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SECOND ORDER TWO–PARAMETRIC QUANTUM BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we study second order two-parametric quantum boundary value problems. The main aims of this paper are presented in two steps. In the first step, we consider second order two-parametric quantum boundary value problems with general nonlinearities and by the use of Krasnoselskii fixed point theorem on positive cones we provide some sufficient conditions to reach the existence, multiplicity and nonexistence of positive solutions. At the and of this step, some illustrative examples are given to show practical implementability of the obtained theoretical results. In the second step, we consider the corresponding two-parametric quantum eigenvalue problems and in the light of Lyapunov inequalities, we present a lower bound estimation for positive eigenvalues. We complete this step with a numerical evaluation to identify validity of the obtained lower bound.

1. Introduction

In this paper we initiate study of two-parametric quantum difference equations of the form

$$-D^2_{p,q}u(t) = f(t,u(t)), \quad 0 \leqslant t \leqslant 1, \tag{1}$$

with Dirichlet-Neumann boundary conditions

$$\alpha u(0) - \beta D_{p,q}u(0) = 0, \tag{2}$$

$$\gamma u(1) + \delta D_{p,q}u(1) = 0, \tag{3}$$

where,

1. $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$ with $\gamma(\alpha \pm \beta) + \delta\alpha, \gamma(\alpha+\beta) - \delta\alpha > 0$,
2. $f \in C([0,1] \times \mathbb{R}; \mathbb{R}_+)$.

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We note that, $D_{p,q}$ stands for the two-parametric quantum difference operator and from now on, we will call the two-parametric quantum boundary value problem (1)-(3) as (p,q)-BVP (1)-(3) as well as two-parametric quantum operators that will be defined in the next section.
A general insight on the theory of quantum calculus or more commonly the q-calculus, indicates its supreme position in science and technology. For instance one can mention q-approximation theory and q-special functions that have been possessed influential roles in theory and in applications of the related topics. For more details we refer to the [1],[3],[9],[11],[12],[17]-[20],[25],[26],[29] and cited bibliography therein. Besides, as an extremely attractive branch of the applied mathematics, we can observe day to day growing investigations on difference equations via fixed point theory. In this way, we suggest the papers [4],[7],[8],[14],[15],[16],[22],[24],[28],[31],[32] for more consultation on the topic.
In this position, we are going to discuss about motivation of the our study. L. H. Erbe and H. Wang in [8], considered the second order differential equation

$$u'' + a(t)f(u) = 0, \quad 0 < t < 1, \tag{4}$$

subject to the Dirichlet-Neumann boundary conditions

$$\alpha u(0) - \beta u'(0) = 0, \tag{5}$$

$$\gamma u(1) + \delta u'(1) = 0, \tag{6}$$

where

$(A.1)$ $f \in C([0,\infty),[0,\infty))$,

$(A.2)$ $a \in C([0,1],[0,\infty))$ and $a(t) \not\equiv 0$ on any subinterval of $[0,1]$,

$(A.3)$ $\alpha,\beta,\gamma,\delta > 0$ and $\gamma\beta + \alpha\gamma + \alpha\delta > 0$.

The authors making use of the Krasnoselskii fixed point theorem on positive cones in Banach spaces, in both of sublinear and superlinear cases obtained one positive solution for the second order boundary value problem (4)-(6).
Inspired by the above work, M. El-Shahed and H. A. Hassan in [7], considered the second order q-difference equation

$$-D_q^2 u(t) = a(t)f(u(t)), \quad 0 < t < 1, \tag{7}$$

subject to the Dirichlet-Neumann boundary conditions (5)-(6) and the same general assumptions $(A.1)-(A.3)$. Consequently, the authors by means of the same fixed point technique obtained one positive solution for the second order q-difference boundary value problem (7), (5)-(6) in the both of sublinear and superlinear cases. So, motivated by the papers [7] and [8] in one hand and abstract generalization of the classic q-calculus to the (p,q)-calculus on the other hand, led to the main problem of this paper, i.e. the more generalized second order (p,q)-BVP (1)-(3) that based on the best of our google scholar based knowledge it can be the first second order (p,q)-BVP in the

