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SQUARE MEAN ALMOST AUTOMORPHIC SOLUTION OF STOCHASTIC EVOLUTION EQUATIONS WITH IMPULSES ON TIME SCALES

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Abstract. In this paper, we study the existence, uniqueness and exponential stability of the square-mean almost automorphic solution for stochastic evolution equation with impulses on time scales. For this purpose, we introduce the concept of equipotentially square-mean almost automorphic sequence and square-mean almost automorphic functions with impulses on time scales. At the end, a numerical example is given to illustrate the effectiveness of the obtained theoretical results.

1. Introduction

Almost automorphy is a natural generalization of almost periodicity and introduced by Bochner. In [1, 2, 3], the theory of almost automorphic functions and their applications to differential equations are given by authors. Existence, uniqueness and stability of almost automorphic solutions of differential equations have been studied at a large scale by several authors, we refer to [4, 5, 6, 7] and references therein. On the other hand, there are many phenomenons in nature whose states change suddenly at certain moments and therefore can be described by impulsive system. The theory of impulsive differential equations are useful for mathematical modeling of many real-world phenomena, such as the neural networks [8] and population dynamic system [9, 10, 11].

Stochastic dynamic equations theory has been developed follows the work of It$\hat{o}$ [12, 13, 14] and others. In the authentic world, lots of dynamic systems have variable structures subject to stochastic sudden and unexpected changes. Stochastic processes have played a significant role in various engineering disciplines like power systems, robotics, automotive technology, signal processing, manufacturing systems, semiconductor manufacturing, communication networks, wireless networks etc. Due to these reasons, the study of stochastic differential equations become consequential. The existence of almost periodic solutions to some stochastic differential equations have been considered in many publications such as [15, 16, 17, 18] and the references therein. Fu and Liu [19], introduced a new concept of square-mean almost automorphic stochastic

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processes with some basic properties. There are several publication on continuous system related to square-mean almost automorphic stochastic process in which existence, uniqueness, stablity and others properties are investigated.

Generally, one study the continuous and discrete cases discretely and there are many discrete sets which are very utilizable. Ergo, this is an arduous task that we study discretely for all cases, so for evading this type quandary Hilger, during his Ph.D in 1988, [20] introduce time scales theory which cumulates continuous and discrete analysis. This theory present a potent implement for applications to population models, economics and quantum physics among others. Ergo, dealing with quandaries of differential equations on time scales becomes very consequential and purposeful in the research field of dynamic systems. The books [21, 22] and papers [23, 24, 25, 26, 27] present a nice description about the theory of time scales and its application.

More recently, there are few authors which have been worked on existence of almost periodic and almost automorphic [28, 29, 30, 31, 32, 33] solution for stochastic equations on time scales. There are some papers [34, 35, 36, 37] which are most useful to study for stochastic calculus on time scale. In [38, 39, 40, 41], the existence of piecewise mean-square almost periodic solution for different types of models on time scales are discussed. As per our knowledge, there is no paper which discuss the almost automorphy of stochastic process with evolution system and impulses on time scales. Motivated by the above discussion, we consider the following non-autonomous stochastic differential equations with impulsive on time scales in the abstract form,

$$\Delta u(t) = (A(t)u(t) + P(t,u(t)))\Delta t + Q(t,u(t))\Delta W(t), \quad t \in \mathbb{T},\, t \neq t_i,\, i \in \mathbb{Z}, \tag{1.1}$$

$$u(t_i^+) - u(t_i^-) = I_i(u(t_i)), \tag{1.2}$$

where $\mathbb{T}$ is an almost periodic time scale and Δu denotes Δ-stochastic differential of u. $A(t) : \mathfrak{D}(A(t)) \subset L^2(\mathbb{P},\mathbb{K}) \to L^2(\mathbb{P},\mathbb{K})$ is a family of linear operator. $\{W(t) : t \in \mathbb{T}\}$ is a Brownian motion indexed by time scale or a standard Wiener process defined on a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$ with a natural filtration $\{\{\mathscr{F}_t\}_{t\geqslant 0}, t \in \mathbb{T}\}$ generated by $W(t)$ and denoted by $\mathscr{F}$, the associated $\sigma-$algebra generated by W(t) with the probability measure $\mathbb{P}$. $u(t_i^+) = \lim_{h\to 0^+} u(t_i + h), u(t_i^-) = \lim_{h\to 0^+} u(t_i - h)$ represent the right and left limits of $u(t)$ at $t = t_i$ in the sense of time scale. $P,Q : \mathbb{T} \times L^2(\mathbb{P},\mathbb{K}) \to L^2(\mathbb{P},\mathbb{K})$ and $I_i : L^2(\mathbb{P},\mathbb{K}) \to L^2(\mathbb{P},\mathbb{K})$ are appropriate functions described later.

In this paper, we introduce the concept of equipotentially square-mean almost automorphic sequence and square-mean almost automorphic functions with impulses on time scales. We establish the existence, uniqueness and exponential stability results of the square-mean almost automorphic solution to (1.1)-(1.2). This paper is organized in the following manners, In the first and second section, we give Introduction, basic definitions, preliminaries and some useful lemmas. In the third section, we establish the existence and uniqueness of square-mean almost automorphic solution. The fourth section is devoted to the exponential stability of the solution. In the last section, an example is given to illustrate the effectiveness of the analytic results.

2. Preliminaries and Definitions

In this section, we give some basic theory and lemmas for time scales, stochastic process, square-mean almost automorphy which is required further.
A time scale is a non empty closed subset of real line $\mathbb{R}$, denoted by $\mathbb{T}$. The real numbers $\mathbb{R}$, integer numbers $\mathbb{Z}$, natural numbers $\mathbb{N}$ and any intervals are some trivial examples of time scales. There are some operators which are important for study of time scale theory. The forward jump operator denoted by, $\sigma(t)$, is define by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. The backward jump operator $\rho(t)$ is define by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t, \quad \forall\, t \in \mathbb{T}$. The right dense point is defined be a point t when $t < \sup \mathbb{T}$ and $\sigma(t) = t$. It is called right scattered if $\sigma(t) > t$. So a left dense point is defined by the points such that $t > \inf \mathbb{T}$ and $\rho(t) = t$. It is called left scattered if $\rho(t) < t$. The notations $\mathbb{T}^k = T\backslash\{m\}$ or $\mathbb{T}_k = T\backslash\{m\}$ if $\mathbb{T}$ has a left scattered maximum or right scattered minimum m respectively, otherwise $\mathbb{T}^k = \mathbb{T}_k = \mathbb{T}$. For any function $\phi : \mathbb{T} \to \mathbb{R}$, we define $\phi^\sigma : \mathbb{T}^k \to \mathbb{R}$ by $\phi^\sigma = \phi o \sigma$. We will denote the interval $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$.
A function $\phi : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right dense points of $\mathbb{T}$ and its left-side limits exist at left dense points and the set of all rd-continuous functions $\phi : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.
Let ϕ is rd-continuous; if $\Phi^\Delta(t) = \phi(t)$, then delta integral is defined by,

$$\int_s^r \phi(t)\Delta t = \Phi(r) - \Phi(s), \quad \forall\, r,s \in \mathbb{T}.$$

DEFINITION 1. ([21]) Let $\phi : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then delta derivative, $\phi^\Delta(t)$ is the number (when it exists) such that given any $\delta > 0$, there is a neighborhood N of t such that

$$\Big|[\phi(\sigma(t)) - \phi(s)] - \phi^\Delta(t)[\sigma(t) - s]\Big| \leqslant \delta|\sigma(t) - s|,\ \forall\, s \in N.$$

DEFINITION 2. ([21]) A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0,\ \forall\, t \in \mathbb{T}$. The set of all regressive function is denoted by $\mathscr{R}$. If $1 + \mu(t)p(t) > 0,\ \forall\, t \in \mathbb{T}$, the function is said to be positive regressive function and denoted by $\mathscr{R}^+$.

DEFINITION 3. The exponential function is defined as

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \quad t,s \in \mathbb{T}, p \in \mathscr{R},$$

for $h > 0$,

$$\xi_h(z) = \frac{1}{h} Log(1 + zh),$$

where Log is the principal logarithm function. For $h = 0$, $\xi_0(z) = z$.

