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GENERALIZED FIRST ORDER DYNAMIC EQUATIONS ON TIME SCALES WITH Δ–CARATHÉODORY FUNCTIONS

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Abstract. In this paper we consider a first order dynamic equation on time scales in which the right hand side is a Δ-Carathéodory function, which is not necessarily continuous. We generalize this discontinuous dynamic equation using Henstock–Kurzweil Δ-integral and establish results concerning existence of solutions using simple analysis. Uniqueness of solutions is obtained using an Osgood type condition. Moreover we introduce the concept of Henstock–Kurzweil Δ-equi-integrability and study continuous dependence and convergence of solutions.

1. Introduction

The theory of time scales was successfully proposed by S. Hilger [14], in order to create a theory that could unify both discrete and continuous calculus. Since then the theory of time scales has received a lot of attention and constitutes quite an active research area. Many researchers have studied dynamic equations on time scale domains and presented some important results, this can be witnessed by the works [1], [7], [9], [11], [15], [16], [21]. Such equations are extensively useful in mathematical models to exhibit several phenomena in physics, control theory, economics etc. To the best of my knowledge, most of the existing theory has been developed and still evolving in the framework of Riemann and Lebesgue type delta and nabla integrals [4], [12], [18] which do not cover general theory. Taking into consideration highly oscillating functions, these integrals fail to integrate all derivatives. Thus it is worthwhile to introduce a theory that gives us the possibility to study more general problems. In this connection, recently, B. Satco [19], [20] tried to extend several important results using Henstock–Kurzweil Δ-integral introduced by A. Peterson and B. Thompson [17]. This work motivates us to study dynamic equations on time scale domains in Henstock–Kurzweil Δ-integral setting. In this paper we have attempted to extend the basic notions and results for the first order dynamic equation on time scales involving a Δ-Carathéodory function using Henstock–Kurzweil Δ-integral.

Throughout this paper we are dealing with the first order dynamic equations defined by

$$\begin{cases} x^{\Delta}(t) = f\big(t, x(t)\big),\ t \in \mathbb{T}, \\ x(a) = x_0 \end{cases} \tag{1.1}$$

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where $x:\mathbb{T}\to\mathbb{R}$, $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a Δ-Carathéodory function and $\mathbb{T}=[a,b]_{\mathbb{T}}$ is a finite time scale interval with $\min\mathbb{T}=a$ and $\max\mathbb{T}=b$. We shall call such equations as Δ-*Carathéodory dynamic equations*.

The paper is organized as follows. In Section 2, we recall some basic concepts and results from time scales calculus and necessary facts about Henstock–Kurzweil Δ-integral. The Section 3 contains generalization of (1.1) using Henstock–Kurzweil Δ-integral. Section 4 deals with existence and uniqueness results pertaining to the equation (1.1). In Section 5, we study continuous dependence of the solution on initial condition and its convergence.

2. Preliminaries

This section contains some preliminary concepts and results that will be used throughout the paper. The following definitions and theorems related to time scales can be found in [2], [3], [5].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $t\in\mathbb{T}$, we define two operators, $\sigma:\mathbb{T}\to\mathbb{T}$, *the forward jump operator* and $\rho:\mathbb{T}\to\mathbb{T}$, *the backward jump operator* as $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$ and $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$ respectively. We assume that $\sigma(M)=M$ and $\rho(m)=m$ if $\mathbb{T}$ has maximum M and minimum m. The points in the time scale $\mathbb{T}$ are classified, with the help of jump operators σ and ρ as follows: A point $t\in\mathbb{T}$ is said to be right-scattered, left-scattered, right-dense, left-dense, dense and isolated if $\sigma(t)>t$, $\rho(t)<t$, $\sigma(t)=t$, $\rho(t)=t$, $\sigma(t)=t=\rho(t)$ and $\rho(t)<t<\sigma(t)$ respectively. For any $x:\mathbb{T}\to\mathbb{R}$, the corresponding forward shift $x^{\sigma}:\mathbb{T}\to\mathbb{R}$ is defined as $x^{\sigma}(t)=x\big(\sigma(t)\big)$ for any $t\in\mathbb{T}$. *The forward graininess function* $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t)=\sigma(t)-t$. If $\mathbb{T}$ has a left-scattered maximum M, then we derive a new set from $\mathbb{T}$ as $\mathbb{T}^{\kappa}=\mathbb{T}\setminus\{M\}$ otherwise $\mathbb{T}^{\kappa}=\mathbb{T}$. That is,

$$\mathbb{T}^{\kappa}=\begin{cases}\mathbb{T}\setminus\big(\rho(\sup\mathbb{T}),\sup\mathbb{T}\big] & \text{if}\sup\mathbb{T}<\infty,\\ \mathbb{T} & \text{if}\sup\mathbb{T}=\infty.\end{cases}$$

DEFINITION 1. [5] Let $f:\mathbb{T}\to\mathbb{R}$ be a given function. Then its *extension*, $\overline{f}$, to $\mathbb{R}$ is defined as

$$\overline{f}(t)=\begin{cases}f(t) & \text{if } t\in\mathbb{T},\\ f(t_i) & \text{if } t\in\big(t_i,\sigma(t_i)\big), \text{ for some } i\in I,\end{cases}$$

where $I\subset\mathbb{N}$ and $\{t_i\}_{i\in I}\subset\mathbb{T}$ is such that $\{t_i\}_{i\in I}=\big\{t\in\mathbb{T}:t<\sigma(t)\big\}$.

DEFINITION 2. [2] A function $x:\mathbb{T}\to\mathbb{R}$ is said to be Δ-*differentiable* at $t\in\mathbb{T}^{\kappa}$ if, for given any $\varepsilon>0$, there is a neighborhood U of t, $U=(t-\delta,t+\delta)\cap\mathbb{T}$ for some $\delta>0$ and a number $x^{\Delta}(t)$, such that for all $s\in U$ we have

$$\Big|x\big(\sigma(t)\big)-x(s)-x^{\Delta}(t)\big(\sigma(t)-s\big)\Big|\leqslant\varepsilon\big|\sigma(t)-s\big|.$$

The number $x^{\Delta}(t)$, if exists, is unique and is called the *delta derivative* of x at t.

THEOREM 1. [2] *Let $x:\mathbb{T}\to\mathbb{R}$ be a strictly increasing function and $x(\mathbb{T})=\widetilde{\mathbb{T}}$ is a time scale. Then $1/x^{\Delta}=(x^{-1})^{\widetilde{\Delta}}\circ x$ at points where x^{Δ} is nonzero.*

DEFINITION 3. [2] A function $x:\mathbb{T}\to\mathbb{R}$ is said to be *rd-continuous* if it is continuous at every right-dense point in $\mathbb{T}$ and its left sided limits exist at left dense points in $\mathbb{T}$. The set of all rd-continuous functions $x:\mathbb{T}\to\mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.

