z_2(t) + \mu_2 \end{aligned}\right\} t = nT \tag{B}$$

3. Preliminaries

In this section, we give some notations, definitions and Lemmas which will be useful for our main results.

Let us consider $f = (f_1, f_2, f_3, f_4)^T$, defined by the right hand side of the first four equation of system $(A - B)$ and let $V_0 = V(t,x) : R_+ \times R_+^4 \to R_+$, where $R_+ = [0, +\infty)$, $R_+^4 = (x \in R^4 : x \geqslant 0)$ then V is said to belong to class V_0 if

(1) V continuous in $(nT, (n+1)T] \times R_+^4$ and for each $x \in R_+^4$, $n \in N$,

$$\lim_{(t,u)\to(nT^+,x)} V(t,u) = V(nT^+, x)$$

exists.

(2) V is locally Lipschitzian in x.

DEFINITION 3.1. Let $V \in V_0$, then for $(t,x) \in ((nT, (n+1)T] \times R_+^4$, $n \in N$. The upper right derivatives of $V(t,x)$ with respect to the impulsive differential system $(A - B)$ is defined as

$$D^+V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h, x+hf(t,x)] - V(t,x)]$$

REMARK. The smoothness properties of f guarantee the global existence and uniqueness of the solutions of system $(A-B)$ [13, 14].

DEFINITION 3.2. The system $(A-B)$ is permanent if there exist $M \geqslant m > 0$ such that, for any solution $(x(t), y(t), z_1(t), z_2(t))$ of system $(A-B)$ with $x_0 > 0$, $y_0 > 0$, $z_{10} > 0$, $z_{20} > 0$

$$m \leqslant \liminf_{t\to\infty} x(t) \leqslant \limsup_{t\to\infty} x(t) \leqslant M.$$
$$m \leqslant \liminf_{t\to\infty} y(t) \leqslant \limsup_{t\to\infty} y(t) \leqslant M.$$
$$m \leqslant \liminf_{t\to\infty} z_1(t) \leqslant \limsup_{t\to\infty} z_1(t) \leqslant M.$$
$$m \leqslant \liminf_{t\to\infty} z_2(t) \leqslant \limsup_{t\to\infty} z_2(t) \leqslant M.$$

4. Boundedness of the system

In this section, we will discuss the boundedness of the system.

THEOREM 1. *For each solution $(x(t), y(t), z_1(t), z_2(t))$ of $(A-B)$ their exists a constant $M > 0$ such that $x(t) \leqslant M$, $y(t) \leqslant M$, $z_1(t) \leqslant M$ and $z_2(t) \leqslant M$ with t being sufficiently large enough.*

Proof. Let $(x(t), y(t), z_1(t), z_2(t))$ be the solution of $(A-B)$

We define a function $V(t) = x(t) + y(t) + z_1(t) + z_2(t)$. Let $0 < \bar{d} < \min(d, d_3)$ then for $t \neq nT$, we obtain that

$$D^+V(t) + \bar{d}V(t) \leqslant (r+\bar{d})x(t) - \frac{r}{k}x^2(t) \leqslant M_0$$

where

$$M_0 = \frac{k(r+\bar{d})^2}{4r}$$

when $t = nT$, $V(t^+) = V(t) + \mu_1 + \mu_2$. For $t \in (nT, (n+1)T]$, we have

$$V(t) \leqslant V(0)\exp(-\bar{d}t) + \int_0^t M_0 \exp(-\bar{d}(t-s))ds + \sum_{0<nT<t} (\mu_1+\mu_2)\exp(-\bar{d}(t-nT))$$

$$= V(0)\exp(-\bar{d}t) + \frac{M_0(1-\exp(-\bar{d}t))}{\bar{d}}$$

$$+(\mu_1+\mu_2)\frac{\exp(-\bar{d}(t-T)) - \exp(-\bar{d}(t-(n+1)T))}{1-\exp(\bar{d}T)}$$

$$< V(0)\exp(-\bar{d}t) + \frac{M_0}{\bar{d}}(1-\exp(-\bar{d}t))$$

$$+\frac{\mu_1+\mu_2\exp(-\bar{d}(t-T))}{1-\exp(\bar{d}T)} + \frac{\mu_1+\mu_2\exp(-\bar{d}T)}{\exp(\bar{d}T)-1}$$

$$\to \frac{M_0}{\bar{d}} + \frac{(\mu_1+\mu_2)\exp(\bar{d}T)}{\exp(\bar{d}T)-1}$$

as $t \to \infty$.

Thus, $V(t)$ is a uniformly bounded. Hence, by the definition of $V(t)$, their exist a constant $M := (M_0/\bar{d}) + (\mu_1+\mu_2)\exp(\bar{d}T)/(\exp(\bar{d}T)-1) > 0$ such that $x(t) \leqslant M$, $y(t) \leqslant M$, $z_1(t) \leqslant M$, $z_2(t) \leqslant M$, for all t large enough. This completes the proof. $\square$

LEMMA 1. *Consider the following impulsive system*

$$\frac{du}{dt} = c - du(t), \quad t \neq nT \tag{5}$$

$$u(t^+) = u(t) + \mu, \quad t = nT, \quad n = 1,2,3,\ldots$$

Then system (5) *has a positive periodic solution* $\tilde{u}(t)$ *and for any solution* $u(t)$ *of the system* (5)*, we have*

$$|u(t) - \tilde{u}(t)| \to 0, \quad \text{as } t \to \infty,$$

where,for $t \in (nT,(n+1)T]$

$$\tilde{u}(t) = \frac{c}{d} + \frac{\mu\exp(-d(t-nT))}{1-\exp(-dT)}$$

with

$$\tilde{u}(0^+) = \frac{c}{d} + \frac{\mu}{1-\exp(-dT)}$$

Proof. We can easily verify that $\tilde{u}(t)$ is a positive periodic solution (5) with the given initial values. Suppose that $u(t)$ is an arbitrary solution of (5), then we can solve that

$$u(t) = (u(0^+) - \tilde{u}(t))e^{-dt} + \tilde{u}(t), \quad t \in (nT,(n+1)T]$$