LEMMA 1. ([22]) *Assume that* ϕ *and* ψ *are two delta differential function at* $t \in \mathbb{T}$*; then:*

(i) $(\phi \pm \psi)^{\Delta}(t) = \phi^{\Delta}(t) \pm \psi^{\Delta}(t)$;

(ii) $(\phi\psi)^{\Delta}(t) = \phi^{\Delta}(t)\psi(t) + \phi(\sigma(t))\psi^{\Delta}(t) = \phi(t))\psi^{\Delta}(t) + \phi^{\Delta}(t)\psi(\sigma(t))$;

(iii) *If* $\psi(t)\psi(\sigma(t)) \neq 0$, *then* $(\frac{\phi}{\psi})^{\Delta}(t) = \frac{\phi^{\Delta}(t)\psi(t)+\phi(t)\psi^{\Delta}(t)}{\psi(t)\psi(\sigma(t))}$;

(iv) $(\int_a^t \phi(t,s)\Delta s)^{\Delta} = \phi(\sigma(t),t) + \int_a^t \phi^{\Delta}(t,s)\Delta s$.

DEFINITION 4. ([22]) Let $p,q : \mathbb{T} \to \mathbb{R}$ are regressive function, define

$$p \oplus q = p + q + \mu pq, \quad \ominus p = \frac{-p}{1+\mu p}, \quad p \ominus q = p \oplus (\ominus q).$$

LEMMA 2. ([22]) *Assume that* $p,q : \mathbb{T} \to \mathbb{R}$ *are regressive function. Then:*

(i) $e_0(t,s) = 1, \quad e_p(t,t) = 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p)e_p(t,s)$;

(iii) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;

(iv) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $(1/e_p(t,s))^{\Delta} = -p(t)/e_p(\sigma(t),s)$.

LEMMA 3. ([22]) *Let* $a,b,c \in \mathbb{T}$ *and* $p \in \mathscr{R}$, *then*

$$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

LEMMA 4. ([4]) *If* $\omega > 0$, *then* $e_{\ominus\omega}(t,s) \leqslant 1$, *for all t,s* $\in \mathbb{T}$ *where* $t \geqslant s$.

For more details on time scales see [21, 22, 23, 24].
Throughout this paper, $(\mathbb{K}, ||\cdot||)$ is assume to be real Hilbert space. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. $L^2(\mathbb{P},\mathbb{K})$ stands for the space of all $\mathbb{K}$-valued random variables u such that

$$E||u||^2 = \int_{\Omega} ||u||^2 d\mathbb{P} < \infty.$$

$L^2(\mathbb{P},\mathbb{K})$ is Hilbert space with the norm

$$||u||_2 = (E||u||^2)^{\frac{1}{2}}.$$

$L^2(\mathbb{P},\mathbb{K})$ is Banach space with the norm

$$||u||_{PC} = \sup_{t\in\mathbb{T}} (E||u||^2)^{\frac{1}{2}}.$$

DEFINITION 5. ([42]) A Brownian motion indexed by a time scale $\mathbb{T}$ is an adapted stochastic process $W = \{W(t) : t \in \mathbb{T}\}$, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the following properties:
(i) $W(t_0) = 0, \quad a.s.$;
(ii) if $t_0 \leqslant s < t$ and $s,t \in \mathbb{T}$, then the increment $W(t) - W(s)$ is independent of $\mathscr{F}(s)$ and is normally distributed with mean zero and variance $t - s$.

DEFINITION 6. ([43]) One can say that the random process $f : \mathbb{T} \times \Omega \to \mathbb{R}$ belongs to the class $L^2([0,1]_{\mathbb{T}})$ if the following conditions hold:
(i) f is adapted, i.e., $f(t,\cdot)$ is $\mathscr{F}_t$ measurable for all $t \in \mathbb{T}$.
(ii) $\mathbb{P}\Big(\int_0^1 |f(t,\omega)|^2 \Delta t < \infty\Big) = 1$.

LEMMA 5. ([42]) *Δ-stochastic integral has the following properties:*
(i) *If $f,\ g \in L^2([0,1]_{\mathbb{T}})$ and $c_1, c_2 \in \mathbb{R}$; then*

$$\int_0^1 (c_1 f(t) + c_2 g(t))\Delta W(t) = c_1 \int_0^1 f(t)\Delta W(t) + c_2 \int_0^1 g(t)\Delta W(t).$$

(ii) $E\Big(\int_0^1 |f(t)|^2 \Delta t\Big) < \infty$, *then* $E\Big(\int_0^1 f(t)\Delta W(t)\Big) = 0$ *and the Itô- isometry holds; that is,*

$$E\Bigg(\Big(\int_0^1 f(t)\Delta W(t)\Big)^2\Bigg) = E\Big(\int_0^1 f^2(t)\Delta t\Big).$$

DEFINITION 7. A stochastic process $u(t) : \mathbb{T} \to L^2(\mathbb{P}, \mathbb{K})$ is said to be stochastically continuous whenever

$$\lim_{t \to s} E||u(t) - u(s)||^2 = 0.$$

DEFINITION 8. A stochastic process $u(t) : \mathbb{T} \to L^2(\mathbb{P}, \mathbb{K})$ is said to be stochastically bounded whenever there exist a positive number M such that

$$E||u(t)||^2 < M, \quad \forall\, t \in \mathbb{T}.$$

$SBC(\mathbb{T}, L^2(\mathbb{P}, \mathbb{K}))$ denotes the collection of all the stochastically bounded and continuous processes.

REMARK 1. $SBC(\mathbb{T}, L^2(\mathbb{P}, \mathbb{K}))$ is Banach space with respect to the norm,

$$||u||_{PC} = \sup_{t \in \mathbb{T}} \Big(E||u(t)||^2\Big)^{\frac{1}{2}}.$$

DEFINITION 9. ([28]) A time scale $\mathbb{T}$ is said to almost periodic if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Let $\eth$ be a collection of subsets of $\mathbb{R}$. A time scale $\mathbb{T}$ is called almost periodic with respect to $\eth$ if

$$\eth^* = \{\pm\tau \in \bigcap_{a\in\eth} a : t\pm\tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \emptyset$$

and $\eth^*$ is called the smallest almost periodic set of $\mathbb{T}$.

DEFINITION 10. ([38]) A function $f : \mathbb{T} \to L^2(\mathbb{P},\mathbb{K})$ is said to be rd-piecewise continuous with respect to a sequence $\{t_k\} \subset \mathbb{T}$ satisfying $t_k < t_{k+1}, k \in \mathbb{Z}$ if $f(t)$ is continuous on $[t_k,t_{k+1})_{\mathbb{T}}$ and rd-continuous on $\mathbb{T}\backslash\{t_k\}$. Furthermore, $[t_k,t_{k+1})_{\mathbb{T}}$ are called intervals of continuity of the function f.

$PC_{rd}(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$ denotes the set of all piecewise continuous functions with respect to a sequence $\{t_k\}, k \in \mathbb{Z}$.
Now, we introduce a set

$$\Im = \left\{ \{t_k\}, t_k \in \mathbb{T} : t_k < t_{k+1}, k \in \mathbb{Z}, \lim_{k\to\pm\infty} t_k = \pm\infty \right\}.$$

This set denotes all unbounded increasing sequences of real numbers. let $\Im$ be the set consisting of all sequences $\{t_k\}, k \in \mathbb{Z}$ such that $\inf_{k\in\mathbb{Z}}(t_{k+1}-t_k) > 0$.

DEFINITION 11. ([44]) Let $\{t_k\} \in \Im, k \in \mathbb{Z}$, we say $\{t_k^j\}$ is a derivative sequence of $\{t_k\}$ and denoted

$$\{t_k^j\} = t_{k+j} - t_k, \quad k, j \in \mathbb{Z}.$$

DEFINITION 12. ([44]) Let $\{t_k^j\} = t_{k+j} - t_k, \quad k, j \in \mathbb{Z}$. We say $\{t_k^j\}$ is equipotentially almost automorphic on an almost periodic time scale $\mathbb{T}$ if for every sequence $\{s_n\}_{n=1}^{\infty} \subset \mathbb{Z}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that, for some sequence $\{\gamma_k\} \subset \Pi$ and for each $t_k \in \Im$,

$$\lim_{n\to\infty} t_k^{\tau_n} = \gamma_k \text{ and } \lim_{n\to\infty} \gamma_k^{-\tau_n} = t_k,$$

hold simultaneously.