Now we recall some notions of measure theory on time scales from [5], [10], [13]. For $\tilde{a},\tilde{b}\in\mathbb{T}$, by the time scale interval $[\tilde{a},\tilde{b}]_{\mathbb{T}}$, we mean $[\tilde{a},\tilde{b}]\cap\mathbb{T}$. Let $\mathfrak{F}$ be the family of all intervals of the form $[\tilde{a},\tilde{b})_{\mathbb{T}}=\{t\in\mathbb{T}:\tilde{a}\leqslant t<\tilde{b}\}$, where $\tilde{a},\tilde{b}\in\mathbb{T}$ with $\tilde{a}\leqslant\tilde{b}$. The interval $[\tilde{a},\tilde{a})_{\mathbb{T}}$ is understood as the empty set. Let $m_1:\mathfrak{F}\to[0,\infty]$ be a function defined as $m_1\big([\tilde{a},\tilde{b})_{\mathbb{T}}\big)=\tilde{b}-\tilde{a}$. That is, m_1 assigns to each interval $[\tilde{a},\tilde{b})_{\mathbb{T}}$ its length. Now using the pair $(\mathfrak{F},m_1)$, we generate the outer measure m_1^* on the family of all subsets of $\mathbb{T}$ as follows.

For each subset E of $\mathbb{T}$, if there exists at least one finite or countable collection of intervals $[a_i,b_i)_{\mathbb{T}}\in\mathfrak{F}$ $(i=1,2,\ldots)$ such that $E\subset\cup_i[a_i,b_i)_{\mathbb{T}}$, then we define $m_1^*(E)=\inf\sum_i m_1\big([a_i,b_i)_{\mathbb{T}}\big)$, where the infimum is taken over all the coverings of E by a finite or countable collection of intervals $[a_i,b_i)_{\mathbb{T}}\in\mathfrak{F}$. If there is no such covering of E, then we set $m_1^*(E)=+\infty$. A subset $A\subset\mathbb{T}$ is said to be Δ-measurable if $m_1^*(E)=m_1^*(E\cap A)+m_1^*(E\setminus A)$ for all subsets $E\subset\mathbb{T}$. Now the restriction of m_1^* to the family $M(m_1^*)=\{A\subset\mathbb{T}:A \text{ is } \Delta\text{-measurable}\}$ defines a countably additive measure, denoted by μ_{Δ}, on $M(m_1^*)$, and is called *Lebesgue Δ-measure*. A property that holds everywhere except for a set of Δ-measure zero is said to hold Δ-almost everywhere or Δ-a.e.

DEFINITION 4. [10] A function $x:\mathbb{T}\to\overline{\mathbb{R}}=[-\infty,+\infty]$ is said to be *Δ-measurable* if for every $\alpha\in\mathbb{R}$, the set $f^{-1}\big([-\infty,\alpha)\big)=\big\{t\in\mathbb{T}:f(t)<\alpha\big\}$ is Δ-measurable.

DEFINITION 5. [5] A function $x:\mathbb{T}\to\mathbb{R}$ is said to be *simple* if it takes only finite number of distinct values, say, $\alpha_1,\alpha_2,\ldots,\alpha_n$, $n\in\mathbb{N}$. If $A_j=\big\{t\in\mathbb{T}:x(t)=\alpha_j\big\}$, then $x=\sum_{j=1}^{n}\alpha_j\chi_{A_j}$, where $\chi_{A_j}:\mathbb{T}\to\mathbb{R}$ is the characteristic function of A_j.

THEOREM 2. [10] *Let $x:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. Then there exists a sequence (x_n) of Δ-measurable simple functions such that $x_n(t)\to x(t)$ for all $t\in\mathbb{T}$.*

DEFINITION 6. [19] A function $x:\mathbb{T}\to\mathbb{R}$ is said to be *absolutely continuous in the restricted sense* on $E\subset\mathbb{T}$ if, for given every $\varepsilon>0$, there exists $\delta>0$ such that, whenever $\big\{[c_i,d_i]_{\mathbb{T}}:1\leqslant i\leqslant n\big\}$ is a finite pairwise disjoint family of subintervals of $\mathbb{T}$ with $c_i,d_i\in E$ satisfying $\sum_{i=1}^{n}\mu_{\Delta}\big([c_i,d_i]_{\mathbb{T}}\big)<\delta$ we have $\sum_{i=1}^{n}\operatorname{osc}\big(x,[c_i,d_i]_{\mathbb{T}}\big)<\varepsilon$. We write $x\in\Delta\text{-}AC_*$. x is said to be a *generalized absolutely continuous function in the restricted sense* if x is continuous on E and E can be written as countable union of sets on each of which x is $\Delta\text{-}AC_*$. In this case we write $x\in\Delta\text{-}ACG_*$. If x is uniformly continuous on E and E can be written as countable union of sets on each of which x is $\Delta\text{-}AC_*$, then we say that x is uniformly $\Delta\text{-}ACG_*$ and write $x\in\Delta\text{-}UACG_*$.

The following definitions and theorem are taken from [17].

DEFINITION 7. We say that $\delta = (\delta_L, \delta_R)$ is a Δ-*gauge* for $[a,b]_{\mathbb{T}}$ provided $\delta_L(t) > 0$ on $[a,b)_{\mathbb{T}}$, $\delta_R(t) > 0$ on $(a,b]_{\mathbb{T}}$, $\delta_L(a) \geqslant 0$, $\delta_R(b) \geqslant 0$ and $\delta_R(t) \geqslant \mu(t)$ for all $t \in [a,b)_{\mathbb{T}}$.

DEFINITION 8. A partition $\mathscr{P}$ of $[a,b]_{\mathbb{T}}$ is a division of $[a,b]_{\mathbb{T}}$ defined by $\mathscr{P} = \{a = t_0 \leqslant \xi_1 \leqslant t_1 \leqslant \xi_2 \leqslant t_2 \ldots t_{n-1} \leqslant \xi_n \leqslant t_n = b\}$ with $t_i > t_{i-1}$ for $1 \leqslant i \leqslant n$ and t_i, $\xi_i \in \mathbb{T}$. We call the points, ξ_i, tag points associated with the subinterval $[t_{i-1}, t_i]_{\mathbb{T}}$ of $[a,b]_{\mathbb{T}}$. We denote such partitions by $\mathscr{P} = \{\xi_i, [t_{i-1}, t_i]_{\mathbb{T}}\}_{i=1}^{n}$.

DEFINITION 9. If δ is a Δ-gauge for $[a,b]_{\mathbb{T}}$, then we say that a partition $\mathscr{P}$ is δ-*fine* if $\xi_i - \delta_L(\xi_i) \leqslant t_{i-1} < t_i \leqslant \xi_i + \delta_R(\xi_i)$ for $1 \leqslant i \leqslant n$.