Thus, $\lim_{t\to\infty}|u(t) - \tilde{u}(t)| = 0$. This completes the proof. $\square$

Again for the pest extinction we can obtain the following impulsive system:

$$\frac{dz_1(t)}{dt} = -(d_3+m)z_1(t), \quad t \neq nT \tag{6}$$

$$z_1(t^+) = z_1(t) + \mu_1, \quad t = nT \tag{7}$$

$$\frac{dz_2(t)}{dt} = mz_1(t) - d_3z_2(t), \quad t \neq nT \tag{8}$$

$$z_2(t^+) = z_2(t) + \mu_2, \quad t = nT \tag{9}$$

For the system (6), using the lemma (1), it is obtained that

$$\tilde{z_1}(t) = \frac{\mu_1\exp(-(d_3+m)(t-nT))}{1-\exp(-(d_3+m)T)}$$

with

$$\tilde{z_1}(0^+) = \frac{\mu_1}{1-\exp(-(d_3+m)T)}$$

is a positive periodic solution of the system (6), which is globally asymptotically stable. Substituting $\tilde{z_1}(t)$ into (8), we obtain the following subsystem:

$$\frac{dz_2(t)}{dt} = m\tilde{z_1}(t) - d_3 z_2(t), \quad t \neq nT \tag{10}$$

$$z_2(t^+) = z_2(t) + \mu_2, \quad t = nT$$

Integrating the first equation of (10) over the interval $t \in (nT, (n+1)T]$, we have

$$z_2(t) = \frac{\mu_1(\exp(-d_3(t-nT)) - \exp(-(d_3+m)(t-nT)))}{1-\exp(-(d_3+m)T)}$$

$$+z_2(nT^+)\exp(-d_3(t-nT)), \quad nT < t \leqslant (n+1)T.$$

After successive pulse,we can obtain the following stroboscopic map of (10):

$$\begin{aligned} z_2((n+1)T^+) = & \frac{\mu_1(\exp(-d_3(t-nT)) - \exp(-(d_3+m)(t-nT)))}{1-\exp(-(d_3+m)T)} \\ & +z_2(nT^+)\exp(-d_3(t-nT)) + \mu_2 \triangleq f(z_2(nT^+)) \end{aligned} \tag{11}$$

It is easy to check that (11) has a unique positive fixed point

$$z_2^* = \frac{\mu_1 \exp(-d_3T)(1-\exp(-mT))}{1-\exp(-(d_3+m)T)(1-\exp(-d_3T))} + \frac{\mu_2}{1-\exp(-d_3T)},$$

which satisfies $z_2 < f(z_2) < z_2^*$ if $0 < z_2 < z_2^*$ and $z_2^* < f(z_2) < z_2$ if $z_2 > z_2^*$. Thus the corresponding periodic solution of the system (10) in the interval $(nT, (n+1)T]$ is

$$\tilde{z_2}(t) = -\frac{\mu_1 \exp(-(d_3+m)(t-nT))}{1-\exp(-(d_3+m)T)} + \frac{(\mu_1+\mu_2)\exp(-d_3(t-nT))}{1-\exp(-d_3T)}$$

with initial value

$$\tilde{z_1}(0^+) = -\frac{\mu_1}{1-\exp(-(d_3+m)T)} + \frac{(\mu_1+\mu_2)}{1-\exp(-d_3T)}$$

which is globally asymptotically stable.

Moreover, due to pest eradication we can also consider the following subsystem of $(A-B)$:

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{k}\right) \tag{12}$$

Which is a logistic differential equation, thus from well known properties of logistic differential equation (12), there exists a unstable equilibrium $x = 0$ and a globally asymptotically stable equilibrium $x = k$. Hence the system $(A-B)$ has following two periodic solutions: plant-pest-extinction periodic solution $(0, 0, \tilde{z_1}(t), \tilde{z_2}(t))$ and pest-extinction periodic solution $(k, 0, \tilde{z_1}(t), \tilde{z_2}(t))$.

5. Local stability

In this section, we will be discussing the local stability of plant-pest eradication periodic solution and pest eradication periodic solution of the system $(A-B)$ in the following theorem.

THEOREM 2. *Let* $(x(t), y(t), z_1(t), z_2(t))$ *be any solution of equation* $(A-B)$ *then,*
(i) The plant-pest eradication periodic solution $(0, 0, \tilde{z_1}(t), \tilde{z_2}(t))$ *is unstable.*
(ii) The pest eradication periodic solution $(k, 0, \tilde{z_1}(t), \tilde{z_2}(t))$ *is locally asymptotically stable if and only if* $T < T_{\max}$*, where*

$$T_{\max} = \frac{1}{\left(\frac{b}{\beta+k} - d\right)} \left(\frac{\alpha d_3 \mu_2 + m\alpha(\mu_1 + \mu_2)}{d_3(d_3+m)} - \log(1-\delta) \right)$$

provided $\frac{b}{\beta+k} > d$*.*

Proof. (i) For the local stability of periodic solution $(0, 0, \tilde{z_1}(t), \tilde{z_2}(t))$, we define $x(t) = \phi_1(t)$, $y(t) = \phi_2(t)$, $z_1(t) = \tilde{z_1}(t) + \phi_3(t)$, $z_2(t) = \tilde{z_2}(t) + \phi_4(t)$ where $\phi_i(t)$, $i = 1, 2, 3, 4$ are small amplitude perturbation of the solution respectively, then the system $(A-B)$ can be expanded in the following linearized form:

$$\left.\begin{aligned} \frac{d\phi_1(t)}{dt} &= \phi_1(t) \\ \frac{d\phi_2(t)}{dt} &= -(d + \alpha\tilde{z_2}(t))\phi_2(t) \\ \frac{d\phi_3(t)}{dt} &= \alpha\tilde{z_2}(t)\phi_2(t) - (d_3+m)\phi_3(t) \\ \frac{d\phi_4(t)}{dt} &= m\phi_3(t) - d_3\phi_4(t) \end{aligned}\right\} t \neq nT \tag{C}$$

$$\left.\begin{aligned} \phi_1(t^+) &= \phi_1(t) \\ \phi_2(t^+) &= (1-\delta)\phi_2(t) \\ \phi_3(t^+) &= \phi_3(t) \\ \phi_4(t^+) &= \phi_4(t) \end{aligned}\right\} t = nT \tag{D}$$