DEFINITION 13. A stochastic process $u \in PC_{rd}(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$ is said to be square-mean almost automorphic if sequences of impulsive $\{t_k\}$ satisfying the derived sequence $\{t_k^j\}$ is equipotentially almost automorphic and for every sequence of real numbers $\{s_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that, for some stochastic process $u^*(t) : \mathbb{T} \to L^2(\mathbb{P},\mathbb{K})$, $\lim_{n\to\infty} E||u(t+\tau_n) - u^*(t)||^2 = 0$ and $\lim_{n\to\infty} E||u^*(t - \tau_n) - u(t)||^2 = 0$ simultaneously hold for each $t \in \mathbb{T}$.

The collection of all the square-mean almost automorphic processes $u(t) : \mathbb{T} \to L^2(\mathbb{P},\mathbb{K})$ is denoted by $SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$.

REMARK 2. $SAA(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$ is a closed subspace of $SBC(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$. Hence, it is also Banach space with norm

$$||u||_{PC} = \sup_{t\in\mathbb{T}}\Big(E||u(t)||^2\Big)^{\frac{1}{2}}.$$

DEFINITION 14. A stochastically process $u(t,x) \in PC_{rd}(\mathbb{T}\times L^2(\mathbb{P},\mathbb{K}), L^2(\mathbb{P},\mathbb{K}))$, which is jointly continuous, is said to be square-mean almost automorphic in $t \in \mathbb{T}$ and for all $x \in L^2(\mathbb{P},\mathbb{K})$ if sequences of impulsive $\{t_k\}$ satisfying the derived sequence $\{t_k^j\}$ is equipotentially almost automorphic and for every sequence of real numbers $\{s_n\}_{n=1}^{\infty} \subset \Pi$, if we can subtract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that, for some stochastic process $u^*(t,x) : \mathbb{T}\times L^2(\mathbb{P},\mathbb{K}) \to L^2(\mathbb{P},\mathbb{K})$, $\lim_{n\to\infty}\mathbb{E}||u(t+\tau_n,x) - u^*(t,x)||^2 = 0$ and $\lim_{n\to\infty}\mathbb{E}||u^*(t-\tau_n,x) - u(t,x)||^2 = 0$ simultaneously hold for each $t \in \mathbb{T}$ and $x \in L^2(\mathbb{P},\mathbb{K})$.

The collection of all such type process is denoted by $SAA(\mathbb{T}\times L^2(\mathbb{P},\mathbb{K}), L^2(\mathbb{P},\mathbb{K}))$.

LEMMA 6. ([19]) *If u_1 and u_2 are two square-mean almost automorphic stochastic process then these three properties hold true,*

1. *$u_1 + u_2$ is square-mean almost automorphic stochastic process.*

2. *λu_1 is square-mean almost automorphic for every $\lambda \in \mathbb{R}$.*

3. *u_1 is bounded in $L^2(\mathbb{P},\mathbb{K})$, i.e., there exist a positive number M such that $||u_1||_{PC} < M$.*

Now, we prove some important lemmas which are useful to establish the main results.

LEMMA 7. *Let $p : \mathbb{T}\times L^2(\mathbb{P},\mathbb{K}) \to L^2(\mathbb{P},\mathbb{K}), (t,u) \to p(t,u)$ be square-mean almost automorphic in $t \in \mathbb{T}$ for each $u \in L^2(\mathbb{P},\mathbb{K})$ and assume that p satisfies Lipschitz condition in the following sense,*

$$E||p(t,u) - p(t,u^*)||^2 \leqslant LE||u-u^*||^2,$$

for all $u, u^* \in L^2(\mathbb{P},\mathbb{K})$ and for each $t \in \mathbb{T}$, where $L > 0$ is independent from t. Then the stochastic process $P : \mathbb{T} \to L^2(\mathbb{P},\mathbb{K})$ given by $P(\cdot) = p(\cdot, u(\cdot))$ is square-mean almost automorphic provided $u : \mathbb{T} \to L^2(\mathbb{P},\mathbb{K})$ is square-mean almost automorphic.

Proof. Since $p \in SAA(\mathbb{T}\times L^2(\mathbb{P},\mathbb{K}), L^2(\mathbb{P},\mathbb{K}))$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty} \subset \Pi$ of $\{s_n\}_{n=1}^{\infty}$ and a stochastic process $p^* : \mathbb{T}\times L^2(\mathbb{P},\mathbb{K}) \to L^2(\mathbb{P},\mathbb{K})$ such that

$$\lim_{n\to\infty}\mathbb{E}||p(t+\tau_n,u) - p^*(t,u)||^2 = 0 \quad \textit{and} \quad \lim_{n\to\infty}\mathbb{E}||p^*(t-\tau_n,u) - p(t,u)||^2 = 0, \tag{2.1}$$

for all $t \in \mathbb{T}$ and $u \in L^2(\mathbb{P},\mathbb{K})$. On the other hand, $u : \mathbb{T} \to L^2(\mathbb{P},\mathbb{K})$ is also square mean almost automorphic, so we can extract a common subsequence $\{\tau_n\}_{n=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ and a stochastic process $u^* \in L^2(\mathbb{P},\mathbb{K})$ such that

$$\lim_{n\to\infty} \mathbb{E}||u(t+\tau_n) - u^*(t)||^2 = 0 \quad \textit{and} \quad \lim_{n\to\infty} \mathbb{E}||u^*(t-\tau_n) - u(t)||^2 = 0, \tag{2.2}$$

for all $t \in \mathbb{T}$. Now, let us consider the function $P^* : \mathbb{T} \to L^2(\mathbb{P},\mathbb{K})$, defined by $P^*(t) = p^*(t,u^*(t)), t \in \mathbb{T}$. Hence, we have

$$\begin{aligned} &E||P(t+\tau_n) - P^*(t)||^2 \\ &= E||p(t+\tau_n, u(t+\tau_n)) - p(t+\tau_n, u^*(t)) + p(t+\tau_n, u^*(t)) - p^*(t,u^*(t))||^2 \\ &\quad \leqslant 2E||p(t+\tau_n, u(t+\tau_n)) - p(t+\tau_n, u^*(t))||^2 + 2E||p(t+\tau_n, u^*(t) - p^*(t,u^*(t))||^2. \end{aligned} \tag{2.3}$$

By (2.1), we get

$$\lim_{n\to\infty} E||p(t+\tau_n, u^*(t)) - p^*(t,u^*(t))||^2 = 0. \tag{2.4}$$

Using the Lipschitz condition on p, we get

$$E||p(t+\tau_n, u(t+\tau_n)) - p(t+\tau_n, u^*(t))||^2 \leqslant LE||u(t+\tau_n) - u^*(t)||^2.$$

Using (2.2), we get

$$\lim_{n\to\infty} E||p(t+\tau_n, u(t+\tau_n)) - p(t+\tau_n, u^*(t))||^2 = 0. \tag{2.5}$$

From (2.4), (2.5) and (2.3) , we get

$$\lim_{n\to\infty} E||P(t+\tau_n) - P^*(t)||^2 = 0.$$

Similarly, using the same method as above, we can prove that $\lim_{n\to\infty} E||P^*(t-\tau_n) - P(t)||^2 = 0$ for each $t \in \mathbb{T}$, which prove that $P(t)$ is square-mean almost automorphic. This completes the proof. □

LEMMA 8. *If $u \in PC_{rd}(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$ is a square-mean almost automorphic function. $\{t_k\} \subset \mathbb{T}$ is equipotentially almost automorphic satisfying $\inf_{i\in\mathbb{Z}} t_i^q > 0, q \in \mathbb{Z}$ then $\{u(t_k)\}$ is a square-mean almost automorphic sequence in $L^2(\mathbb{P},\mathbb{K})$.*

Proof. $t_i^j = t_{i+j} - t_i, i, j \in \mathbb{Z}$. Obviously, from the definition of Π, it is easy to know that $t_i^j \in \Pi$. Since $u \in SAA(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$ and $\{t_k\} \subset \mathbb{T}$, using definitions 12 and 13, for any sequence $\{s_n\} \subset \mathbb{Z}$, we find that there exists a subsequence $\{\tau_n\}$ such that

$$\begin{aligned} \lim_{n\to\infty} E||u(t_{k+\tau_n}) - u^*(t_k)||^2 &= \lim_{n\to\infty} E||u(t_k + t_k^{\tau_n}) - u^*(t_k)||^2 \\ &= E||u(t_k + \gamma_k) - u^*(t_k)||^2 \\ &= 0. \end{aligned}$$

Similarly, we can prove that $\lim_{n\to\infty} E||u^*(t_{k-\tau_n}) - u(t_k)||^2 = 0$ for each $\{t_k\} \subset \mathbb{T}$. This completes the proof. □

DEFINITION 15. A sequence of continuous functions, $I_k : L^2(\mathbb{P},\mathbb{K}) \to L^2(\mathbb{P},\mathbb{K})$ is square-mean almost automorphic, if for integer sequence $\{k_n^{'}\}$, there exists a subsequence $\{k_n\}$ such that $\lim_{n\to\infty} E||I_{k+k_n}(u) - I_k^*(u)||^2 = 0$ and $\lim_{n\to\infty} E||I_{k-k_n}^*(u) - I_k(u)||^2 = 0$ for each $k \in \mathbb{Z}$ and $u \in L^2(\mathbb{P},\mathbb{K})$.