literature.
At the end of this section, we state the organization of the paper as follows. Section 2, includes some basic definitions and lemmas of the (p,q)-calculus that will be needed in the next sections. Also, in this section we state desired elements of the fixed point theory and corresponding Banach spaces. We finalize this section with an interesting brief discussion about importance of positivity of solutions of boundary value problems from view point of mathematics. In Section 3, we present existence, multiplicity and nonexistence results of positive solutions of the (p,q)-BVP (1)-(3). At the end of this section, some illustrative examples are given to justify the obtained theoretical results. Finally, we have the Section 4, where we use the Lyapunov inequalities to present a lower bound for positive eigenvalues of the second order (p,q)-eigenvalue problem corresponding to the (p,q)-BVP (1)-(3).

2. Technical background

2.1. Brief discussion on (p,q)-calculus

This section begins with a quick overview on the (p,q)-calculus. So, we first define the concept of (p,q)-geometric sets as follows.

DEFINITION 2.1. Assume $p,q \in \mathbb{C}$ are fixed. A set $\mathscr{Q} \subseteq \mathbb{C}$ is called (p,q)-geometric provided that for each $x \in \mathscr{Q}$, $px, qx \in \mathscr{Q}$.

Now we are ready to define (p,q)-difference operator as follows.

DEFINITION 2.2. ([3],[26]). Assume $f : \mathscr{Q} \to \mathbb{R}$. Then, the (p,q)-derivative of f is defined by

$$D_{p,q}f(t) := \frac{f(pt) - f(qt)}{(p-q)t}, \quad 0 < q < p \leqslant 1,\ t \in \mathscr{Q} - \{0\}. \tag{8}$$

If $t = 0$, we say that f has (p,q)-derivative at zero provided that

$$\lim_{n\to\infty} \frac{f\left(\frac{q^n}{p^{n+1}}t\right) - f(0)}{\frac{q^n}{p^{n+1}}t}, \quad t \in \mathscr{Q}, \tag{9}$$

exists and dos not depend on t. We then denote this limit by $D_{p,q}f(0)$.

REMARK 2.3. The higher order (p,q)-derivatives for $f : \mathscr{Q} \to \mathbb{R}$ are defined by

$$D^n_{p,q}f(t) := D_{p,q}\left(D^{n-1}_{p,q}f(t)\right), \quad n \in \mathbb{Z}^+, \tag{10}$$

where by convention, $D^0_{p,q}f(t) := f(t)$. In this case, with a straightforward calculation one has:

$$D^n_{p,q}\left(\sum_{k=0}^{n-1} c_k t^k\right) = 0, \quad c_k \in \mathbb{R},\ n \in \mathbb{Z}^+. \tag{11}$$

As is expected, it is time to define the mutual inversions of the (p,q)-difference operators that we call them (p,q)-integral operators and define them as follows.

DEFINITION 2.4. ([3],[26]). Assume that $f:\mathscr{Q}\to\mathbb{R}$. Then, the (p,q)-integral of f is defined by

$$I_{p,q}f(t)=\int_0^t f(s)d_{p,q}s:=(p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}t\right). \tag{12}$$

Accordingly,

$$\int_a^b f(s)d_{p,q}s:=I_{p,q}f(b)-I_{p,q}f(a), \quad a,b\in\mathscr{Q}. \tag{13}$$

Here we present some basic properties of the (p,q)-operators defined by (8) and (12).

THEOREM 2.5. *Assume* $f,g:\mathscr{Q}\to\mathbb{R}$ *and* $\lambda,\lambda_i\in\mathbb{R}$, $i=1,2$. *Then*

$(B.1)$ $D_{p,q}\lambda=0$;

$(B.2)$ $D_{p,q}(\lambda_1 f+\lambda_2 g)(t)=\lambda_1 D_{p,q}f(t)+\lambda_2 D_{p,q}g(t)$;

$(B.3)$

$$\begin{aligned}(D_{p,q}fg)(t)&=f(pt)D_{p,q}g(t)+g(qt)D_{p,q}f(t)\\&=f(qt)D_{p,q}g(t)+g(pt)D_{p,q}f(t);\end{aligned}$$