Let $\phi(t)$ be the fundamental matrix of $(C-D)$, then $\phi(t)$ must satisfy

$$\frac{d\phi(t)}{dt} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(d+\alpha\tilde{z_2}(t)) & 0 & 0 \\ 0 & \alpha\tilde{z_2}(t) & -(d_3+m) & 0 \\ 0 & 0 & m & -d_3 \end{bmatrix} \phi(t) = A\phi(t) \tag{13}$$

Thus, the monodromy matrix of $(C-D)$ is

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \phi(t)$$

From (13), we obtain that $\phi(T) = \phi(0)\exp(\int_0^T A dt)$, where $\phi(0)$ is the identity matrix. Then the monodromy matrix M have the following eigen values:

$$\lambda_1 = \exp(T) > 1$$

$$\lambda_2 = (1-\delta)\exp\left(-\int_0^T (d+\alpha\tilde{z_2}(t))dt\right) < 1$$

$$\lambda_3 = \exp(-(d_3+m)T) < 1$$

$$\lambda_4 = \exp(-d_3 T) < 1$$

Thus from the Floquet theory of impulsive differential equation,the plant pest eradication periodic solution of the system $(C-D)$ is unstable since $|\lambda_1| > 1$.

(ii) The local stability of periodic solution $(k, 0, \tilde{z_1}(t), \tilde{z_2}(t))$ may be establish in the similar manner as in the previous case. Now, we define $x(t) = 1+\phi_1(t)$, $y(t) = \phi_2(t)$, $z_1(t) = \tilde{z_1}(t)+\phi_3(t)$, $z_2(t) = \tilde{z_2}(t)+\phi_4(t)$, then the system $(A-B)$ can be expanded in the following linearized form:

$$\left.\begin{aligned} \frac{d\phi_1(t)}{dt} &= -r\phi_1(t) - \frac{b}{\beta+k}\phi_2(t) \\ \frac{d\phi_2(t)}{dt} &= (\frac{b}{\beta+k} - d - \alpha\tilde{z_2}(t))\phi_2(t) \\ \frac{d\phi_3(t)}{dt} &= \alpha\tilde{z_2}(t)\phi_2(t) - (d_3+m)\phi_3(t) \\ \frac{d\phi_4(t)}{dt} &= m\phi_3(t) - d_3\phi_4(t) \end{aligned}\right\} t \neq nT \qquad (E)$$

$$\left.\begin{aligned} \phi_1(t^+) &= \phi_1(t) \\ \phi_2(t^+) &= (1-\delta)\phi_2(t) \\ \phi_3(t^+) &= \phi_3(t) \\ \phi_4(t^+) &= \phi_4(t) \end{aligned}\right\} t = nT \qquad (F)$$

The fundamental matrix $\phi(t)$ of $(E-F)$ must satisfy

$$\frac{d\phi(t)}{dt} = \begin{bmatrix} -r & \frac{-b}{\beta+k} & 0 & 0 \\ 0 & (\frac{b}{\beta+k} - d - \alpha\tilde{z_2}(t)) & 0 & 0 \\ 0 & \alpha\tilde{z_2}(t) & -(d_3+m) & 0 \\ 0 & 0 & m & -d_3 \end{bmatrix} \phi(t) \qquad (14)$$

Thus, the monodromy matrix of $(E-F)$ is given as:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \phi(T),$$

which has the eigenvalues

$$\lambda_1 = \exp(-rT) < 1,$$

$$\lambda_2 = (1-\delta)\exp\left(\int_0^T \left(\frac{b}{\beta+k} - d - \alpha\tilde{z_2}(t)\right)dt\right),$$

$$\lambda_3 = \exp(-(d_3+m)T) < 1,$$

$$\lambda_4 = \exp(-d_3T) < 1.$$

Therefore,according to the Floquet theory of impulsive differential equation, the pest eradication periodic solution of the system $(E-F)$ is locally asymptotically stable if and only if $|\lambda_2| \leqslant 1$, i.e, $T \leqslant T_{\max}$,since correspondance to λ_2, $T_{\max}$ expression has a simple elementary divisor. Hence the proof of the theorem. □

6. Global attractivity

In this section, we will be discussing the global attractivity of pest eradication periodic solution in the following theorem.

THEOREM 3. *Let $(x(t),y(t),z_1(t),z_2(t))$ be any solution of equation $(A-B)$. Then, the pest extinction periodic solution $(k,0,\tilde{z_1}(t),\tilde{z_2}(t))$ of $(A-B)$ is globally attractive provided that $T < T_{\max}$ holds.*

Proof. If $(x(t),y(t),z_1(t),z_2(t))$ is any solution of $(A-B)$, then equation (1) can be rewritten as

$$\frac{dx(t)}{dt} \leqslant rx(t)\left(1-\frac{x(t)}{k}\right)$$

which implies that $\lim_{t\to\infty}\sup x(t) = k$. Thus there exists an integer $k_1 > 0$ such that if $t > k_1$ then $x(t) < k+\varepsilon_0$. Now, if we consider third and fourth equation of the system A,

$$\frac{dz_1(t)}{dt} \geqslant -(d_3+m)z_1(t), \quad t \neq nT, \tag{15}$$

$$z_1(t^+) = z_1(t)+\mu_1, \quad t = nT$$

Then by considering the following comparison system:

$$\frac{dw_1(t)}{dt} = -(d_3+m)w_1(t), \quad t \neq nT, \tag{16}$$

$$w_1(t^+) = w_1(t)+\mu_1, \quad t = nT \tag{17}$$

Using the lemma (1), we obtain that the system (16)–(17) has a periodic solution

$$\tilde{w_1}(t) = \frac{\mu_1\exp(-(d_3+m))(t-nT)}{1-\exp(-(d_3+m)T)}, \quad nT < t \leqslant (n+1)T, \quad n \in Z_+,$$

which is globally asymptotically stable. In view of lemma (1) and comparison theorem of impulsive equation [13], we have $z_1(t) \geqslant w_1(t)$ and $w_1(t) \to \tilde{w_1}(t)$ as $t \to \infty$. Then there exists an integer $k_2 > k_1$, $t > k_2$ such that

$$z_1(t) \geqslant w_1(t) > \tilde{z_1}(t)-\varepsilon_0, \quad nT < t < (n+1)T, \quad n > k_2. \tag{18}$$