LEMMA 9. *Let $I_k : L^2(\mathbb{P},\mathbb{K}) \to L^2(\mathbb{P},\mathbb{K})$ is a sequence of square-mean almost automorphic function and $u \in SAA(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$. If I_k satisfies Lipschitz condition i.e.,*

$$E||I_k(x) - I_k(y)||^2 \leqslant LE||x-y||^2, \ \forall x,y \in L^2(\mathbb{P},\mathbb{K}), \ \forall\, k \in \mathbb{Z},$$

$L > 0$ is Lipschitz constant, then the sequence $\{I_k(u(t_k))\}$ is square-mean almost automorphic.

Proof. Using Definition 15 and Lemma 8, we get,

$$\begin{aligned} &E||I_{k+k_n}(u(t_{k+k_n})) - I_k^*(u(t_k))||^2 \\ &\quad \leqslant 2E||I_{k+k_n}(u(t_{k+k_n})) - I_{k+k_n}(u(t_k))||^2 + 2E||I_{k+k_n}(u(t_k)) - I_k^*(u(t_k))||^2 \\ &\quad \leqslant 2LE||u(t_{k+k_n}) - u(t_k)||^2 + 2E||I_{k+k_n}(u(t_k)) - I_k^*(u(t_k))||^2, \end{aligned}$$

in the last inequality, first term tends to zero as $u \in SAA(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$ and second term also tend to zero as $I_k \in SAA(L^2(\mathbb{P},\mathbb{K}), L^2(\mathbb{P},\mathbb{K}))$, when $n \to \infty$. Hence

$$\lim_{n\to\infty} E||I_{k+k_n}(u(t_{k+k_n})) - I_k^*(u(t_k))||^2 = 0.$$

Similarly, using the same method as above, we can prove that

$$\lim_{n\to\infty} E||I_{k+k_n}(u(t_{k+k_n})) - I_k^*(u(t_k))||^2 = 0.$$

The proof is now complete. □

DEFINITION 16. The equation (1.1)-(1.2) is said to be exponentially stable if, for all $\varepsilon > 0$, there exist $\lambda = \lambda(\varepsilon) > 0$ and $L > 0$ such that if $||u(a) - v(a)|| \leqslant \varepsilon$, then, for all $t \geqslant a$,

$$E||u(t) - v(t)||^2 \leqslant LE||u(a) - v(a)||^2 e_{\ominus\lambda}(t,a).$$

The following lemma can be easily prove.

LEMMA 10. *If $u \in PC_{rd}(\mathbb{T}, (L^2(\mathbb{P},\mathbb{K}))$ satisfies the following inequality*

$$u(t) \leqslant \zeta + \int_a^t p(s)u(s)\Delta s + \sum_{t_k<t} \beta_k u(t_k), \quad \forall t \in \mathbb{T},$$

then

$$u(t) \leqslant \zeta \prod_{t_k<t} (1+\beta_k) e_p(t,a), \quad \forall t \in \mathbb{T}.$$

DEFINITION 17. A two parameter family $S(t,s) : \mathbb{T}\times\mathbb{T} \to L^2(\mathbb{P},\mathbb{K})$ is said to be linear evolution operator if it satisfies the following conditions:
(i) $S(t,t) = Id,$ where Id is identity operator in $L^2(\mathbb{P},\mathbb{K})$.
(ii) $S(t,s)S(s,r) = S(t,r)$.
(iii) The mapping $(t,s) \to S(t,s)u$ is continuous for each $u \in L^2(\mathbb{P},\mathbb{K})$.

DEFINITION 18. A stochastic process $u(t) \in PC_{rd}(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$ is said to be mild solution of problem (1.1) – (1.2) on $\mathbb{T}$ if it satisfies the following integral equation,

$$u(t) = S(t,a)u(a) + \int_a^t S(t,\sigma(s))P(s,u(s))\Delta s + \int_a^t S(t,\sigma(s))Q(s,u(s))\Delta W(s) + \sum_{a<t_i<t} S(t,t_i)I_i(u(t_i)), \qquad \forall t \geqslant a.$$

3. Existence and uniqueness

To prove our main result, we consider the following assumptions:
[**HA**]. The family $\{A(t) : t \in \mathbb{T}\}$ of operators generates an exponential stable evolution system $\{S(t,s) : t \geqslant s\}$ i.e., there exist a $K_0 > 0$ and $\omega > 0$ such that

$$||S(t,s)|| \leqslant K_0 e_{\ominus\omega}(t,s) \quad t \geqslant s$$

and for any sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, there exist a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that for any $\varepsilon > 0, \exists N \in \mathbb{N}$ such that

$$||S(t+\tau_n, s+\tau_n) - S(t,s)|| \leqslant \varepsilon e_{\ominus\omega(t,s)}$$

and

$$||S(t-\tau_n, s-\tau_n) - S(t,s)|| \leqslant \varepsilon e_{\ominus\omega(t,s)},$$

for all $n > N$, for each $t,s \in \mathbb{T}, t \geqslant s$.
[**HP**]. The function $P \in PC_{rd}(\mathbb{T}\times L^2(\mathbb{P},\mathbb{K}), L^2(\mathbb{P},\mathbb{K}))$ is square-mean almost automorphic in $t \in \mathbb{T}$ for each $x \in L^2(\mathbb{P},\mathbb{K})$ and satisfy Lipschitz condition in x uniformly in t, i.e., there exist a positive number L_P such that

$$E||P(t,x) - P(t,y)||^2 \leqslant L_P E||x-y||^2,$$

for all $t \in \mathbb{T}$ and $x,y \in L^2(\mathbb{P},\mathbb{K})$.
[**HQ**]. The function $Q \in PC_{rd}(\mathbb{T}\times L^2(\mathbb{P},\mathbb{K}), L^2(\mathbb{P},\mathbb{K}))$ is square-mean almost automorphic in $t \in \mathbb{T}$ for each $x \in L^2(\mathbb{P},\mathbb{K})$ and satisfy Lipschitz condition in x uniformly in t, i.e., there exist a positive number L_Q such that

$$E||Q(t,x) - Q(t,y)||^2 \leqslant L_Q E||x-y||^2,$$

for all $t \in \mathbb{T}$ and $x,y \in L^2(\mathbb{P},\mathbb{K})$.
[**HI**]. $I_i \in PC_{rd}(L^2(\mathbb{P},\mathbb{K}), L^2(\mathbb{P},\mathbb{K}))$ is square-mean almost automorphic sequence and satisfy Lipschitz condition i.e., there exist a positive number L_I such that

$$E||I_i(x) - I_i(y)||^2 \leqslant L_I E||x-y||^2.$$

To investigate the existence and uniqueness of a square-mean almost automorphic solution to $(1.1)-(1.2)$, we need the following lemmas:

LEMMA 11. *Assume that condition (HA) holds, $p \in SAA(\mathbb{T}, L^2(\mathbb{P}, \mathbb{K}))$ and $u : \mathbb{T} \to L^2(\mathbb{P}, \mathbb{K})$ is defined by:*

$$u(t) = \int_{-\infty}^{t} S(t, \sigma(s))p(s)\Delta s, \quad t \geqslant \sigma(s),$$

then $u \in SAA(\mathbb{T}, L^2(\mathbb{P}, \mathbb{K}))$.