DEFINITION 10. We say that $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is *Henstock–Kurzweil* Δ-*integrable* on $[a,b]_{\mathbb{T}}$ provided there is a real number I such that given any $\varepsilon > 0$ there is a Δ-gauge δ, for $[a,b]_{\mathbb{T}}$ such that for all δ-fine partitions $\mathscr{P} = \{\xi_i, [t_{i-1}, t_i]_{\mathbb{T}}\}_{i=1}^{n}$ of $[a,b]_{\mathbb{T}}$ we have $\left|I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})\right| < \varepsilon$. The number I is called as Henstock–Kurzweil Δ-integral of f on $[a,b]_{\mathbb{T}}$ and we write $I = (HK)\int_a^b f(t)\Delta t$.

THEOREM 3. (Monotone Convergence Theorem) *Let (f_n) be a sequence of Henstock–Kurzweil Δ-integrable functions on $[a,b]_{\mathbb{T}}$ such that $f_n \leqslant f_{n+1}$ Δ-a.e. in $[a,b]_{\mathbb{T}}$. Then f is Henstock–Kurzweil Δ-integrable on $[a,b]_{\mathbb{T}}$ and* $\lim_{n\to\infty} \int_a^b f_n(t)\Delta t = \int_a^b f(t)\Delta t$.

3. Generalized dynamic equations

H. Gilbert [12] has given the notion of Carathéodory function on time scales in the following way.

A function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is said to be a Δ-*Carathéodory function* if it satisfies the following conditions:

(C-i) The map $t \mapsto f(t,x)$ is Δ-measurable for every $x \in \mathbb{R}$.

(C-ii) The map $x \mapsto f(t,x)$ is continuous Δ-a.e. $t \in \mathbb{T}$.

(C-iii) For every real number $r > 0$ there exists a function $h_r \in L^1_\Delta(\mathbb{T}, [0,\infty))$ such that $|f(t,x)| \leqslant h_r(t)$ Δ-a.e. $t \in \mathbb{T}$ and for $x \in \mathbb{R}$ with $x \in \overline{B}_r(x_0) = \{x \in \mathbb{R} : |x - x_0| \leqslant r\}$.

With this function f, the equation (1.1) is said to be Δ-*Carathéodory dynamic equation*.

According to I. L. D. Santos [18], a function $x : \mathbb{T} \to \mathbb{R}$ is said to be the *solution* of Δ-Carathéodory dynamic equation (1.1) if $x(t)$ satisfies the following conditions:

(i) $x(t)$ is absolutely continuous on each compact time scale subinterval J of $\mathbb{T}$.

(ii) $x^\Delta(t) = f(t, x(t))$ Δ-a.e. $t \in \mathbb{T}$ and $x(a) = x_0$.

On the lines of T. S. Chew and F. Flordeliza [6], we generalize the Δ-Carathéodory dynamic equation (1.1) using Henstock–Kurzweil Δ-integral as follows.

Equation (1.1) is said to be *generalized* Δ-*Carathéodory dynamic equation* if the function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(C-i) The map $t \mapsto f(t,x)$ is Δ-measurable for every $x \in \mathbb{R}$.

(C-ii) The map $x \mapsto f(t,x)$ is continuous Δ-a.e. $t \in \mathbb{T}$.

(GC-iii) For every real number $r > 0$ there exist two Henstock–Kurzweil Δ-integrable functions $g_r(t)$ and $h_r(t)$ on $\mathbb{T}$ such that $g_r(t) \leqslant f(t,x) \leqslant h_r(t)$ Δ-a.e. $t \in \mathbb{T}$ and for $x \in \mathbb{R}$ with $x \in \overline{B}_r(x_0) = \left\{x \in \mathbb{R} : |x - x_0| \leqslant r\right\}$.

In this case, the function f is said to be the *generalized Δ-Carathéodory function*.

DEFINITION 11. A function $x : \mathbb{T} \to \mathbb{R}$ is said to be a *solution* of generalized Δ-Carathéodory dynamic equation (1.1) if $x(t)$ satisfies the following conditions:

(i) $x(t)$ is Δ-ACG_* function on each compact time scale subinterval J of $\mathbb{T}$.

(ii) $x^{\Delta}(t) = f\big(t, x(t)\big)$ Δ-a.e. $t \in \mathbb{T}$ and $x(a) = x_0$.

We observe that if the functions $g_r(t)$ and $h_r(t)$ are Δ-Lebesgue integrable functions on $\mathbb{T}$, then equation (1.1) reduces to the dynamic equation considered by I. L. D. Santos [18] and the function $x : \mathbb{T} \to \mathbb{R}$ reduces to the corresponding Δ-Carathéodory solution. Also, condition (GC-iii) can be written as $0 \leqslant f(t,x) - g_r(t) \leqslant h_r(t) - g_r(t)$ Δ-a.e. $t \in \mathbb{T}$ and $x \in \overline{B}_r(x_0)$. Then $h_r(t) - g_r(t)$ is nonnegative Henstock–Kurzweil Δ-integrable function and by Theorem 2.19 of [17] it is a Δ-Lebesgue integrable function, and $f(t,x) - g_r(t)$ satisfies condition (C-iii). Thus our generalized Δ-Carathéodory dynamic equation (1.1) is essentially first order dynamic equation of the form $x^{\Delta}(t) = F(t,x) + G(t)$, where $F(t,x)$ is Δ-Lebesgue integrable function and $G(t)$ is Henstock–Kurzweil Δ-integrable function on $\mathbb{T}$, that is, *a first order dynamic equation perturbed by Henstock–Kurzweil Δ-integrable function*.

4. Existence and uniqueness of solutions

In this section we investigate the results concerning existence of solution of the generalized Δ-Carathéodory dynamic equation (1.1) and its uniqueness. The following theorem establishes existence of solutions, the proof of which is analogous to the Theorem 3.1 of [6].

THEOREM 4. *Let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be a generalized Δ-Carathéodory function. Then the dynamic equation (1.1) has generalized Δ-Carathéodory solution in $\mathbb{T}$.*

Proof. Since f is a generalized Δ-Carathéodory function, by (GC-iii) there exist two Henstock–Kurzweil Δ-integrable functions $g_r(t)$ and $h_r(t)$ on $\mathbb{T}$ such that for all $x \in \overline{B}_r(x_0)$, $g_r(t) \leqslant f(t,x) \leqslant h_r(t)$ Δ-a.e. $t \in \mathbb{T}$. The function $h_r(t) - g_r(t)$ is Henstock–Kurzweil Δ-integrable on $\mathbb{T}$ and $h_r(t) - g_r(t) \geqslant 0$, by Theorem 2.19 of [17], $h_r(t) - g_r(t)$ is Δ-Lebesgue integrable on $\mathbb{T}$.