Incorporating (18) with the equation (3), we obtain the following subsystem:

$$\frac{dz_2(t)}{dt} \geqslant m\left(\frac{\mu_1\exp(-(d_3+m))(t-nT)}{1-\exp(-(d_3+m)T)} - \varepsilon_0\right) - d_3 z_2(t), \quad t \neq nT, \tag{19}$$

$$z_2(t^+) = z_2(t) + \mu_2, \quad t = nT.$$

Now, we consider the comparison system of (19) as follows:

$$\frac{dw_2(t)}{dt} = m\left(\frac{\mu_1\exp(-(d_3+m))(t-nT)}{1-\exp(-(d_3+m)T)} - \varepsilon_0\right) - d_3 w_2(t), \quad t \neq nT, \tag{20}$$

$$w_2(t^+) = w_2(t) + \mu_2, \quad t = nT.$$

In the previous manner, we obtain that the system (19)–(20) has a periodic solution

$$\tilde{w_2}(t) = -\frac{\mu_1\exp(-(d_3+m))(t-nT)}{1-\exp(-(d_3+m)T)} + \frac{(\mu_1+\mu_2)\exp(-d_3(t-nT))}{1-\exp(-d_3T)} - \frac{\alpha m\varepsilon_0}{d_3},$$
$$nT < T \leqslant (n+1)T,$$

which is globally asymptotically stable. By using comparison theorem of impulse equation[13], we have $z_2(t) \geqslant w_2(t)$ and $w_2(t) \to \tilde{w_2}(t)$ as $t \to \infty$. Thus there exists an integer $k_3 > k_2$, $t > k_3$ such that

$$z_2(t) \geqslant w_2(t) > \tilde{z_2}(t) - \varepsilon_0, \quad nT < t < (n+1)T, \quad t > k_3. \tag{21}$$

From the system $(A-B)$, we get

$$\frac{dy(t)}{dt} \leqslant y(t)\left[\frac{b\varepsilon_0}{\beta} - d - \alpha(\tilde{z_2}(t) - \varepsilon_0)\right], \quad t \neq nT, \tag{22}$$

$$y(t^+) = (1-\delta)y(t), \quad t = nT$$

Integrating the equation (22) between the pulse, we have

$$y(t) \leqslant y(nT^+)\exp\left(\int_{nT}^{(n+1)T}\left[\frac{b\varepsilon_0}{\beta} - d - \alpha(\tilde{z_2}(t) - \varepsilon_0)\right]dt\right)$$

After the successive pulse, we can obtain the following stroboscopic map

$$y((n+1)T^+) \leqslant (1-\delta)y(nT^+)\exp\left(\int_{nT}^{(n+1)T}\left[\frac{b\varepsilon_0}{\beta} - d - \alpha(\tilde{z_2}(t) - \varepsilon_0)\right]dt\right) \tag{23}$$

$$= (1-\delta)y(nT^+)\sigma \tag{24}$$

where

$$\sigma = \exp\left(\int_{nT}^{(n+1)T}\left[\frac{b\varepsilon_0}{\beta} - d - \alpha(\tilde{z_2}(t) - \varepsilon_0)\right]dt\right) < 1,$$

as $T < T_{\max}$, therefore, for $\varepsilon_0 > 0$, we get that

$$(1-\delta)\exp\left(\left(\frac{b\varepsilon_0}{\beta} - d + \frac{\alpha m\varepsilon_0}{d_3} + \alpha\varepsilon_0\right)T + \frac{\alpha\mu_1}{d_3+m} - \frac{\alpha(\mu_1+\mu_2)}{d_3}\right) < 1$$

Thus $y(nT^+) \leqslant (1-\delta)y(0^+)\sigma^n$ and $y(nT^+) \to 0$ as $n \to \infty$. Therefore $y(t) \to 0$ as $n \to \infty$. Therefore, there exists an $\varepsilon_1 > 0$ (sufficiently small) such that $0 < y(t) < \varepsilon_1$ for all t large enough.

Again, for $(A-B)$, we have

$$\frac{dx(t)}{dt} \geqslant x\left(r - \frac{rx(t)}{k} - \frac{b\varepsilon_1}{\beta}\right), \quad t \neq nT$$

which implies that $\lim_{t\to\infty}\inf x(t) = k$ i.e $x(t) \to k$ as $t \to \infty$. Thus from the equation $(A-B)$, we have

$$\frac{dz_1(t)}{dt} \leqslant \alpha\varepsilon_1 M - (d_3+m)z_1(t), \quad t \neq nT, \tag{25}$$

$$z_1(t^+) = z_1(t) + \mu_1, \quad t = nT$$

In the previous manner, using the comparison theorem for impulsive equations [13] and lemma (1), there exists an $\varepsilon_2 > 0$ (sufficiently small) such that

$$z_1(t) < \frac{\alpha\varepsilon_1 M}{d_3+m} + \frac{\mu_1\exp(-(d_3+m))(t-nT)}{1-\exp(-(d_3+m)T)} - \varepsilon_2$$

for t large enough. From the equation $(A-B)$, we obtain the following subsystem

$$\frac{dz_2(t)}{dt} \leqslant m\left(\frac{\alpha\varepsilon_1 M}{d_3+m} + \frac{\mu_1\exp(-(d_3+m))(t-nT)}{1-\exp(-(d_3+m)T)} - \varepsilon_2\right) - d_3 z_2(t), \quad t \neq nT \tag{26}$$

$$z_2(t^+) = z_2(t) + \mu_2, \quad t = nT$$

In the similar, we obtain that there exists an $\varepsilon_3 > 0$ such that

$$\begin{aligned} z_2(t) < &-\frac{\mu_1\exp(-(d_3+m))(t-nT)}{1-\exp(-(d_3+m)T)} + \frac{(\mu_1+\mu_2)\exp(-d_3(t-nT))}{1-\exp(-d_3T)} \\ &+\frac{m}{d_3}\left(\frac{\alpha\varepsilon_1 M}{d_3+m} + \varepsilon_2\right) + \varepsilon_3, \end{aligned} \tag{27}$$