Proof. Since $p(\cdot) \in SAA(\mathbb{T}, L^2(\mathbb{P}, \mathbb{K}))$, there exist a subsequence $\{\tau_n\}_{n=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty} \subset \Pi$ and a stochastic process $p^* \in L^2(\mathbb{P}, \mathbb{K})$ such that

$$\lim_{n\to\infty} E||p(t+\tau_n) - p^*(t)||^2 = 0 \quad \textit{and} \quad \lim_{n\to\infty} E||p^*(t-\tau_n) - p(t)||^2 = 0. \tag{3.1}$$

Moreover, if we let $u^*(t) = \int_{-\infty}^{t} S(t, \sigma(s))p^*(s)\Delta s, \quad t \geqslant \sigma(s)$, using Cauchy-Schwartz inequality, we have

$$\begin{aligned}
E||u(t+\tau_n) - u^*(t)||^2 &= E\Big\|\int_{-\infty}^{t+\tau_n} S(t+\tau_n, \sigma(s))p(s)\Delta s - \int_{-\infty}^{t} S(t, \sigma(s))p^*(s)\Delta s\Big\|^2 \\
&= E\Big\|\int_{-\infty}^{t} S(t+\tau_n, \sigma(s)+\tau_n)p(s+\tau_n)\Delta s - \int_{-\infty}^{t} S(t, \sigma(s))p^*(s)\Delta s\Big\|^2 \\
&\leqslant 2E\Big\|\int_{-\infty}^{t} \Big[S(t+\tau_n, \sigma(s)+\tau_n) - S(t, \sigma(s))\Big]p(s+\tau_n)\Delta s\Big\|^2 \\
&\quad + 2E\Big\|\int_{-\infty}^{t} S(t, \sigma(s))\Big[p(s+\tau_n) - p^*(s)\Big]\Delta s\Big\|^2 \\
&\leqslant 2\varepsilon^2 E\Big(\int_{-\infty}^{t} e_{\ominus\omega}(t, \sigma(s))||p(s+\tau_n)||\Delta s\Big)^2 \\
&\quad + 2K_0^2 E\Big(\int_{-\infty}^{t} e_{\ominus\omega}(t, \sigma(s))||p(s+\tau_n) - p^*(s)||\Delta s\Big)^2 \\
&\leqslant 2\varepsilon^2 \Big(\int_{-\infty}^{t} e_{\ominus\omega}(t, \sigma(s))\Delta s\Big)\Big(\int_{-\infty}^{t} e_{\ominus\omega}(t, \sigma(s))E||p(s+\tau_n)||^2\Delta s\Big) \\
&\quad + 2K_0^2 \Big(\int_{-\infty}^{t} e_{\ominus\omega}(t, \sigma(s))\Delta s\Big)\Big(\int_{-\infty}^{t} e_{\ominus\omega}(t, \sigma(s))E||p(s+\tau_n) - p^*(s)||^2\Delta s\Big) \\
&\leqslant \frac{-2\varepsilon^2}{\ominus\omega}\Big(\int_{-\infty}^{t} e_{\ominus\omega}(t, \sigma(s))\Delta s\Big)\sup_{t\in\mathbb{T}} E||p(t)||^2 \\
&\quad + \frac{-2K_0^2}{\ominus\omega}\Big(\int_{-\infty}^{t} e_{\ominus\omega}(t, \sigma(s))\Delta s\Big)\sup_{t\in\mathbb{T}} E||p(t+\tau_n) - p^*(t)||^2 \\
&= \frac{2\varepsilon^2}{(\ominus\omega)^2}\sup_{t\in\mathbb{T}} E||p(t)||^2 + \frac{2K_0^2}{(\ominus\omega)^2}\sup_{t\in\mathbb{T}} E||p(t+\tau_n) - p^*(t)||^2,
\end{aligned}$$

for all $t \geqslant s$ and all $n > \mathbb{N}$. Since $p(\cdot)$ is bounded and satisfies (3.1), then we obtain that

$$\lim_{n\to\infty} E||u(t+\tau_n) - u^*(t)||^2 = 0.$$

Similarly, using the same method we can easily get $\lim_{n\to\infty} E||u^*(t-\tau_n) - u(t)||^2 = 0$. Thus we conclude that $u \in SAA(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$. This complete the proof. □

LEMMA 12. *Assume that condition* (HA) *holds,* $q \in SAA(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$ *and* $u : \mathbb{T} \to L^2(\mathbb{P},\mathbb{K})$ *is defined by:*

$$u(t) = \int_{-\infty}^{t} S(t,\sigma(s))q(s)\Delta W(s), \quad t \geqslant \sigma(s),$$

then $u \in SAA(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$.

Proof. Since $q(\cdot) \in SAA(\mathbb{T}, L^2(\mathbb{P},\mathbb{K}))$, there exist a subsequence $\{\tau_n\}_{n=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty} \subset \Pi$ and a stochastic process $q^* \in L^2(\mathbb{P},\mathbb{K}))$ such that

$$\lim_{n\to\infty} E||q(t+\tau_n) - q^*(t)||^2 = 0 \quad \textit{and} \quad \lim_{n\to\infty} E||q^*(t-\tau_n) - q(t)||^2 = 0. \tag{3.2}$$

Let $\widetilde{W}(r) = W(r+\tau_n) - W(r)$ for all $r \in \mathbb{T}$. $\widetilde{W}$ is also a Brownian motion and has the same distribution as W. Moreover, if we let $u^*(t) = \int_{-\infty}^{t} S(t,\sigma(s))q^*(s)\Delta W(s), \quad t \geqslant \sigma(s)$, by changing of variable $r = s - \tau_n$ and using estimation on Ito integral, we get

$$\begin{aligned}
&E||u(t+\tau_n) - u^*(t)||^2 \\
&\quad = E\Big\| \int_{-\infty}^{t+\tau_n} S(t+\tau_n,\sigma(s))q(s)\Delta W(s) - \int_{-\infty}^{t} S(t,\sigma(s))q^*(s)\Delta W(s)\Big\|^2 \\
&\quad = E\Big\| \int_{-\infty}^{t} S(t+\tau_n,\sigma(r)+\tau_n)q(r+\tau_n)\Delta\widetilde{W}(r) - \int_{-\infty}^{t} S(t,\sigma(r))q^*(r)\Delta\widetilde{W}(r)\Big\|^2 \\
&\quad \leqslant 2E\Big\| \int_{-\infty}^{t} \Big[S(t+\tau_n,\sigma(r)+\tau_n) - S(t,\sigma(r))\Big] q(r+\tau_n)\Delta\widetilde{W}(r)\Big\|^2 \\
&\qquad + 2E\Big\| \int_{-\infty}^{t} S(t,\sigma(r))\Big[q(r+\tau_n) - q^*(r)\Big]\Delta\widetilde{W}(r)\Big\|^2 \\
&\quad \leqslant 2\varepsilon^2 \int_{-\infty}^{t} \Big(e_{\ominus\omega}(t,\sigma(r))E||q(r+\tau_n)||\Big)^2 \Delta r \\
&\qquad + 2K_0^2 \int_{-\infty}^{t} \Big(e_{\ominus\omega}(t,\sigma(r))E||q(r+\tau_n) - q^*(r)||\Big)^2 \Delta r \\
&\quad \leqslant 2\varepsilon^2 \left(\int_{-\infty}^{t} \Big(e_{\ominus\omega}(t,\sigma(r))\Big)^2 \Delta r\right) \sup_{t\in\mathbb{T}} E||q(t)||^2 \\
&\qquad + 2K_0^2 \left(\int_{-\infty}^{t} \Big(e_{\ominus\omega}(t,\sigma(r))\Big)^2 \Delta r\right) \sup_{t\in\mathbb{T}} E||q(t+\tau_n) - q^*(t)||^2
\end{aligned}$$

$$\leqslant 2\varepsilon^2\left(\int_{-\infty}^{t} e_{(\ominus\omega)\oplus(\ominus\omega)}(t,\sigma(r))\Delta r\right)\sup_{t\in\mathbb{T}} E||q(t)||^2$$

$$+2K_0^2\left(\int_{-\infty}^{t} e_{(\ominus\omega)\oplus(\ominus\omega)}(t,\sigma(r))\Delta r\right)\sup_{t\in\mathbb{T}} E||q(t+\tau_n)-q^*(t)||^2$$

$$=\frac{-2\varepsilon^2}{(\ominus\omega)\oplus(\ominus\omega)}\sup_{t\in\mathbb{T}} E||q(t)||^2+\frac{-2K_0^2}{(\ominus\omega)\oplus(\ominus\omega)}\sup_{t\in\mathbb{T}} E||q(t+\tau_n)-q^*(t)||^2$$

$$\leqslant \frac{2\varepsilon^2(1+\bar{\mu})^2}{2\omega+\underline{\mu}\omega^2}\sup_{t\in\mathbb{T}} E||q(t)||^2+\frac{2K_0^2(1+\bar{\mu})^2}{2\omega+\underline{\mu}\omega^2}\sup_{t\in\mathbb{T}} E||q(t+\tau_n)-q^*(t)||^2,$$

for all $t\geqslant s$ and all $n>\mathbb{N}$, here $\bar{\mu}=\sup_{t\in\mathbb{T}}\mu(t)$ and $\underline{\mu}=\inf_{t\in\mathbb{T}}\mu(t)$. Since $q(\cdot)$ is bounded and satisfies (3.2), then we obtain that

$$\lim_{n\to\infty} E||u(t+\tau_n)-u^*(t)||^2=0.$$

Similarly, using the same method we can easily get $\lim_{n\to\infty} E||u^*(t-\tau_n)-u(t)||^2=0$. Thus we conclude that $u\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$. This complete the proof. □

LEMMA 13. *Assume that condition (HA) holds, t_i is square-mean almost automorphic sequence; $u(t_i)\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$ and $u:\mathbb{T}\to L^2(\mathbb{P},\mathbb{K})$ is defined by:*

$$u(t)=\sum_{t_i<t} S(t,t_i)u(t_i),$$

then $u\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$.