Define $F : \mathbb{T} \times \overline{B}_r(x_0) \to \mathbb{R}$ by

$$F(t,x) = f\left(t, x + \int_a^t g_r(s)\Delta s\right) - g_r(t).$$

This function F is a Δ-Carathéodory function satisfying $0 \leqslant F(t,x) \leqslant h_r(t) - g_r(t)$ for all $(t,x) \in \Omega$, where $\Omega = \left\{(t,x) \in \mathbb{T} \times \overline{B}_r(x_0) : \left|x + \int_a^t g_r(\tau)\Delta\tau - x_0\right| \leqslant r\right\}$. By Theorem 5 of [18], there is a function $y : \mathbb{T} \to \mathbb{R}$ such that $y^\Delta(t) = F\big(t, y(t)\big)$ Δ-a.e. $t \in \mathbb{T}$ and $y(a) = x_0$.

Define $\phi(t) = y(t) + \int_a^t g_r(\tau)\Delta\tau$ for $t \in \mathbb{T}$. Then

$$\begin{aligned}
\phi^\Delta(t) &= y^\Delta(t) + g_r(t) \;\; \Delta\text{-a.e. } t \in \mathbb{T} \\
&= F\big(t, y(t)\big) + g_r(t) \\
&= f\left(t, y + \int_a^t g_r(\tau)\Delta\tau\right) - g_r(t) + g_r(t) \\
&= f\left(t, y + \int_a^t g_r(\tau)\Delta\tau\right)
\end{aligned}$$

and $\phi(a) = y(a) = x_0$.

Note that the function ϕ is Δ-ACG_* on $\mathbb{T}$ because y is Δ-ACG_* on $\mathbb{T}$ and g_r is Henstock–Kurzweil Δ-integrable there. This completes the proof. $\square$

EXAMPLE 1. Let $\mathbb{T} = \left\{t = \frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$.

Let $f(t,x) = h(t,x) + g(t)$, where $\left|h(t,x)\right| \leqslant H(t)$, $\forall\, t \in \mathbb{T}$ and $x \in \overline{B}_r(x_0)$ and $H(t)$ is Lebesgue Δ-integrable on $\mathbb{T}$.

Define $g : \mathbb{T} \to \mathbb{R}$ by

$$g(t) = \begin{cases} (-1)^n\, n & \text{if } t = \frac{1}{n}, \\ L & \text{if } t = 0, \end{cases}$$

where L is any constant.

g is neither Δ-integrable on $\mathbb{T}$ nor Lebesgue Δ-integrable on $\mathbb{T}$. But it is Henstock–Kurzweil Δ-integrable on $\mathbb{T}$.

Note that $\forall\, t \in \mathbb{T}$ and $x \in \overline{B}_r(x_0)$, $g(t) - H(t) \leqslant f(t,x) \leqslant g(t) + H(t)$.

Thus $f(t,x)$ is generalized Δ-Carathéodory function and by Theorem 4 the dynamic equation (1.1) has a solution in $\mathbb{T}$.

When $h(t,x) = 0$, the function $x : \mathbb{T} \to \mathbb{R}$ defined by

$$x(t) = \begin{cases} 0 & \text{if } t = 1, \\ \sum_{k=2}^n \frac{(-1)^{k+1}}{k-1} & \text{if } t = \frac{1}{n}, \\ -\ln(2) & \text{if } t = 0 \end{cases}$$

is a solution of (1.1) with f as defined above and $x(0) = -\ln(2)$.

A new existence result for solution of (1.1) is established as follows.

THEOREM 5. *Let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be a Δ-Carathéodory function. Assume that (F_n) is a sequence of Δ-UACG$_*$ functions on $\mathbb{T}$ such that $F_n \to x$ uniformly Δ-a.e. on $\mathbb{T}$, where $F_n(t) = \int_a^t \phi_n(\tau)\Delta\tau$, $\big(\phi_n(\tau)\big)$ is a sequence of Δ-measurable simple functions. Then the dynamic equation (1.1) has generalized Δ-Carathéodory solution $x(t)$.*

Proof. Since $f\big(t,x(t)\big)$ is Δ-measurable in t, by Theorem 3.13 of [10], there exist a nondecreasing sequence of Δ-measurable simple functions $\big(\phi_n(t)\big)$ on $\mathbb{T}$ such that $\phi_n(t) \to f\big(t,x(t)\big)$ for all $t \in \mathbb{T}$. Since $\phi_n(t)$ is Henstock–Kurzweil Δ-integrable and $\phi_n(t) \leqslant \phi_{n+1}(t)$ Δ-a.e. on $\mathbb{T}$ and for all $n \in \mathbb{N}$, by monotone convergence theorem, $f\big(t,x(t)\big)$ is Henstock–Kurzweil Δ-integrable on $\mathbb{T}$ and

$$\int_a^t f\big(\tau,x(\tau)\big)\Delta\tau = \lim_{n\to\infty}\int_a^t \phi_n(\tau)\Delta\tau.$$

Define $F_n(t) = \int_a^t \phi_n(\tau)\Delta\tau + x_0$. Then $\big(F_n\big)$ is a sequence of Δ-*UACG*$_*$ functions and $F_n(t) \to x(t)$, $t \in \mathbb{T}$. Therefore

$$\lim_{n\to\infty} F_n(t) = \int_a^t f\big(\tau,x(\tau)\big)\Delta\tau + x_0.$$

That is,

$$x(t) = \int_a^t f\big(\tau,x(\tau)\big)\Delta\tau + x_0,$$

which gives equation (1.1). Now since F_n is a sequence of uniformly Δ-*ACG*$_*$ functions converges to x and x is primitive of Henstock–Kurzweil Δ-integrable function f on $\mathbb{T}$, it follows that x is Δ-*ACG*$_*$ on $\mathbb{T}$. This completes the proof. □

The inequality which is useful for further analysis is given in the following result.

THEOREM 6. *Suppose* $g : [0,\infty) \to [0,\infty)$ *is continuous nondecreasing function and* $y : \mathbb{T} \to [0,\infty)$ *is bounded function such that* $g \circ y : \mathbb{T} \to [0,\infty)$ *is rd-continuous. Let* $G : \mathbb{T} \to \mathbb{R}$ *be a strictly increasing function such that* $G^{\Delta}(u) = 1/g(u) \neq 0$ *for* $u \in \mathbb{T}$. *Then for all* $t \in \mathbb{T}$ *and for some constant* $l \geqslant 0$

$$y(t) \leqslant l + \int_a^t g\big(y(\tau)\big)\Delta\tau \tag{4.1}$$

implies $y(t) \leqslant G^{-1}\big(G^{\sigma}(l) + t - a\big)$, *where* $G^{-1} : \widetilde{\mathbb{T}} = G(\mathbb{T}) \to \mathbb{T}$ *is the inverse of* $G :$ $\mathbb{T} \to \mathbb{R}$.

Proof. Let $\alpha = \min\widetilde{\mathbb{T}}$, $\beta = \max\widetilde{\mathbb{T}}$. Denote $G(a) = \alpha$, $G(b) = \beta$ and $y(t) = l + \int_a^t g\big(y(\tau)\big)\Delta\tau$. Let $T \in \mathbb{T}$ be such that $\alpha - (t-a) < G(T) < \beta - (t-a)$ for all $t \in \mathbb{T}$. For $s \in \big[\alpha - G(T), \beta - G(T)\big]_{\widetilde{\mathbb{T}}}$, set $G^{-1}\big(G(T)+s\big) = u$. Then $u \in \mathbb{T}$ and therefore $G^{\Delta}(u) = 1/g(u) \neq 0$, that is, $G^{\Delta}\big(G^{-1}(G(T)+s)\big) = 1/g\Big(G^{-1}\big(G(T)+s\big)\Big)$.