$nT < t \leqslant (n+1)T$, $t > k_3$. Which evidences that $z_1(t) \to \tilde{z_1}(t)$ and $z_2(t) \to \tilde{z_2}(t)$ as $t \to \infty$. This completes the proof. □

7. Permanence

The permanence of the system is $(A-B)$ is stated below with the theorem:

THEOREM 4. *The system $(A-B)$ is permanent if $T > T_{\max}$.*

Proof. Suppose $(p(t), q(t), r_1(t), r_2(t))$ is the solution of the system $(A-B)$. From the boundedness of the solution, we saw that $x(t) \leqslant M$, $y(t) \leqslant M$, $z_1(t) \leqslant M$ and $z_2(t) \leqslant M$ for all large t. From equation (1), we have $\frac{dx}{dt} \geqslant (r - \frac{bM}{\beta} - \frac{rx}{k})x$ which implies that $x(t) > r - \frac{bM}{\beta} \triangleq m_1$ for all t large enough. For sufficiently small $\varepsilon_4 > 0$, we choose $m_1 = r - \varepsilon_4 > 0$ and also define

$$m_2 = \frac{-\mu_1 \exp(-(d_3+m)(t-nT))}{1-\exp(-(d_3+m)T)} - \varepsilon_4 > 0$$

$$m_3 = \frac{-\mu_1 \exp(-(d_3+m)(t-nT))}{1-\exp(-(d_3+m)T)} + \frac{(\mu_1+\mu_2)\exp(-d_3(t-nT))}{1-\exp(-d_3T)} - \frac{\varepsilon_4 m}{d_3} - \varepsilon_4 > 0$$

Now, the system can be rewritten as:

$$\frac{dz_1(t)}{dt} = -(d_3+m)z_1(t), \quad t \neq nT \tag{28}$$

$$\frac{dz_2(t)}{dt} = mz_1(t) - D_3 z_2(t), \quad t \neq nT \tag{29}$$

$$z_1(t^+) = z_1(t) + \mu_1, \quad t = nT, \quad n = 1,2,3\dots \tag{30}$$

$$z_2(t^+) = z_2(t) + \mu_2, \quad t = nT, \quad n = 1,2,3\dots \tag{31}$$

The system (28)–(31) is same as (6)–(9). Thus, using the same approach, we can easily obtain that $z_1(t) > m_2$ and $z_2(t) > m_3$ for all t large enough. Therefore, for the permanence of the system, we only need to find $m_4 > 0$, such that $q(t) \geqslant m_4$ for large enough t. We prove this result in two steps.

Step 1. Firstly assume that $y(t) \geqslant m_4$ is not true, then there exists a $t_1 \in (0,\infty)$ such that $y(t) < m_4$ for all $t > t_1$. Using this assumption, we get the following subsystem of the system $(A-B)$:

$$\frac{dz_1(t)}{dt} \leqslant \alpha M m_4 - (d_3+m)z_1(t), \quad t \neq nT$$

$$z_1(t^+) = z_1(t) + \mu_1, \quad t = nT, \quad n = 1,2,3\dots$$

Consider the following Comparison system:

$$\frac{d\bar{w}_1(t)}{dt} \leqslant \alpha M m_4 - (d_3+m)\bar{w}_1(t), \quad t \neq nT \tag{32}$$

$$\bar{w}_1(t^+) = \bar{w}_1(t) + \mu_1, \quad t = nT, \quad n = 1,2,3\dots \tag{33}$$

Using lemma (1), the system (32)–(33) has a periodic solution

$$\tilde{\bar{w}}_1(t) = \frac{\alpha m_4 M}{d_3+m} + \frac{\mu_1 \exp(-(d_3+m)(t-nT))}{1-\exp(-(d_3+\mu_0)T)}$$

which is globally asymptotically stable. Then, there exists an $\varepsilon_5 > 0$ such that

$$z_1(t) \leqslant \tilde{\bar{w}}_1(t) < \frac{\alpha m_4 M}{d_3+m} + \frac{\mu_1 \exp(-(d_3+m)(t-nT))}{1-\exp(-(d_3+m)T)} + \varepsilon_5 > 0$$

for t large enough. Thus we obtain the following subsystem of $(A-B)$:

$$\frac{dz_2(t)}{dt} = \mu_0\left(\frac{\alpha m_4 M}{d_3+m} + \frac{\mu_1 \exp(-(d_3+m)(t-nT))}{1-\exp(-(d_3+m)T)} + \varepsilon_5\right)(t) - d_3 z_2(t), \quad t \neq nT \tag{34}$$

$$z_2(t^+) = z_2(t) + \mu_2, \quad t = nT, \quad n = 1,2,3\ldots \tag{35}$$

Consider the comparison system (34)–(35) as follows:

$$\frac{d\bar{w}_2(t)}{dt} = m\left(\frac{\alpha m_4 M}{d_3+m} + \frac{\mu_1 \exp(-(d_3+m)(t-nT))}{1-\exp(-(d_3+m)T)} + \varepsilon_5\right)(t) - d_3 \bar{w}_2(t), \quad t \neq nT \tag{36}$$

$$\bar{w}_2(t^+) = \bar{w}_2(t) + \mu_2, \quad t = nT, \quad n = 1,2,3\ldots \tag{37}$$