Proof. Since $u(t_i)\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$, there exist a subsequence $\{\tau_n\}_{n=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}\subset\Pi$ and a stochastic process $u^*(t_i)\in L^2(\mathbb{P},\mathbb{K})$ such that

$$\lim_{n\to\infty} E||u(t_i+\tau_n)-u^*(t_i)||^2=0 \quad and \quad \lim_{n\to\infty} E||u^*(t_i-\tau_n)-u(t_i)||^2=0, \tag{3.3}$$

for all $t_i\in\mathbb{T}, i\in\mathbb{Z}$. Moreover, if we let $u^*(t)=\sum_{t_i<t} S(t,\sigma(s))u^*(t_i)$,

$$E||u(t+\tau_n)-u^*(t)||^2=E\Big\|\sum_{t_i<t+\tau_n} S(t+\tau_n,t_i)u(t_i)-\sum_{t_i<t} S(t,t_i)u^*(t_i)\Big\|^2$$

$$=E\Big\|\sum_{t_i<t} S(t+\tau_n,t_i+\tau_n)u(t_i+\tau_n)-\sum_{t_i<t} S(t,t_i)u^*(t_i)\Big\|^2$$

$$\leqslant 2E\Big\|\sum_{t_i<t}\Big[S(t+\tau_n,t_i+\tau_n)-S(t,t_i)\Big]u(t_i+\tau_n)\Big\|^2$$

$$+2E\Big\|\sum_{t_i<t} S(t,t_i)\Big[u(t_i+\tau_n)-u^*(t_i)\Big]\Big\|^2$$

$$\leqslant 2\varepsilon^2\Big(\sum_{t_i<t} e_{\ominus\omega}(t,t_i)\Big)\Big(\sum_{t_i<t} e_{\ominus\omega}(t,t_i)E||u(t_i+\tau_n)||^2\Big)$$

$$+2K_0^2\Big(\sum_{t_i<t} e_{\ominus\omega}(t,t_i)\Big)\Big(\sum_{t_i<t} e_{\ominus\omega}(t,t_i)E||u(t_i+\tau_n)-u^*(t_i)||^2\Big)$$

$$\leqslant 2\varepsilon^2\Big(\sum_{t_i<t} e_{\ominus\omega}(t,t_i)\Big)^2 \sup_{t_i,i\in\mathbb{Z}} E||u(t_i+\tau_n)||^2\Big)$$

$$+2K_0^2\Big(\sum_{t_i<t} e_{\ominus\omega}(t,t_i)\Big)^2 \sup_{t_i,i\in\mathbb{Z}} E||u(t_i+\tau_n)-u^*(t_i)||^2$$

$$\leqslant \frac{2\varepsilon^2}{(1-e_{\ominus\omega}(\theta,0))^2}\sup_{t_i,i\in\mathbb{Z}} E||u(t_i+\tau_n)||^2$$

$$+\frac{2K_0^2}{(1-e_{\ominus\omega}(\theta,0))^2}\sup_{t_i,i\in\mathbb{Z}} E||u(t_i+\tau_n)-u^*(t_i)||^2,$$

for all $t_i\in\mathbb{Z}$ and all $n>\mathbb{N}$, here $\theta=\inf_{i\in\mathbb{Z}}(t_{i+1}-t_i)>0$. Since $e_{\ominus\omega}(\theta,0)<1$, $u(t_i)$ is bounded and satisfies (3.3), then we obtain that

$$\lim_{n\to\infty} E||u(t+\tau_n)-u^*(t)||^2=0.$$

Similarly, using the same method we can easily get $\lim_{n\to\infty} E||u^*(t-\tau_n)-u(t)||^2=0$. Thus we conclude that $u\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$. This complete the proof. □

THEOREM 1. *Under the assumptions* $(HA)-(HI)$, *the problem* (1.1)–(1.2) *has a unique square mean almost automorphic solution* $u(\cdot)\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$, *provided*

$$M=\left[\frac{3K_0^2L_P(1+\bar{\mu}\omega)^2}{\omega^2}+\frac{3K_0^2L_Q(1+\bar{\mu}\omega)^2}{2\omega+\underline{\mu}\omega^2}+\frac{3K_0^2L_I}{[1-e_{\ominus\omega}(\theta,0)]^2}\right]<1. \tag{3.4}$$

Proof. Consider the nonlinear operator

$$\Upsilon u(t)$$
$$=\int_{-\infty}^t S(t,\sigma(s))P(s,u(s))\Delta s+\int_{-\infty}^t S(t,\sigma(s))Q(s,u(s))\Delta W(s)+\sum_{t_i<t}S(t,t_i)I_i(u(t_i)).$$

First we show that it is well defined, i.e., $\Upsilon u(\cdot): SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))\to SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$. Indeed, let $u\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$ then by Lemma 7, the function $s\to P(s,u(s))\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$, $s\to Q(s,u(s))\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$ and by Lemma 9, the function $t_i\to I_i(u(t_i))\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$. Clearly, by Lemmas 11, 12, 13, the operator $\Upsilon\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$. Now to complete our prove, we have to show that Υ is a contraction mapping on $SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$. Indeed, if we have $u(t),v(t)\in SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$, then

$$E||\Upsilon u(t)-\Upsilon v(t)||^2=E\Big\|\int_{-\infty}^t S(t,\sigma(s))\Big[P(s,u(s))-P(s,v(s))\Big]\Delta s$$
$$+\int_{-\infty}^t S(t,\sigma(s))\Big[Q(s,u(s))-Q(s,v(s))\Big]\Delta W(s)$$
$$+\sum_{t_i<t}S(t,t_i)\Big[I_i(u(t_i))-I_i(v(t_i))\Big]\Big\|^2$$