But, by derivative of inverse, we have $\big(G^{-1}(p)\big)^{\Delta} = \big(\frac{1}{G^{\Delta}} \circ G^{-1}\big)(p)$ for $p \in [\alpha,\beta]_{\widetilde{\mathbb{T}}}$. Therefore $\big(G^{-1}(p)\big)^{\Delta} = \big(g \circ G^{-1}\big)(p)$.

Now for $t \in \mathbb{T}$

$$\begin{aligned}\int_a^t g\Big(G^{-1}\big(G(T)+\tau-a\big)\Big)\Delta\tau &= \int_a^t (g\circ G^{-1})\big(G(T)+\tau-a\big)\Delta\tau \\ &= \int_a^t (G^{-1})^{\Delta}\big(G(T)+\tau-a\big)\Delta\tau \\ &= G^{-1}\big(G(T)+t-a\big)-G^{-1}\big(G(T)+a-a\big) \\ &= G^{-1}\big(G(T)+t-a\big)-T.\end{aligned}$$

Therefore $T+\int_a^t g\Big(G^{-1}\big(G(T)+\tau-a\big)\Big)\Delta\tau = G^{-1}\big(G(T)+t-a\big)$. Let $T=\sigma(l)$ for some $l\in\mathbb{T}^{\kappa}$ such that $G^{\sigma}(l)<\alpha-b+a$. Then

$$\sigma(l)+\int_a^t g\Big(G^{-1}\big(G^{\sigma}(l)+\tau-a\big)\Big)\Delta\tau = G^{-1}\big(G^{\sigma}(l)+t-a\big). \tag{4.2}$$

The function G^{-1} is now bounded. So combing (4.1) and (4.2), we obtain $y(t)-G^{-1}\big(G^{\sigma}(l)+t-a\big)$

$$\begin{aligned}&\leqslant l+\int_a^t g\big(y(\tau)\big)\Delta\tau-\sigma(l)-\int_a^t g\Big(G^{-1}\big(G^{\sigma}(l)+\tau-a\big)\Big)\Delta\tau \\ &= l-\sigma(l)+\int_a^t \big[g\big(y(\tau)\big)-g\big(G^{-1}(G^{\sigma}(l)+\tau-a)\big)\big]\Delta\tau.\end{aligned}$$

By boundedness of y and G^{-1}, and since g is nondecreasing, we get

$$\left|g\big(y(\tau)\big)-g\Big(G^{-1}\big(G^{\sigma}(l)+\tau-a\big)\Big)\right|<M$$

for some $M>0$ and for all $\tau\in\mathbb{T}$.

Therefore $y(t)-G^{-1}\big(G^{\sigma}(l)+t-a\big)\leqslant l-\sigma(l)+M(t-a)$.

If $\Lambda=\big\{t\in\mathbb{T}: y(t)<G^{-1}\big(G^{\sigma}(l)+\tau-a\big)$ for $a\leqslant\tau\leqslant b\big\}$, then one can see that $\sup\Lambda=\max\mathbb{T}=b$. Hence $y(t)\leqslant G^{-1}\big(G^{\sigma}(l)+t-a\big)$ for $t\in\mathbb{T}$. This completes the proof. $\square$

We present local uniqueness of solution of (1.1), using an Osgood type condition.

THEOREM 7. *Assume that the function $f:\mathbb{T}\times\overline{B}_r(x_0)\to\mathbb{R}$ satisfies the condition $|f(t,x)-f(t,y)|\leqslant g\big(|x-y|\big)$ where $g:\mathbb{T}\to[0,\infty)$ is a function such that its extension $\overline{g}:[0,\infty)\to[0,\infty)$ is an increasing function with $\overline{g}(0)=0$ and $\overline{g}(s)>0$ for $s\in(0,\infty)$ and for every $k>0$ $\int_0^k 1/\overline{g}(s)ds=\infty$. Then the generalized Δ-Carathéodory dynamic equation (1.1) has unique solution in $\mathbb{T}$.*

Proof. Assume that $x(t)$ and $y(t)$ are two solutions of the generalized Δ-Carathéodory dynamic equation (1.1). Then $x,y:\mathbb{T}\to\mathbb{R}$ are such that $x(t)=x_0+\int_a^t f\big(\tau,x(\tau)\big)\Delta\tau$

and $y(t)=x_0+\int_a^t f\big(\tau,y(\tau)\big)\Delta\tau$. Let $z(t)=x(t)-y(t)$ for all $t\in\mathbb{T}$. Therefore

$$\begin{aligned}|z(t)| &= \left|\int_a^t \Big[f\big(\tau,x(\tau)\big)-f\big(\tau,y(\tau)\big)\Big]\Delta\tau\right| \\ &\leqslant \int_a^t \big|f\big(\tau,x(\tau)\big)-f\big(\tau,y(\tau)\big)\big|\Delta\tau \\ &\leqslant \int_a^t g\big(|z(\tau)|\big)\Delta\tau \\ &\leqslant \int_a^b g\big(|z(\tau)|\big)\Delta\tau.\end{aligned}$$

If $G^\Delta(u)=1/g(u)$, then by Theorem 6 we can write

$$\begin{aligned}\big|z(t)\big| &\leqslant G^{-1}\big(G^\sigma(0)+b-a\big) \\ G\Big(\big|z(t)\big|\Big) &\leqslant G^\sigma(0)+b-a \\ G\Big(\big|z(t)\big|\Big)-G^\sigma(0) &\leqslant b-a \\ \int_{\sigma(0)}^{|z(t)|} G^\Delta(r)\Delta r &\leqslant b-a \\ \int_{\sigma(0)}^{|z(t)|} \frac{1}{g(r)}\Delta r &\leqslant b-a.\end{aligned}$$

But from Theorems 19 and 20 of [22], $\int_{\sigma(0)}^{|z(t)|}\frac{1}{g(r)}\Delta r=\int_0^{|z(t)|}\frac{1}{\overline{g}(s)}ds$.

Therefore $\int_0^{|z(t)|} 1/\overline{g}(s)ds\leqslant b-a$ which gives $\infty\leqslant b-a$, a contradiction.