In the similar manner, the system (36)–(37) also has a periodic solution

$$\tilde{\bar{w}}_2(t) < \frac{-\mu_1 \exp(-(d_3+m)(t-nT))}{1-\exp(-(d_3+m)T)} + \frac{(\mu_1+\mu_2)\exp(-d_3(t-nT))}{1-\exp(-d_3 T)} + \frac{m}{d_3}\left(\frac{\alpha m_4 M \exp(-d_1\tau)}{(d_3+m)} + \varepsilon_5\right) \tag{38}$$

which is globally asymptotically stable and there exists an $\varepsilon_6 > 0$ such that,

$$z_2(t) < \tilde{\bar{w}}_2(t) < \frac{-\mu_1 \exp(-(d_3+m)(t-nT))}{1-\exp(-(d_3+m)T)} + \frac{(\mu_1+\mu_2)\exp(-d_3(t-nT))}{1-\exp(-d_3 T)} + \frac{m}{d_3}\left(\frac{\alpha m_4 M}{(d_3+m)} + \varepsilon_5\right) + \varepsilon_6.$$

It follows that there exists a $T_1 > 0$ such that for $nT < t \leqslant (n+1)T$, we have the following subsystem of $(A-B)$:

$$\frac{dy(t)}{dt} \geqslant \left[\frac{bm_1}{\beta} - d - \alpha(\tilde{\bar{w}}_2(t) + \varepsilon_6)\right] y(t), \quad t \neq nT \tag{39}$$

$$y(t^+) = (1-\delta)q(t), \quad t = nT, \quad \text{and}, \quad t > T_1 \tag{40}$$

Integrating the equation (39) on $(nT,(n+1)T]$, $n \geqslant N_1$ (here, N_1 is the nonnegative integer and $N_1 T \geqslant T_1$), then we obtain that

$$\begin{aligned} y((n+1)T) &\geqslant (1-\delta)q(nT^+)\exp\left(\int_{nT}^{(n+1)T}\left(\frac{bm_1}{\beta} - d - \alpha(\tilde{\bar{w}}_2(t)+\varepsilon_6)dt\right)\right) \\ &= q(nT^+)\bar{\sigma} \end{aligned}$$

where $\bar{\sigma} = (1-\delta)\exp(\int_{nT}^{(n+1)T}(\frac{bm_1}{\beta} - d - \alpha(\tilde{\bar{w}}_2(t) + \varepsilon_6)dt) > 1$, as, $T > T_{\max}$, therefore, for ε_5, where, $\varepsilon_5 > 0$, it is obtained that,

$$\left(\frac{bm_1}{\beta} - d - \alpha\varepsilon_6\right)T - \frac{\alpha m}{d_3}\left(\frac{\alpha m_4 M}{d_3+m} + \varepsilon_5\right)T - \alpha\left(\frac{\mu_1+\mu_2}{d_3} - \frac{m_1}{d_3+m}\right) - \log\left(\frac{1}{1-\delta}\right) > 1.$$

Thus, $y((N_1+k)T) \geqslant y(N_1T^+)\bar{\sigma}^k \to \infty$ as $k \to \infty$, which is a contradiction of our assumption that $y(t) < m_4$, for every $t > t_2$. Hence there exists a $t_2 > t_1$ such that $y(t_2) \geqslant m_4$.

Step 2. If $y(t) \geqslant m_4$ for all $t \geqslant t_2$, then our aim will be fulfilled. Otherwise $y(t) < m_4$ for some $t > t_2$. Let $t^* = \inf\{t \mid y(t) < m_4, t > t_2\}$, then there will be two cases:

Case 1. Let $t^* = n_1T$, n_1 is some positive integer. In this case $q(t) \geqslant m_4$ for $t \in [t_2, t^*)$ and $(1-\delta)m_4 \leqslant y\ (t^{*+} = (1-\delta)y(t^*) < m_4)$. Assume that $T_2 = n_2T + n_3T$, where $n_2 = n_2' + n_2''$, n_2', n_2'' and n_3 satisfy the following inequalities:

$$n_2'T > -\frac{1}{D_3+\mu_0} ln \frac{\varepsilon_5}{M+\mu_1},$$

$$n_2''T > -\frac{1}{D_3+\mu_0} ln \frac{\varepsilon_6}{M+\mu_2},$$

$$(1-\delta)^{n_2+n_3}\exp(\eta n_2 T)\exp(n_3\sigma) > 1,$$

$\eta = \frac{bm_1}{\beta} - d - \alpha M < 0$. Now, we claim that there exists a time $t_2' \in (t^*, t^* + T_2)$ such that $y(t_2') \geqslant m_4$, if it is not true, then $y(t_2') < m_4$, $t_2' \in (t^*, t^* + T_2)$. If the system (32)–(33) is considered with initial value $\bar{w}_1(t^{*+}) = z_1(t^{*+})$, then using lemma 1 for $t \in (nT, (n+1)T]$, we have

$$\bar{w}_1(t) = \left(\bar{w}_1(t^{*+}) - \frac{\alpha m_4 M}{D_3+m} + \frac{\mu_1}{1-\exp(-(D_3+m)T)}\right)\exp(-(D_3+m)(t-t^*)) + \tilde{\bar{w}}_1(t),$$

for $n_1 \leqslant n \leqslant n_1 + n_2 + n_3$.

Which evidences that $\mid \bar{w}_1(t) - \tilde{\bar{w}}_1(t) \mid \leqslant (M+\mu_1)\exp(-(D_3+m)(t-n_1T)) < \varepsilon_5$, and $z_1(t) \leqslant \bar{w}_1(t) < \tilde{\bar{w}}_1(t) + \varepsilon_5$ for $t^* + n_2'T \leqslant t \leqslant t^* + T_2$.

Now, we consider the system (32)–(33) with initial values $\bar{w}_2(t^* + n_2'T) = q_2(t^* + n_2'T) \geqslant 0$. Again using lemma 1, we obtain that $\mid \bar{w}_1(t) - \tilde{\bar{w}}_1(t) \mid < (M+\mu_2)\exp(D_3(t - (n_1 + N_2')T)) < \varepsilon_6$, and $r_2(t) \leqslant \bar{w}_2(t) < \tilde{\bar{w}}_2(t) + \varepsilon_6$ for $t^* + n_2'T + n_2''T \leqslant t \leqslant t^* + T_2$, which leads that system (39)–(40) holds for $[t^* + n_2T, t^* + T_2]$.