$$
\begin{aligned}
&\leqslant 3E\Big\|\int_{-\infty}^{t} S(t,\sigma(s))\Big[P(s,u(s))-P(s,v(s))\Big]\Delta s\Big\|^2\\
&\quad+3E\Big\|\int_{-\infty}^{t} S(t,\sigma(s))\Big[Q(s,u(s))-Q(s,v(s))\Big]\Delta W(s)\Big\|^2\\
&\quad+3E\Big\|\sum_{t_i<t} S(t,t_i)\Big[I_i(u(t_i))-I_i(v(t_i))\Big]\Big\|^2\\
&\leqslant 3K_0^2E\Bigg[\Big(\int_{-\infty}^{t} e_{\ominus\omega}(t,\sigma(s))\Delta s\Big)\Big(\int_{-\infty}^{t} e_{\ominus\omega}(t,\sigma(s))\Big\|P(s,u(s))-P(s,v(s))\Big\|^2\Delta s\Big)\Bigg]\\
&\quad+3K_0^2\int_{-\infty}^{t}\Big(e_{\ominus\omega}(t,\sigma(s))E\Big\|Q(s,u(s))-Q(s,v(s))\Big\|\Big)^2\Delta s\\
&\quad+3K_0^2E\Bigg[\Big(\sum_{t_i<t} e_{\ominus\omega}(t,t_i)\Big)\Big(\sum_{t_i<t} e_{\ominus\omega}(t,t_i)\|I_i(u(t_i))-I_i(v(t_i))\Big)\Big\|^2\Big)\Bigg]\\
&\leqslant 3K_0^2\Big(\int_{-\infty}^{t} e_{\ominus\omega}(t,\sigma(s))\Delta s\Big)\Big(\int_{-\infty}^{t} e_{\ominus\omega}(t,\sigma(s))E\Big\|P(s,u(s))-P(s,v(s))\Big\|^2\Delta s\Big)\\
&\quad+3K_0^2\int_{-\infty}^{t}\Big(e_{\ominus\omega}(t,\sigma(s))E\Big\|Q(s,u(s))-Q(s,v(s))\Big\|\Big)^2\Delta s\\
&\quad+3K_0^2\Big(\sum_{t_i<t} e_{\ominus\omega}(t,t_i)\Big)\Big(\sum_{t_i<t} e_{\ominus\omega}(t,t_i)E\Big\|I_i(u(t_i))-I_i(v(t_i))\Big\|^2\Big)\\
&\leqslant \frac{-3K_0^2L_P}{\ominus\omega}\int_{-\infty}^{t} e_{\ominus\omega}(t,\sigma(s))E\Big\|u(s))-v(s)\Big\|^2\Delta s\\
&\quad+3K_0^2L_Q\int_{-\infty}^{t}\Big(e_{\ominus\omega}(t,\sigma(s))\Big)^2E\Big\|u(s)-v(s)\Big\|^2\Delta s\\
&\quad+\frac{3K_0^2L_I}{1-e_{\ominus\omega}(\theta,0)}\sum_{t_i<t} e_{\ominus\omega}(t,t_i)E\Big\|u(s))-v(s)\Big\|^2\Delta s\\
&\qquad\leqslant\Bigg[\frac{3K_0^2L_P(1+\bar{\mu}\omega)^2}{\omega^2}+\frac{3K_0^2L_Q(1+\bar{\mu}\omega)^2}{2\omega+\underline{\mu}\omega^2}+\frac{3K_0^2L_I}{[1-e_{\ominus\omega}(\theta,0)]^2}\Bigg]\\
&\qquad\qquad\times\sup_{t\in\mathbb{T}}E\Big\|u(t)-v(t)\Big\|^2,
\end{aligned}
$$

that is,

$$
||\Upsilon u(t)-\Upsilon v(t)||_2^2\leqslant M\sup_{t\in\mathbb{T}}||u(t)-v(t)||_2^2\leqslant M\Big(\sup_{t\in\mathbb{T}}||u(t)-v(t)||_2\Big)^2
$$

and $||\Upsilon u(t)-\Upsilon v(t)||_2\leqslant\sqrt{M}||u-v||_{PC}$, $||\Upsilon u-\Upsilon v||_{PC}=\sup_{t\in\mathbb{T}}||\Upsilon u(t)-\Upsilon v(t)||_2$. Hence we obtain,

$$
||\Upsilon u-\Upsilon v||_{PC}\leqslant\sqrt{M}||u-v||_{PC},
$$

which implies Υ is a contraction mapping by (3.4). So by the Banach contraction principle, we conclude that there exists a unique fixed point $u(\cdot)$ for Υ in $SAA(\mathbb{T},L^2(\mathbb{P},\mathbb{K}))$, such that $\Upsilon u = u$, that is,

$$u(t) = \int_{-\infty}^{t} S(t,\sigma(s))P(s,u(s))\Delta s + \int_{-\infty}^{t} S(t,\sigma(s))Q(s,u(s))\Delta W(s) + \sum_{t_i<t} S(t,t_i)I_i(u(t_i)),$$

for all $t \in \mathbb{T}$. If we let $a \in \mathbb{T}$, then

$$\begin{aligned} &u(a) \\ &= \int_{-\infty}^{a} S(a,\sigma(s))P(s,u(s))\Delta s + \int_{-\infty}^{a} S(a,\sigma(s))Q(s,u(s))\Delta W(s) + \sum_{t_i<a} S(a,t_i)I_i(u(t_i)) \end{aligned}$$

and

$$\begin{aligned} &S(t,a)u(a) \\ &= \int_{-\infty}^{a} S(t,\sigma(s))P(s,u(s))\Delta s + \int_{-\infty}^{a} S(t,\sigma(s))Q(s,u(s))\Delta W(s) + \sum_{t_i<a} S(t,t_i)I_i(u(t_i)). \end{aligned}$$

But for all $t \geqslant a$,

$$\begin{aligned} &\int_{a}^{t} S(t,\sigma(s))Q(s,u(s))\Delta W(s) \\ &\qquad = \int_{-\infty}^{t} S(t,\sigma(s))Q(s,u(s))\Delta W(s) - \int_{-\infty}^{a} S(t,\sigma(s))Q(s,u(s))\Delta W(s) \\ &\qquad = u(t) - \int_{-\infty}^{t} S(t,\sigma(s))P(s,u(s))\Delta s - \sum_{t_i<t} S(t,t_i)I_i(u(t_i)) - S(t,a)u(a) \\ &\qquad\qquad + \int_{-\infty}^{a} S(t,\sigma(s))P(s,u(s))\Delta s + \sum_{t_i<a} S(t,t_i)I_i(u(t_i)) \\ &\qquad = u(t) - S(t,a)u(a) - \int_{a}^{t} S(t,\sigma(s))P(s,u(s))\Delta s - \sum_{a<t_i<t} S(t,t_i)I_i(u(t_i)). \end{aligned}$$

In conclusion,

$$\begin{aligned} u(t) = S(t,a)u(a) + \int_{a}^{t} S(t,\sigma(s))P(s,u(s))\Delta s + \int_{a}^{t} S(t,\sigma(s))Q(s,u(s))\Delta W(s) \\ + \sum_{a<t_i<t} S(t,t_i)I_i(u(t_i)), \end{aligned}$$

is unique mild solution of problem (1.1)-(1.2). This completes our proof. □

4. Stability

In this section, we establish the exponential stability of almost automorphic solution.

THEOREM 2. *Let the assumptions of theorem 1 hold and* $(\ominus\omega)\oplus c<0$, *where*

$$c=4K_0^2(1+\bar{\mu}\omega)\Big[\frac{L_P(1+\bar{\mu}\omega)}{\omega}+L_Q\Big], \tag{4.1}$$

$\bar{\mu}=\sup_{t\in\mathbb{T}}\mu(t)$. *Then, problem (1.1)-(1.2) has a unique square-mean almost automorphic mild solution which is exponentially stable.*

Proof. By Theorem 1, we see that problem (1.1)-(1.2) has a unique square-mean almost automorphic mild solution whose integral form is given by,

$$u(t)=S(t,a)u(a)+\int_a^t S(t,\sigma(s))P(s,u(s))\Delta s+\int_a^t S(t,\sigma(s))Q(s,u(s))\Delta W(s) + \sum_{a<t_i<t} S(t,t_i)I_i(u(t_i)),$$

for all $t>a, a\neq t_i, i\in\mathbb{Z}$. Let $u(t)$ and $v(t)$ are two solution of problem (1.1)-(1.2), then

$$\begin{aligned}
E||u(t)-v(t)||^2 &= E\Big\|S(t,a)\Big[u(a)-v(a)\Big]+\int_a^t S(t,\sigma(s))\Big[P(s,u(s))-P(s,v(s))\Big]\Delta s\\
&+\int_a^t S(t,\sigma(s))\Big[Q(s,u(s))-Q(s,v(s))\Big]\Delta W(s)+\sum_{a<t_i<t}S(t,t_i)\Big[I_i(u(t_i))-I_i(v(t_i))\Big]\Big\|^2\\
&\leqslant 4E\Big\|S(t,a)\Big[u(a)-v(a)\Big]\Big\|^2+4E\Big\|\int_a^t S(t,\sigma(s))\Big[P(s,u(s))-P(s,v(s))\Big]\Delta s\Big\|^2\\
&+4E\Big\|\int_a^t S(t,\sigma(s))\Big[Q(s,u(s))-Q(s,v(s))\Big]\Delta W(s)\Big\|^2\\
&+4E\Big\|\sum_{a<t_i<t}S(t,t_i)\Big[I_i(u(t_i))-I_i(v(t_i))\Big]\Big\|^2\leqslant 4K_0^2\Big(e_{\ominus\omega}(t,a)\Big)^2E\Big\|u(a)-v(a)\Big\|^2\\
&+4K_0^2E\Big[\Big(\int_a^t e_{\ominus\omega}(t,\sigma(s))\Delta s\Big)\Big(\int_a^t e_{\ominus\omega}(t,\sigma(s))\Big\|P(s,u(s))-P(s,v(s))\Big\|^2\Delta s\Big)\Big]\\
&+4K_0^2\int_a^t\Big(e_{\ominus\omega}(t,\sigma(s))E\Big\|Q(s,u(s))-Q(s,v(s))\Big\|\Big)^2\Delta s\\
&+4K_0^2E\Big[\Big(\sum_{t_i<t}e_{\ominus\omega}(t,t_i)\Big)\Big(\sum_{a<t_i<t}e_{\ominus\omega}(t,t_i)\|I_i(u(t_i))-I_i(v(t_i))\Big)\Big\|^2\Big)\Big]\\
&\leqslant 4K_0^2e_{\ominus\omega}(t,a)E\Big\|u(a)-v(a)\Big\|^2\\
&+4K_0^2\Big(\int_{-\infty}^t e_{\ominus\omega}(t,\sigma(s))\Delta s\Big)\Big(\int_a^t e_{\ominus\omega}(t,\sigma(s))E\Big\|P(s,u(s))-P(s,v(s))\Big\|^2\Delta s\Big)\\
&+4K_0^2\int_a^t\Big(e_{\ominus\omega}(t,\sigma(s))E\Big\|Q(s,u(s))-Q(s,v(s))\Big\|\Big)^2\Delta s\\
&+4K_0^2\Big(\sum_{-\infty<t_i<t}e_{\ominus\omega}(t,t_i)\Big)\Big(\sum_{a<t_i<t}e_{\ominus\omega}(t,t_i)E\Big\|I_i(u(t_i))-I_i(v(t_i))\Big\|^2\Big)
\end{aligned}$$