Hence we must have $\big|z(t)\big|=0$. and therefore the dynamic equation (1.1) has unique solution in $\mathbb{T}$. This completes the proof. □

REMARK 1. If the conditions given in Theorem 7 do not hold, then the dynamic equation (1.1) may have more than one solutions. For example:

Let $f:\mathbb{T}\times\overline{B}_r(0)\to\mathbb{R}$ be defined by

$$f(t,x)=\begin{cases}\dfrac{1}{2\left(\sqrt{t}+\sqrt{\sigma(t)}\right)} & \text{for } x<t,\\[2ex] \dfrac{1}{\left(\sqrt{t}+\sqrt{\sigma(t)}\right)} & \text{for } x\geqslant t,\end{cases}$$

where $\overline{B}_r(0)=\big\{x\in\mathbb{R}:|x|\leqslant r\big\}$, $r>1$ and $\mathbb{T}=\big\{t=\frac{1}{n}:n\in\mathbb{N}\big\}\cup\{0\}$. The function f does not satisfies conditions given in Theorem 7 as $\int_0^1 f\big(t,x(t)\big)\Delta t=1$. In this case $x_1(t)=\sqrt{t}$ and $x_1(t)=\sqrt{t}/2$ are two solutions of equation (1.1).

REMARK 2. The conditions given in the Theorem 7 do not guarantee the existence of a solution of equation (1.1), as shown in the next example.

Let $[0,1)_{\mathbb{T}}$ be a time scale interval that contains a countable infinite subset $\cup_{i=1}^{\infty}\{r_i\}$ with $\sigma(r_i)=r_i$.

Let $f:[0,1]_{\mathbb{T}}\times\overline{B}_r(0)\to\mathbb{R}$ defined by

$$f(t,x)=\begin{cases}0 & \text{for } x\neq 0,\ t\neq r_i\\ 1 & \text{for } x\neq 0,\ t=r_i \quad \text{or} \quad x=0,\ t\neq r_i\\ 2 & \text{for } x=0,\ t=r_i\ .\end{cases}$$

f is generalized Δ-Carathéodory function and satisfies condition given in Theorem 7 with $\overline{g}\equiv 0$. For this function f, dynamic equation (1.1) has no solution.

5. Continuous dependence on initial condition and convergence of solutions

In this section we first obtain a result which describes the continuous dependence of solution of (1.1) on the initial condition.

THEOREM 8. *Assume that the function $f:\mathbb{T}\times\overline{B}_r(x_0)\to\mathbb{R}$ satisfies the condition $\big|f(t,x)-f(t,y)\big|\leqslant g\big(|x-y|\big)$ where $g:\mathbb{T}\to[0,\infty)$ is an increasing continuous function such that $G^{\Delta}(u)=1/g(u)\neq 0$ for $u\in\mathbb{T}$. Let x and y be the solution of $x^{\Delta}(t)=f\big(t,x(t)\big)$, $x(a)=x_0$ and $y^{\Delta}(t)=f\big(t,y(t)\big)$, $y(a)=y_0$ respectively in $\mathbb{T}$. Then $\big|x(t)-y(t)\big|\leqslant G^{-1}\Big(G\big(|x_0-y_0|\big)+t-a\Big)$ for all $t\in\mathbb{T}$.*

Proof. Since x and y are the solutions of $x^{\Delta}(t)=f\big(t,x(t)\big)$, $x(a)=x_0$ and $y^{\Delta}(t)=f\big(t,y(t)\big)$, $y(a)=y_0$ in $\mathbb{T}$ respectively, we have

$$x(t)=x_0+\int_a^t f\big(\tau,x(\tau)\big)\Delta\tau \ \text{ and } \ y(t)=y_0+\int_a^t f\big(\tau,y(\tau)\big)\Delta\tau\ .$$

Therefore for all $t\in\mathbb{T}$

$$\begin{aligned}\big|x(t)-y(t)\big| &\leqslant |x_0-y_0|+\int_a^t\Big|f\big(\tau,x(\tau)\big)-f\big(\tau,y(\tau)\big)\Big|\Delta\tau\\ &\leqslant |x_0-y_0|+\int_a^t g\Big(\big|x(\tau)-y(\tau)\big|\Big)\Delta\tau.\end{aligned}$$

By Theorem 6 we have $\big|x(t)-y(t)\big|\leqslant G^{-1}\Big(G\big(|x_0-y_0|\big)+t-a\Big)$, which shows that the solutions of equation (1.1) depend continuously on the initial condition. To emphasize the dependence on the initial condition, we denote the solutions of (1.1) by $\phi(t,a,x_0)$. Thus $\big|\phi(t,a,x_0)-\phi(t,a,y_0)\big|\leqslant G^{-1}\Big(G\big(|x_0-y_0|\big)+t-a\Big)$ for all $t\in\mathbb{T}$. This completes the proof. $\square$

COROLLARY 1. *Under the assumption of the above theorem we have the following.*

$$\begin{aligned}\left|\phi(t,a,x_0)-\phi(s,\widetilde{a},y_0)\right| \leqslant\ & G^{-1}\Big(G\big(|x_0-y_0|\big)+t-a\Big)\\ &+G^{-1}\Big(G\big(M\,|\widetilde{a}-a|\big)+t-\widetilde{a}\Big)+M\,|t-s|,\end{aligned}$$

where $M=\sup_{t\in\mathbb{T}}\left|f(t,\widetilde{a},y_0)\right|$.

Proof. We write

$$\begin{aligned}\left|\phi(t,a,x_0)-\phi(s,\widetilde{a},y_0)\right| \leqslant\ & \left|\phi(t,a,x_0)-\phi(t,a,y_0)\right|+\left|\phi(t,a,y_0)-\phi(t,\widetilde{a},y_0)\right|\\ &+\left|\phi(t,\widetilde{a},y_0)-\phi(s,\widetilde{a},y_0)\right|.\end{aligned} \tag{5.1}$$

Consider

$$\begin{aligned}\left|\phi(t,a,x_0)-\phi(t,a,y_0)\right| &\leqslant |x_0-y_0|+\int_a^t\Big|f\big(\tau,\phi(\tau,a,x_0)\big)-f\big(\tau,\phi(\tau,a,y_0)\big)\Big|\Delta\tau\\ &\leqslant |x_0-y_0|+\int_a^t g\Big(\left|\phi(\tau,a,x_0)-\phi(\tau,a,y_0)\right|\Big)\Delta\tau\ ,\end{aligned}$$

which yields

$$\left|\phi(t,a,x_0)-\phi(t,a,y_0)\right| \leqslant G^{-1}\Big(G\big(|x_0-y_0|\big)+t-a\Big). \tag{5.2}$$