Integrating system (39)–(40) on $[t^* + n_2T, t^* + T_2]$, we have,

$$y((n_1+n_2+n_3)T) \geqslant y((n_1+n_2)T)(1-\delta)^{n_3}\exp(n_3\sigma) \tag{41}$$

In addition from the system, we also have,

$$\frac{dy(t)}{dt} \geqslant \left[\frac{bm_1}{\beta} - d - \alpha M\right] y(t), \quad t \neq nT \tag{42}$$

$$y(t^+) = (1-\delta)y(t), \quad t = nT, \quad n = 1,2,3\ldots \tag{43}$$

On integrating (42)–(43) in the interval $[T^*,(n_1+n_2)T]$, it is obtained that,

$$y((n_1+n_2)T) \geqslant m_4(1-\delta)^{n_2}\exp(\eta n_2 T) \tag{44}$$

Now substitute (44) into (41), we get that

$$y((n_1+n_2+n_3)T) \geqslant m_4(1-\delta)^{n_2+n_3}\exp(n_3\sigma)\exp(\eta n_2 T) > m_4 \tag{45}$$

which is a contradiction, so there exists a tim $t_2{}' \in [t^*, t^*+T_2]$ such that $y_2{}' \geqslant m_4$. Let $\hat{t} = \inf\{t \mid t \geqslant t^*, y(t) \geqslant m_4\}$, since $0 < \delta < 1$, $y(nT^+) = (1-\delta)y(nT) < y(nT)$ and $y(t) < m_4$, $t \in (t^*,\hat{t})$. Thus, $q(\hat{t}) = m_4$. Suppose $t \in (t^*+(l-1)T, T^*+lT]$ (l is a positive integer) and $l \leqslant n_2+n_3$, from the system (42)–(43), we have,

$$y(t) \geqslant y(t^*+(l-1)T)\exp(\eta(t-t^*-(l-1))T)$$

$$y(t) \geqslant y(nT^+)\exp(\eta T(l-1))(1-\delta)^{l-1}\exp(\eta T)$$

$$y(t) \geqslant m_4(1-\delta)^l\exp(l\eta T)$$

$$y(t) \geqslant m_4(1-\delta)^{n_2+n_3}\exp((n_2+n_3)\eta T) \triangleq \bar{m}_4$$

for $t > \hat{t}$. The same argument can be continued since $q(\hat{t}) \geqslant m_4$. Hence $y(t) \geqslant \bar{m}_4$ for all $t > t_2$.

Case 2. If $t^* \neq nT$, then $y(t^*) = m_4$ and $y(t) \geqslant m_4$, $t \in [t_2, t^*]$. Suppose $t^* \in (n_1{}'T, (n_1{}'+1)T]$, we also have two subcases for $t \in [t^*, (n_1{}'+1)T]$ as follows:

Case a. $q(t) \leqslant m_4$, $t \in [t^*, (n_1{}'+1)T]$, we claim that there exists a $t_3 \in [(n_1{}'+1)T, (n_1{}'+1)T+T_2]$ such that $y(t_3) > m_4$. Otherwise, integrating system (42)–(43) on the interval $[(n_1{}'+1+n_2)T, (n_1{}'+1+n_2+n_3)T]$, we have

$$y((n_1{}'+1+n_2+n_3)T) \geqslant y((n_1{}'+1+n_2)T)(1-\delta)^{n_3}\exp(n_3\sigma)$$

Since $y(t) \leqslant m_4$, $t \in [t^*, (n_1{}'+1)T]$, and therefore, (53) holds on $[t^*, (n_1{}'+n_2+n_3)T]$.

Thus,

$$y((n_1{}'+1+n_2)T) = y(t^*)(1-\delta)^{n_2}\exp(\eta(n_1{}'+1+n_2)T - t^*)$$

$$y((n_1{}'+1+n_2)T) \geqslant m_4(1-\delta)^{n_2}\exp(\eta n_2 T)$$

and

$$y((n_1{}'+1+n_2+n_3)T) \geqslant m_4(1-\delta)^{n_2+n_3}\exp(\eta n_2 T)\exp(n_3\sigma) > m_4$$

which is a contradiction. Let $\check{t} = \inf\{t \mid y(t) \geqslant m_4, t > t^*\}$, then $y(\check{t}) = m_4$ and $q(t) < m_4$, $t \in (t^*, \check{t})$. Choose $t \in (n_1{}'T + (l'-1)T, n_1{}'T + l'T] \subset (t^*, \check{t})$, l' is a positive integer and $l' < 1+n_2+n_3$, we have,

$$y(t) \geqslant y((n_1{}'+l'-1)T^+)\exp(\eta(t-(n_1{}'+l'-1)T))$$

$$y(t) \geqslant (1-\delta)^{l'-1}y(t^*)\exp(\eta(t-t^*))$$

$$y(t) \geqslant m_4(1-\delta)^{n_2+n_3}\exp(\eta(n_2+n_3+1)T)$$

so we have $y(t) \geqslant \bar{m_4}$ for $t \in (t^*, \breve{t})$. For $t > \breve{t}$, the same argument can be continued since $y(\breve{t}) \geqslant m_4$.