$(B.4)$

$$\begin{aligned}\left(D_{p,q}\frac{f}{g}\right)(t)&=\frac{g(qt)D_{p,q}f(t)-f(qt)D_{p,q}g(t)}{g(pt)g(qt)}\\&=\frac{g(pt)D_{p,q}f(t)-f(pt)D_{p,q}g(t)}{g(pt)g(qt)}, \quad g(t)\neq 0;\end{aligned}$$

$(B.5)$ $D_{p,q}I_{p,q}f(t)=f(t)$;

$(B.6)$ $I_{p,q}D_{p,q}f(t)=f(t)-f(0)$, *(f is continuous at t=0)*;

$(B.7)$ $\int_a^t D_{p,q}f(s)d_{p,q}s=f(t)-f(a)$, *(f is continuous at t=0)*.

Proof. Following definitions of the (p,q)-operators (8) and (12), leads us directly to the proofs of $(B.1)-(B.4)$. Furthermore, $(B.7)$ can be proved via (13) and $(B.6)$.

So, it just enough to prove $(B.5)$ and $(B.6)$, that we do it as follows:

$$
\begin{aligned}
D_{p,q}I_{p,q}f(t) &= (p-q)D_{p,q}\left\{t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}t\right)\right\}\\
&= \frac{pt\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}pt\right)-qt\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}qt\right)}{t}\\
&= \sum_{n=0}^{\infty}\frac{q^n}{p^n}f\left(\frac{q^n}{p^n}t\right)-\sum_{n=1}^{\infty}\frac{q^n}{p^n}f\left(\frac{q^n}{p^n}t\right)\\
&= f(t).
\end{aligned}
$$

As well as, we have

$$
\begin{aligned}
I_{p,q}D_{p,q}f(t) &= (p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left\{\frac{f\left(\frac{q^n}{p^{n+1}}pt\right)-f\left(\frac{q^n}{p^{n+1}}qt\right)}{(p-q)\left(\frac{q^n}{p^{n+1}}t\right)}\right\}\\
&= \sum_{n=0}^{N}\left\{f\left(\frac{q^n}{p^n}t\right)-f\left(\frac{q^{n+1}}{p^{n+1}}t\right)\right\}, \quad N\to\infty\\
&= f(t)-f\left(\frac{q^{N+1}}{p^{N+1}}t\right), \quad N\to\infty.
\end{aligned}
$$

The continuity of f at $t=0$, implies that $f\left(\frac{q^{N+1}}{p^{N+1}}t\right)\to f(0)$ as $N\to\infty$. This completes the proof. □

As will be seen in the Section 3, the Abel formula for repeated (p,q)-integrals will play the most important role to obtain a symmetric Green function of the homogenous second order (p,q)-difference equation $-D_{p,q}^2u(t)=0$, subject to the boundary conditions (2)-(3). So, we prove it here for 2-fold (p,q)-integration as follows.

LEMMA 2.6. *Assume* $f:\mathcal{Q}\to\mathbb{R}$. *Then,*

$$
\int_0^t\int_0^s f(r)d_{p,q}rd_{p,q}s=\int_0^{\frac{t}{p}}(t-pqs)f(s)d_{p,q}s. \tag{14}
$$

Proof. According to Definition 2.4, it follows that

$$
\int_0^t\int_0^s f(r)d_{p,q}rd_{p,q}s=(p-q)^2t^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n+2m}}{p^{n+2m+3}}f\left(\frac{q^{n+m}}{p^{n+m+2}}t\right). \tag{15}
$$