Assuming $0\leqslant a<\widetilde{a}<t$,

$$\begin{aligned}\left|\phi(t,a,y_0)-\phi(t,\widetilde{a},y_0)\right| &\leqslant \int_a^t\Big|f\big(\tau,\phi(\tau,a,y_0)\big)-f\big(\tau,\phi(\tau,\widetilde{a},y_0)\big)\Big|\Delta\tau\\ &=\int_a^{\widetilde{a}}\Big|f\big(\tau,\phi(\tau,a,y_0)\big)-f\big(\tau,\phi(\tau,\widetilde{a},y_0)\big)\Big|\Delta\tau\\ &\quad+\int_{\widetilde{a}}^t\Big|f\big(\tau,\phi(\tau,a,y_0)\big)-f\big(\tau,\phi(\tau,\widetilde{a},y_0)\big)\Big|\Delta\tau\\ &\leqslant\int_a^{\widetilde{a}}\Big|f\big(\tau,\phi(\tau,a,y_0)\big)\Big|\Delta\tau\\ &\quad+\int_{\widetilde{a}}^t\Big|f\big(\tau,\phi(\tau,a,y_0)\big)-f\big(\tau,\phi(\tau,\widetilde{a},y_0)\big)\Big|\Delta\tau\\ &\leqslant M\,|\widetilde{a}-a|+\int_{\widetilde{a}}^t\Big|f\big(\tau,\phi(\tau,a,y_0)\big)-f\big(\tau,\phi(\tau,\widetilde{a},y_0)\big)\Big|\Delta\tau\\ &\leqslant M\,|\widetilde{a}-a|+\int_{\widetilde{a}}^t g\Big(\left|\phi(\tau,a,y_0)-\phi(\tau,\widetilde{a},y_0)\right|\Big)\Delta\tau,\end{aligned}$$

which yields

$$\left|\phi(t,a,y_0)-\phi(t,\widetilde{a},y_0)\right| \leqslant G^{-1}\Big(G\big(M\left|\widetilde{a}-a\right|\big)+t-\widetilde{a}\Big). \tag{5.3}$$

Now

$$\begin{aligned}\left|\phi(t,\widetilde{a},y_0)-\phi(s,\widetilde{a},y_0)\right| &= \left|\int_s^t f(\tau,\widetilde{a},y_0)\Delta\tau\right| \\ &\leqslant \int_s^t \left|f(\tau,\widetilde{a},y_0)\right|\Delta\tau.\end{aligned}$$

That is,

$$\left|\phi(t,\widetilde{a},y_0)-\phi(s,\widetilde{a},y_0)\right| \leqslant M\left|t-s\right|, \tag{5.4}$$

where $M=\sup_{\tau\in\mathbb{T}}\left|f(\tau,\widetilde{a},y_0)\right|$. So using (5.2), (5.3) and (5.4), (5.1) becomes

$$\begin{aligned}\left|\phi(t,a,x_0)-\phi(s,\widetilde{a},y_0)\right| \leqslant G^{-1}\Big(&G\big(\left|x_0-y_0\right|\big)+t-a\Big) \\ &+G^{-1}\Big(G\big(M\left|\widetilde{a}-a\right|\big)+t-a\Big)+M\left|t-s\right|.\end{aligned} \tag{5.5}$$

This completes the proof. □

Now we introduce the concept of Henstock–Kurzweil Δ-equi-integrability and a result needed for the convergence of solutions of equation (1.1).

DEFINITION 12. A family $\left(f_n\right)$ of functions is said to be Henstock–Kurzweil Δ-equi-integrable on $\mathbb{T}$ provided given any $\varepsilon>0$, there exists a Δ-gauge, δ, for $\mathbb{T}$ such that for all $n\in\mathbb{N}$

$$\left|\int_a^b f_n-\sum_{i=1}^{k} f_n(\xi_i)(t_i-t_{i-1})\right|<\varepsilon$$

for all δ-fine partitions $\mathscr{P}=\left\{\xi_i,[t_{i-1},t_i]_{\mathbb{T}}\right\}_{i=1}^{k}$ of $\mathbb{T}$.

THEOREM 9. *If $\left(f_n\right)$ is the sequence of Henstock–Kurzweil Δ-equi-integrable functions on $\mathbb{T}$ such that $f_n\to f$ on $\mathbb{T}$, then f is Henstock–Kurzweil Δ-integrable on $\mathbb{T}$ and* $\lim_{n\to\infty}\int_a^b f_n(t)\Delta t=\int_a^b f(t)\Delta t$.

Proof. Since the sequence $\left(f_n\right)$ is of Henstock–Kurzweil Δ-equi-integrable functions on $\mathbb{T}$, for given $\varepsilon>0$, there is a Δ-gauge, δ, for $\mathbb{T}$ such that for all $n\in\mathbb{N}$

$$\left|\int_a^b f_n-\sum_{i=1}^{k} f_n(\xi_i)(t_i-t_{i-1})\right|<\varepsilon$$

for all δ-fine partitions $\mathscr{P}=\left\{\xi_i,[t_{i-1},t_i]_{\mathbb{T}}\right\}_{i=1}^{k}$ of $\mathbb{T}$.

We claim that the sequence $\left(\int_a^b f_n(t)\Delta t\right)$ is convergent. We have

$$\begin{aligned}\left|\int_a^b f_n(t)\Delta t-\int_a^b f_m(t)\Delta t\right| &= \Bigg|\int_a^b f_n(t)\Delta t-\sum_{i=1}^k f_n(\xi_i)(t_i-t_{i-1})\\ &\quad+\sum_{i=1}^k f_n(\xi_i)(t_i-t_{i-1})-\sum_{i=1}^k f_m(\xi_i)(t_i-t_{i-1})\\ &\quad+\sum_{i=1}^k f_m(\xi_i)(t_i-t_{i-1})-\int_a^b f_m(t)\Delta t\Bigg|\\ &\leqslant \left|\int_a^b f_n(t)\Delta t-\sum_{i=1}^k f_n(\xi_i)(t_i-t_{i-1})\right|\\ &\quad+\left|\sum_{i=1}^k f_n(\xi_i)(t_i-t_{i-1})-\sum_{i=1}^k f_m(\xi_i)(t_i-t_{i-1})\right|\\ &\quad+\left|\sum_{i=1}^k f_m(\xi_i)(t_i-t_{i-1})-\int_a^b f_m(t)\Delta t\right|\\ &< \varepsilon+\varepsilon+\left|\sum_{i=1}^k f_n(\xi_i)(t_i-t_{i-1})-\sum_{i=1}^k f_m(\xi_i)(t_i-t_{i-1})\right|\\ &= 2\varepsilon+\sum_{i=1}^k \left|f_n(\xi_i)-f_m(\xi_i)\right|\left|t_i-t_{i-1}\right|.\end{aligned}$$

Since $f_n \to f$ on $\mathbb{T}$, for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m,\ n \geqslant n_0$ and for all $\xi_i \in \mathbb{T}$, we have $\left|f_n(\xi_i)-f_m(\xi_i)\right| < \varepsilon/(b-a)$. Therefore

$$\begin{aligned}\left|\int_a^b f_n(t)\Delta t-\int_a^b f_m(t)\Delta t\right| &< 2\varepsilon+\sum_{i=1}^k \frac{\varepsilon}{(b-a)}\left|t_i-t_{i-1}\right|\\ &< 2\varepsilon+\varepsilon=3\varepsilon.\end{aligned}$$

That is, $\left|\int_a^b f_n(t)\Delta t-\int_a^b f_m(t)\Delta t\right| < 3\varepsilon$. Therefore $\left(\int_a^b f_n(t)\Delta t\right)$ is a Cauchy sequence in $\mathbb{R}$ and hence it is convergent.