Case b. If there exists a $t \in \{t^*, (n_1{}' + 1)T\}$ such that $y(t) \geqslant m_4$. Let $\breve{t} = \inf\{t \mid y(t) \geqslant m_4, t > t^*\}$, then $y(t) < m_4$ for $t \in [t^*, \bar{t})$ and $y(\bar{t}) = m_4$. For $t \in [t^*, \bar{t})$ (42)–(43) holds and integrating (42)–(43) on $t^*, \breve{t}$, we have

$$y(t) \geqslant y(t^*) \geqslant \exp(\eta(t-t^*)) \geqslant m_4 \exp(\eta T) > \bar{m_4}$$

Since $y(\hat{t}) \geqslant m_4$ for $t > \hat{t}$, the same argument can be continued. Hence, we have $y(t) \geqslant \bar{m}_4$ for all $t > t_2$. Thus in both cases, we can conclude that $y(t) \geqslant \bar{m}_4$ for all $t \geqslant t_2$. □

8. Numerical simulation and discussions

In this section, we have taken the data's per week as we are dealing with insect population having a very short life cycle. Our aim is to validate our analytical results numerically. We have considered numerical values for the following set of parameters. The parameter values are chosen in some natural realistic senses:

p	v
r	0.1
K	1
d	0.5
b	0.2
α	0.6
d_3	0.4
m	0.2

Then, the threshold point $(T_{\max})$ for the parameters per week is $T_{\max} = 0.8$. Pest extinction periodic solution $(k, 0, \tilde{z}_1(t), \tilde{z}_1(t))$ is globally stable if $T = 0.5 < T_{\max}$ as stated above in the theorem 3 (Fig 1–4). Further, for the permanence of the system $(A-B)$ is justified as $T = 4 > T_{\max}$ (Fig 5–8) as stated in theorem 4. Due to Holling Type 2 functional response the pests are eradicating quiet early in comparison to Holling Type 1 functional response. Further, if we increase the amount of natural enemies impulsively then pest population will gradually extinct again.

Additionally, when chemical pesticides are not capable enough to suppress the pests, then in an adequate amount we release mature and immature natural enemies after every fixed interval of time with chemical pesticides. Thus, when chemical pesticides and natural enemies are fused together to release in an integrated form, then there is an reduction in the pests rather than practicing any of it.

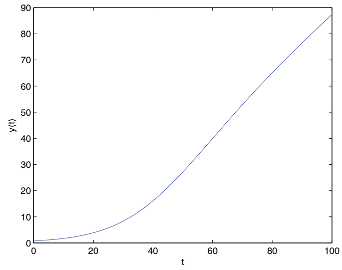

Figure 1: *The graph for the population of plants* $(x(t))$ *is stable*

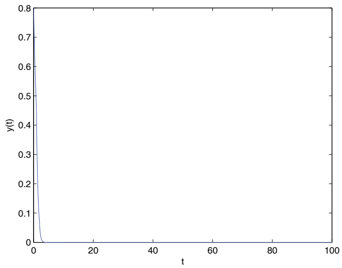

Figure 2: *Pest* $(y(t))$ *perishes away*

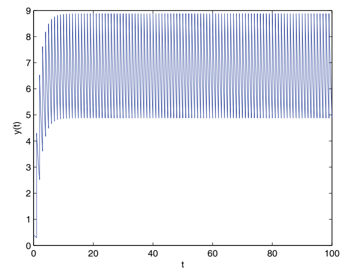

Figure 3: *Bifurcating behaviour for the* $z_1(t)$ *depicting the existence of the immature natural enemy*

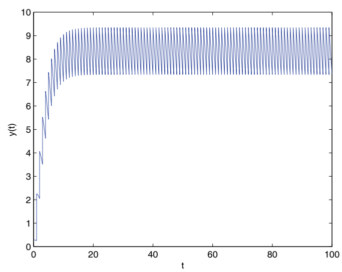

Figure 4: *Bifurcating periodic solution occurs for mature natural enemy* $(z_2(t))$

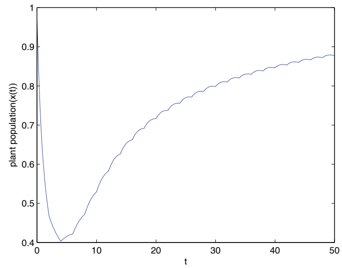

Figure 5: *Plant population* $(x(t))$ *exists for the permanence of the system*

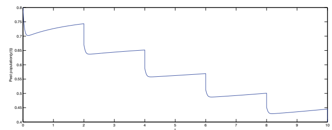

Figure 6: *Pest population* $(y(t))$ *survives for the permanence of the system*

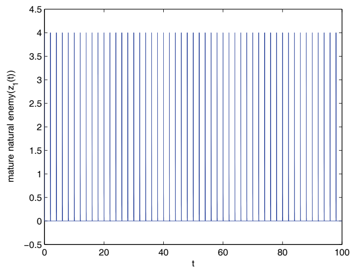

Figure 7: *For the permanence of the system immature natural enemies* $(z_1(t))$ *exists*

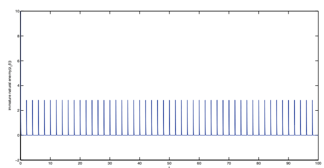

Figure 8: *Bifurcating behaviour occurs for mature natural enemies* $(z_2(t))$ *depicting the existence of the solution*

9. Conclusion

In this paper, we have studied the dynamics of plant-pest-natural enemy using functional response 2 and impulsive perturbation. We obtained the threshold point for pest extinction and permanence of the system depending on the pulse releasing amount of natural enemies and chemical pesticides. When $T < T_{\max} = 0.8$, then the pest extinction equilibrium point $(k, 0, \tilde{z}_1(t), \tilde{z}_1(t))$ is globally stable (Fig 1–4). If $T > T_{\max} = 0.8$, then the system is permanent (Fig 5–8). Thus, we can conclude that various control measures should be applied collectively for the eradication of pest. This will also help us economically as it will be more cost efficient.

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Vandana Kumari
Amity Institute of Applied Science
Amity University
Sector-125, Noida, India
e-mail: vandanakumari@gmail.com

Sudipa Chauhan
Amity Institute of Applied Science
Amity University
Sector-125, Noida, India
e-mail: sudipachauhan@gmail.com

Sumit Kaur Bhatia
Amity Institute of Applied Science
Amity University
Sector-125, Noida, India
e-mail: sumit2212@gmail.com

Joydip Dhar
Atal Bihari Vajpayee-IIITM
Gwalior
e-mail: jdhar@iiitm.ac.in