$$\leqslant 4K_0^2 e_{\ominus\omega}(t,a)E\Big\|u(a)-v(a)\Big\|^2 + \frac{-4K_0^2L_P(1+\bar{\mu}\omega)}{\ominus\omega}\int_a^t e_{\ominus\omega}(t,s)E\Big\|u(s))-v(s)\Big\|^2\Delta s + 4K_0^2L_Q(1+\bar{\mu}\omega)\int_a^t e_{\ominus\omega}(t,s)E\Big\|u(s)-v(s)\Big\|^2\Delta s + \frac{4K_0^2L_I}{1-e_{\ominus\omega}(\theta,0)}\sum_{a<t_i<t} e_{\ominus\omega}(t,t_i)E\Big\|u(s))-v(s)\Big\|^2\Delta s.$$

Let $e_{\ominus\omega}(t,a)E||x(t)||^2 = E||u(t)-v(t)||^2$, then,

$$e_{\ominus\omega}(t,a)E||x(t)||^2 \leqslant c_1 e_{\ominus\omega}(t,a)E||x(a)||^2 + c_2\int_a^t e_{\ominus\omega}(t,a)E||x(s)||^2\Delta s + c_3\int_a^t e_{\ominus\omega}(t,a)E||x(s)||^2\Delta s + c_4\sum_{a<t_i<t} e_{\ominus\omega}(t,a)E||x(t_i)||^2$$

$$E||x(t)||^2 \leqslant c_1 E||x(a)||^2 + (c_2+c_3)\int_a^t E||x(s)||^2\Delta s + c_4\sum_{a<t_i<t} E||x(t_i)||^2,$$

where $c_1 = 4K_0^2$, $c_2 = \frac{4K_0^2L_P(1+\bar{\mu}\omega)^2}{\omega}$, $c_3 = 4K_0^2L_Q(1+\bar{\mu}\omega)$, $c_4 = \frac{4K_0^2L_I}{1-e_{\ominus\omega}(\theta,0)}$. From Lemma 10, we have

$$E||x(t)||^2 \leqslant c_1 E||x(a)||^2 \prod_{a<t_i<t} c_4 e_{c_2+c_3}(t,a),$$

that is,

$$E||u(t)-v(t)||^2 e_\omega(t,a) \leqslant c_1 E||u(a)-v(a)||^2 \prod_{a<t_i<t}(1+c_4)e_{c_2+c_3}(t,a)$$

$$E||u(t)-v(t)||^2 \leqslant c_1 E||u(a)-v(a)||^2 \prod_{a<t_i<t}(1+c_4)e_{(\ominus\omega)\oplus c}(t,a),$$

Where $c = c_2+c_3 = \frac{4K_0^2L_P(1+\bar{\mu}\omega)^2}{\omega} + 4K_0^2L_Q(1+\bar{\mu}\omega)$. Now, by Definition and (4.1) we conclude that problem (1.1)-(1.2) has a square-mean almost automorphic mild solution which is exponentially stable. This completes the proof. □

5. Example

Consider the following stochastic impulsive partial differential equation on an almost periodic time scale $\mathbb{T}$ with $\mu < 4/3$,

$$\frac{\partial}{\Delta_1 t}Z(t,x) = \frac{\partial^2}{\Delta_2 x^2}Z(t,x) + \frac{1}{100}\cos\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right)\sin Z(t,x)$$
$$+\frac{1}{100}\sin\left(\frac{1}{1+\cos t+\sin\sqrt{2}t}\right)\sin Z(t,x)\frac{\partial W(t)}{\Delta_1 t}, \quad t\in\mathbb{T},\ t\neq t_i,\ x\in[0,\pi]_{\mathbb{T}},$$
$$\Delta_2 Z(t_i,x) = \frac{1}{100}\left(\cos i+\sin\sqrt{5}t\right)Z(t_i,x), \quad i\in\mathbb{Z},\ x\in[0,\pi]_{\mathbb{T}},$$
$$Z(t,0)=Z(t,\pi)=0, \quad t\in\mathbb{T}, \tag{5.1}$$

where $t_i = i+\frac{1}{16}|\cos(i+1)-\sin\sqrt{3}t|$, $i\in\mathbb{Z}$ and $W(t)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathscr{F},\mathbb{P},\mathscr{F}_t)$. Let $u(t)=Z(t,\cdot)$ and $X=L^2[0,\pi]_{\mathbb{T}}$, we define the operator A by

$$Au = \frac{\partial^2}{\Delta_2 x^2}u, \quad u\in\mathfrak{D}(A)=\{H_0^1[0,\pi]_{\mathbb{T}}\cap H_0^2[0,\pi]_{\mathbb{T}}\}.$$

Clearly from the same discussion as in Section 3.1 in [45] that one can easily see that the evolution system $\{S(t,s): t\geqslant s\}$ satisfies $||S(t,s)||\leqslant e_{\ominus\frac{1}{2}}(t,s)$, $t\geqslant s$, with $K_0=1$ and $\omega=\frac{1}{2}$. Now, problem (5.1) can be formulated in abstract form as (1.1)-(1.2), where,

$$P(t,u) = \frac{1}{100}\cos\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right)\sin u(t),$$

$$Q(t,u) = \frac{1}{100}\sin\left(\frac{1}{1+\cos t+\sin\sqrt{2}t}\right)\sin u(t),$$

$$I_i(u(t_i)) = \frac{1}{100}\left(\cos i+\sin\sqrt{5}t\right)u(t_i), \quad t_i = i+\frac{1}{16}|\cos(i+1)-\sin\sqrt{3}t|.$$

Now, $\{t_i^j\}$, $i,j\in\mathbb{Z}$ is an equipotentially square-mean almost automorphic sequence and $t_i^1 = t_{i+1}-t_i > 17/20$. Hence $\theta = \inf_{i\in\mathbb{Z}}(t_{i+1}-t_i) > \frac{17}{20} > 0$.
Clearly P,Q,I_i satisfies all assumptions with $L_P = L_Q = \frac{1}{100}, L_I = \frac{1}{50}$. Moreover,

$$\left[\frac{3K_0^2L_P(1+\bar{\mu}\omega)^2}{\omega^2} + \frac{3K_0^2L_Q(1+\bar{\mu}\omega)^2}{2\omega+\underline{\mu}\omega^2} + \frac{3K_0^2L_I}{[1-e_{\ominus\omega}(\theta,0)]^2}\right] = 0.600152 < 1,$$

and since $\mu < 4/3$, then $\bar{\mu} = 4/3$ and $\underline{\mu}=0$, so,

$$c = 4K_0^2(1+\bar{\mu}\omega)\left[\frac{L_P(1+\bar{\mu}\omega)}{\omega}+L_Q\right] = \frac{13}{45},$$

and $(\ominus\omega)\oplus c = -0.544 < 0$. Therefore, the Equation (5.1) has a unique square-mean almost automorphic mild solution which is exponential stable.

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