On the other, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n+2m}}{p^{n+2m+3}}f\left(\frac{q^{n+m}}{p^{n+m+2}}t\right)\\
=&\sum_{n=0}^{\infty}\left\{\frac{q^{n}}{p^{n+3}}f\left(\frac{q^{n}}{p^{n+2}}t\right)+\frac{q^{n+2}}{p^{n+5}}f\left(\frac{q^{n+1}}{p^{n+3}}t\right)+\frac{q^{n+4}}{p^{n+7}}f\left(\frac{q^{n+2}}{p^{n+4}}t\right)+\frac{q^{n+6}}{p^{n+9}}f\left(\frac{q^{n+3}}{p^{n+5}}t\right)\ldots\right\}\\
=&\left\{\frac{1}{p^{3}}f\left(\frac{1}{p^{2}}t\right)+\frac{q^{2}}{p^{5}}f\left(\frac{q}{p^{3}}t\right)+\frac{q^{4}}{p^{7}}f\left(\frac{q^{2}}{p^{4}}t\right)+\frac{q^{6}}{p^{9}}f\left(\frac{q^{3}}{p^{5}}t\right)+\frac{q^{8}}{p^{11}}f\left(\frac{q^{4}}{p^{6}}t\right)+\ldots\right\}\\
&+\left\{\frac{q}{p^{4}}f\left(\frac{q}{p^{3}}t\right)+\frac{q^{3}}{p^{6}}f\left(\frac{q^{2}}{p^{4}}t\right)+\frac{q^{5}}{p^{8}}f\left(\frac{q^{3}}{p^{5}}t\right)+\frac{q^{7}}{p^{10}}f\left(\frac{q^{4}}{p^{6}}t\right)+\frac{q^{9}}{p^{12}}f\left(\frac{q^{5}}{p^{7}}t\right)+\ldots\right\}\\
&+\left\{\frac{q^{2}}{p^{5}}f\left(\frac{q^{2}}{p^{4}}t\right)+\frac{q^{4}}{p^{7}}f\left(\frac{q^{3}}{p^{5}}t\right)+\frac{q^{6}}{p^{9}}f\left(\frac{q^{4}}{p^{6}}t\right)+\frac{q^{8}}{p^{11}}f\left(\frac{q^{5}}{p^{7}}t\right)+\frac{q^{10}}{p^{13}}f\left(\frac{q^{6}}{p^{8}}t\right)+\ldots\right\}\\
&+\ldots\\
=&\frac{1}{p^{3}}f\left(\frac{1}{p^{2}}t\right)+\frac{q}{p^{4}}\left(1+\frac{q}{p}\right)f\left(\frac{q}{p^{3}}t\right)+\frac{q^{2}}{p^{5}}\left(1+\frac{q}{p}+\frac{q^{2}}{p^{2}}\right)f\left(\frac{q^{2}}{p^{4}}t\right)\\
&+\frac{q^{3}}{p^{6}}\left(1+\frac{q}{p}+\frac{q^{2}}{p^{2}}+\frac{q^{3}}{p^{3}}\right)f\left(\frac{q^{3}}{p^{5}}t\right)+\ldots\\
=&\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+3}}\left(1+\frac{q}{p}+\frac{q^{2}}{p^{2}}+\ldots+\frac{q^{n}}{p^{n}}\right)f\left(\frac{q^{n}}{p^{n+2}}t\right).
\end{aligned}$$

Since,

$$1+\frac{q}{p}+\frac{q^{2}}{p^{2}}+\ldots+\frac{q^{n}}{p^{n}}=\frac{1}{p-q}\left(p-\frac{q^{n+1}}{p^{n}}\right),$$

then, the right hand side of (15) becomes

$$\begin{aligned}
&(p-q)^{2}t^{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n+2m}}{p^{n+2m+3}}f\left(\frac{q^{n+m}}{p^{n+m+2}}t\right)\\
=&(p-q)\frac{t}{p}\sum_{n=0}^{\infty}\left\{\frac{q^{n}}{p^{n+1}}t-\frac{q^{n}}{p^{n+1}}\left(\frac{q^{n}}{p^{n+1}}qt\right)\right\}f\left(\frac{q^{n}}{p^{n+1}}\frac{t}{p}\right)\\
=&(p-q)w\sum_{n=0}^{\infty}\left\{\frac{q^{n}}{p^{n+1}}t-\frac{q^{n}}{p^{n+1}}\left(\frac{q^{n}}{p^{n+1}}pqw\right)\right\}f\left(\frac{q^{n}}{p^{n+1}}w\right)\\
=&\int_{0}^{w}(t-pqs)f(s)d_{p,q}s.
\end{aligned}$$

So, we come to the conclusion that

$$\int_{0}^{t}\int_{0}^{s}f(r)d_{p,q}rd_{p,q}s=\int_{0}^{\frac{t}{p}}(t-pqs)f(s)d_{p,q}s.$$

This completes the proof. □

In this position, we define so-called (p,q)-basic numbers that will play a crucial role in the last section.