Let $\lim_{n\to\infty}\int_a^b f_n(t)\Delta t = A$. Now

$$\begin{aligned}\left|\sum_{i=1}^k f(\xi_i)(t_i-t_{i-1})-A\right| &= \Bigg|\sum_{i=1}^k f(\xi_i)(t_i-t_{i-1})-\sum_{i=1}^k f_n(\xi_i)(t_i-t_{i-1})\\ &\quad+\sum_{i=1}^k f_n(\xi_i)(t_i-t_{i-1})-A\Bigg|\\ &\leqslant \left|\sum_{i=1}^k f(\xi_i)(t_i-t_{i-1})-\sum_{i=1}^k f_n(\xi_i)(t_i-t_{i-1})\right|\\ &\quad+\left|\sum_{i=1}^k f_n(\xi_i)(t_i-t_{i-1})-A\right|\\ &< \varepsilon+\varepsilon=2\varepsilon.\end{aligned}$$

That is, $\int_a^b f(t)\Delta t = A$. Hence f is Henstock–Kurzweil Δ-integrable on $\mathbb{T}$ and $\lim_{n\to\infty}\int_a^b f_n(t)\Delta t = \int_a^b f(t)\Delta t$. This completes the proof. $\square$

Using the above result we have the following.

THEOREM 10. *Assume that (f_n) is the sequence of Henstock–Kurzweil Δ-equi-integrable functions on $\mathbb{T}$ such that $f_n \to f$ on $\mathbb{T}$. For each $n\in\mathbb{N}$, $f_n : \mathbb{T}\times\overline{B}_r(x_0)\to\mathbb{R}$ is such that $\big|f_n(t,x)-f_n(t,y)\big| \leqslant g\big(|x-y|\big)$, where $g:\mathbb{T}\to[0,\infty)$ is a continuous increasing function with $g(0)=0$ and $g(t)>0$ for $t\in\mathbb{T}$. For each $n\in\mathbb{N}$, (x_n) is solution of*

$$\begin{cases} x_n^{\Delta}(t) = f_n\big(t,x_n(t)\big), \ t\in\mathbb{T}, \\ x_n(a) = x_0. \end{cases} \tag{5.6}$$

Then there is a subsequence (y_n) of (x_n) and a function $x:\mathbb{T}\to\mathbb{R}$ such that $y_n\to x$ and x satisfies (1.1) in $\mathbb{T}$.

Proof. From the assumption we have

$$x_n(t) = x_n(a) + \int_a^t f_n\big(\tau,x_n(\tau)\big)\Delta\tau,$$

which yields

$$\big|x_n(t)-x_n(a)\big| \leqslant \int_a^t \Big|f_n\big(\tau,x_n(\tau)\big)\Big|\Delta\tau. \tag{5.7}$$

Now

$$\begin{aligned} \Big|f_n\big(\tau,x_n(\tau)\big)\Big| &= \Big|f_n\big(\tau,x_n(a)\big)+f_n\big(\tau,x_n(\tau)\big)-f_n\big(\tau,x_n(a)\big)\Big| \\ &\leqslant \Big|f_n\big(\tau,x_n(a)\big)\Big| + \Big|f_n\big(\tau,x_n(\tau)\big)-f_n\big(\tau,x_n(a)\big)\Big| \\ &\leqslant \Big|f_n\big(\tau,x_n(a)\big)\Big| + g\Big(\big|x_n(\tau)-x_n(a)\big|\Big). \end{aligned}$$

Then

$$\int_a^t \Big|f_n\big(\tau,x_n(\tau)\big)\Big|\Delta\tau \leqslant M + \int_a^t g\Big(\big|x_n(\tau)-x_n(a)\big|\Big)\Delta\tau,$$

where $M = \int_a^b \Big|f_n\big(\tau,x_n(a)\big)\Big|\Delta\tau$. By Theorem 6 we have

$$\int_a^t \Big|f_n\big(\tau,x_n(\tau)\big)\Big|\Delta\tau \leqslant G^{-1}\big(G(M)+t-a\big),$$

where $G^{\Delta}(u) = 1/g(u)$. So equation (5.7) becomes

$$\big|x_n(t)-x_n(a)\big| \leqslant G^{-1}\big(G(M)+t-a\big).$$

Therefore

$$\begin{aligned}|x_n(t)| &\leqslant |x_n(t)-x_n(a)|+|x_n(a)|\\ &\leqslant G^{-1}\left(G(M)+t-a\right)+x_0\\ &=\alpha,\end{aligned}$$

where $\alpha = G^{-1}\big(G(M)+b-a\big)+x_0$. Therefore (x_n) is uniformly bounded on $\mathbb{T}$.

Now for any given $\varepsilon > 0$ we can find $\delta > 0$ such that if $|t-t_1| < \delta$, then

$$\begin{aligned}\big|x_n(t)-x_n(t_1)\big| &= \left|x_n(a)+\int_a^t f_n\big(\tau,x_n(\tau)\big)\Delta t\,\tau - x_n(a) - \int_a^{t_1} f_n\big(\tau,x_n(\tau)\big)\Delta\tau\right|\\ &= \left|\int_{t_1}^t f_n\big(\tau,x_n(\tau)\big)\Delta\tau\right|\\ &\leqslant \int_{t_1}^t \Big|f_n\big(\tau,x_n(\tau)\big)\Big|\Delta\tau\\ &\leqslant L|t-t_1| < L\,\delta\end{aligned}$$

$\big|x_n(t)-x_n(t_1)\big| < \varepsilon$ where $\varepsilon = L\,\delta$. Therefore (x_n) is equicontinuous on $\mathbb{T}$.

Thus by Arzelá–Ascoli's theorem there exists a subsequence (y_n) of (x_n) such that $y_n \to x$ uniformly on $\mathbb{T}$.

Also by hypothesis,

$$y_n(t) = x_0 + \int_a^t f_n(\tau, y_n(\tau))\Delta\tau.$$

Since $f_n \to f$ on $\mathbb{T}$ and (f_n) is Henstock–Kurzweil Δ-equi-integrable on $\mathbb{T}$, by Theorem 9 it follows that

$$\begin{aligned}\lim_{n\to\infty} y_n(t) &= x_0 + \lim_{n\to\infty}\int_a^t f_n\big(\tau,y_n(\tau)\big)\Delta\tau\\ x(t) &= x_0 + \int_a^t \lim_{n\to\infty} f_n\big(\tau,y_n(\tau)\big)\Delta\tau\\ x(t) &= x_0 + \int_a^t f\big(\tau,x(\tau)\big)\Delta\tau.\end{aligned}$$

That is,

$$\begin{cases} x^{\Delta}(t) = f\big(t,x(t)\big),\ t\in\mathbb{T},\\ x(a) = x_0.\end{cases}$$

This completes the proof. □

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