DEFINITION 2.7. [26]. The (p,q)-basic number is defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p-q} = \sum_{k=0}^{n-1} p^{n-k-1} q^k. \tag{16}$$

Accordingly, the (p,q)-factorial is defined by

$$[n]_{p,q}! := \prod_{k=1}^{n} [k]_{p,q}, \quad [0]_{p,q}! := 1. \tag{17}$$

In the following definition we present so-called (p,q)-shifted factorials.

DEFINITION 2.8. [19]. The (p,q)-shifted factorial is defined by

$$((a,b);(p,q))_n := \begin{cases} 1; & n=0, \\ \prod_{k=0}^{n-1} \left(p^k a - q^k b\right); & n=1,2,\dots \end{cases} \tag{18}$$

As special cases, we shall mention the following:

$$((p,q);(p,q))_n := \begin{cases} 1; & n=0, \\ \prod_{k=0}^{n-1} \left(p^{k+1} - q^{k+1}\right) = (p-q)^n [n]_{p,q}!, & n=1,2,\dots \end{cases}, \tag{19}$$

$$((1,a);(1,q))_n := (a;q)_n = \begin{cases} 1; & n=0, \\ \prod_{k=0}^{n-1} \left(1 - q^k a\right); & n=1,2,\dots \end{cases} \tag{20}$$

Also, $(a;q)_\infty := \lim_{n\to\infty} (a;q)_n$.

Our overview on the (p,q)-calculus will be completed by definitions of the (p,q)-trigonometric functions.

DEFINITION 2.9. The (p,q)-basic *sine* and *cosine* functions are defined by

$$\sin_{p,q} t := \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} \frac{t^{2n+1}}{(p-q)^{2n+1}}, \tag{21}$$

$$\cos_{p,q} t := \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} \frac{t^{2n}}{(p-q)^{2n}}. \tag{22}$$

A direct calculation indicates that for each $\lambda \in \mathbb{R}$,

$$-D_{p,q}^2 \sin_{p,q}(\lambda t) := \frac{\lambda^2}{(p-q)^2} \sin_{p,q}(\lambda t), \quad -D_{p,q}^2 \cos_{p,q}(\lambda t) := \frac{\lambda^2}{(p-q)^2} \cos_{p,q}(\lambda t). \tag{23}$$

REMARK 2.10. Let us mention this fact that if we take $p := 1$ in all of the above definitions, lemmas and theorems, then all of the presented results will be reduced to the corresponding results in the classic q-calculus.

2.2. Fixed point theory on Banach spaces

This part of the technical preliminaries is devoted to the fixed point theory on positive cones in appropriate Banach spaces. More precisely, in this paper we restrict ourselves to the Krasnoselskii fixed point theorem that will be used as the essential solvability tool for the (p,q)-BVP (1)-(3). This fixed point theorem can be characterized as one of the most applied fixed point techniques in solvability of differential/difference equations, as can be observed in [4],[7],[8],[14],[15],[16],[22],[24],[28],[31],[32] and considerable similar bibliography cited therein. Thus, we first state it in the following theorem.

THEOREM 2.11. *([21], Ths. 4.12, 4.14). Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume $\Omega_i \subset E$, $i = 1,2$ with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$.

Then, A has a fixed point in $K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

In what follows, we will use the following Banach space:

$$E := (C[0,1]; \|.\|), \quad \|u\| := \max\{u(t) |\, t \in [0,1]\}. \tag{24}$$

Having the above information in hand, it is time to consider the second order (p,q)-BVP

$$-D^2_{p,q}u(t) = f(t,u(t)), \quad 0 \leqslant t \leqslant 1, \tag{25}$$

$$\alpha u(0) - \beta D_{p,q}u(0) = 0, \tag{26}$$

$$\gamma u(1) + \delta D_{p,q}u(1) = 0. \tag{27}$$

Indeed, in the sequel we use the Green function technique to transform the boundary value problem (25)-(27) into its equivalent (p,q)-integral equation. This procedure is given in the following lemma.

LEMMA 2.12. *Assume $f(s,u(s)) \equiv 0$ for each $s \in \left[1, \frac{1}{p}\right]$, $0 < p < 1$. If $u(t)$ be a solution of the (p,q)-BVP (25)-(27), then it uniquely solves the (p,q)-integral equation*

$$u(t) := \int_0^1 G(t,pqs)f(s,u(s))d_{p,q}s, \tag{28}$$

where

$$G(t,pqs) := \frac{1}{\gamma(\alpha+\beta)+\delta\alpha}\begin{cases}(\gamma+\delta-\gamma t)(\beta+\alpha pqs); & 0 \leqslant pqs \leqslant t,\\ (\gamma+\delta-\gamma pqs)(\beta+\alpha t); & t \leqslant pqs \leqslant 1,\end{cases} \tag{29}$$

denotes the Green function of the second order (p,q)*-difference equation* $-D^2_{p,q}u(t) = 0$ *subject to the boundary conditions (26)-(27).*

Proof. According to (11), it follows that

$$D_{p,q}u(t) = c_2 - \int_0^t f(s,u(s))d_{p,q}s, \tag{30}$$

$$u(t) = c_1 + c_2 t - \int_0^t \int_0^s f(r,u(r))d_{p,q}r d_{p,q}s. \tag{31}$$

Relying on the Abel formula (14), we get

$$D_{p,q}u(t) = c_2 - \int_0^t f(s,u(s))d_{p,q}s, \tag{32}$$

$$u(t) = c_1 + c_2 t - \int_0^t (t-pqs)f(s,u(s))d_{p,q}s - \int_t^{\frac{t}{p}} (t-pqs)f(s,u(s))d_{p,q}s. \tag{33}$$

Here we apply the boundary conditions (26), (27) and the assumption $f(s,u(s)) \equiv 0$ for each $s \in \left[1, \frac{1}{p}\right]$, $0 < p < 1$ to obtain

$$c_1 = \frac{\alpha}{\gamma(\alpha+\beta)+\delta\alpha}\int_0^1 \{\gamma+\delta-pqs\}f(s,u(s))d_{p,q}s, \tag{34}$$

$$c_2 = \frac{\beta}{\gamma(\alpha+\beta)+\delta\alpha}\int_0^1 \{\gamma+\delta-pqs\}f(s,u(s))d_{p,q}s. \tag{35}$$

Now, we substitute these coefficients into the equality (33) to reach the following

$$u(t) = \int_0^1 G(t,pqs)f(s,u(s))d_{p,q}s, \tag{36}$$

in which

$$G(t,pqs) := \frac{1}{\gamma(\alpha+\beta)+\delta\alpha}\begin{cases}(\gamma+\delta-\gamma pqs)(\beta+\alpha t)-(t-pqs); & 0 \leqslant pqs \leqslant t,\\ (\gamma+\delta-\gamma pqs)(\beta+\alpha t); & t \leqslant pqs \leqslant 1.\end{cases}$$

On the other hand, since

$$(\beta+\alpha t)(\gamma+\delta-\gamma pqs)-(\beta+\alpha pqs)(\gamma+\delta-\gamma t) := (\gamma(\alpha+\beta)+\delta\alpha)(t-pqs),$$

we conclude that

$$G(t,pqs) := \frac{1}{\gamma(\alpha+\beta)+\delta\alpha}\begin{cases}(\gamma+\delta-\gamma t)(\beta+\alpha pqs); & 0 \leqslant pqs \leqslant t,\\ (\gamma+\delta-\gamma pqs)(\beta+\alpha t); & t \leqslant pqs \leqslant 1.\end{cases}$$

The proof is completed now. $\square$

In order to discuss to positivity of solutions of the (p,q)-BVP (25)-(27), we first analyze the Green function $G(t,pqs)$ defined by (29) in the next lemma.

LEMMA 2.13. *Let* $0<q<p\leqslant 1$ *and* $p+q>1$. *Then, the Green function* $G(t,pqs)$ *defined by (29) satisfies the following properties:*

$(C.1)$ $G(t,pqs)>0$, *for each* $(t,pqs)\in[0,1]\times[0,1]$;

$(C.2)$ $\min\limits_{\frac{1}{\xi}\leqslant t\leqslant\frac{\xi-1}{\xi}} G(t,pqs)\geqslant M\max\limits_{0\leqslant pqs\leqslant 1} G(pqs,pqs)$, *where*

$$M:=\min\left\{\frac{\gamma+\xi\delta}{\xi(\gamma+\delta)},\frac{\alpha+\xi\beta}{\xi(\alpha+\beta)}\right\},\quad \xi\in(2,\infty);$$

$(C.3)$

$$\max_{0\leqslant t,pqs\leqslant 1} G(t,pqs)=\Theta$$
$$:=\frac{(\alpha(\gamma+\delta)(p+q-1)+\beta\gamma)(\beta\gamma(p+q-1)+\alpha(\gamma+\delta))}{\alpha\gamma(p+q)^2(\gamma(\alpha+\beta)+\delta\alpha)}.$$

Proof. Since the property $(C.1)$ is obvious and the proof of the property $(C.2)$ can be found in detail in [7] and [8](with setting $\xi=4$), so, we just prove the property $(C.3)$. Let us define

$$\phi(t):=\gamma+\delta-\gamma t,\quad \psi(t):=\beta+\alpha t,\qquad 0\leqslant t\leqslant 1.$$

So, we have

$$\frac{G(t,pqs)}{G(pqs,pqs)}=\begin{cases}\dfrac{\phi(t)}{\phi(pqs)}\leqslant 1; & pqs\leqslant t,\\[2ex] \dfrac{\psi(t)}{\psi(pqs)}\leqslant 1; & t\leqslant pqs.\end{cases}$$

This implies that

$$\max_{0\leqslant t\leqslant 1} G(t,pqs)=G(pqs,pqs):=\frac{(\gamma+\delta-\gamma pqs)(\beta+\alpha pqs)}{\gamma(\alpha+\beta)+\delta\alpha}.$$

Since,

$$D_{p,q}(\phi(t)\psi(t))=\gamma(\alpha-\beta)+\delta\alpha-\alpha\gamma(p+q)t,$$

it is easy to check that $G(t,t)$ takes its maximum at the point $t:=\frac{\gamma(\alpha-\beta)+\delta\alpha}{\gamma\alpha(p+q)}\in(0,1)$. Hence, we conclude that

$$\max_{0\leqslant t,pqs\leqslant 1} G(t,pqs)=G\left(\frac{\gamma(\alpha-\beta)+\delta\alpha}{\gamma\alpha(p+q)},\frac{\gamma(\alpha-\beta)+\delta\alpha}{\gamma\alpha(p+q)}\right)$$
$$=\frac{(\alpha(\gamma+\delta)(p+q-1)+\beta\gamma)(\beta\gamma(p+q-1)+\alpha(\gamma+\delta))}{\alpha\gamma(p+q)^2(\gamma(\alpha+\beta)+\delta\alpha)}.$$

This completes the proof. □

As explained previously, we are going to find positive solutions of the (p,q)-BVP (25)-(27), via fixed point theory on cones in Banach spaces. By now, both of the desired fixed point technique and Banach space are identified. So, it suffices to define an appropriate cone in the Banach space E and consequently a completely continuous operator $A : E \to E$. To this aim, we first define the positive cone $K \subset E$ as below:

$$K := \left\{ u \in E \,\middle|\, u(t) \geqslant 0,\ t \in [0,1], \qquad \min_{\frac{1}{\xi} \leqslant t \leqslant \frac{\xi-1}{\xi}} u(t) \geqslant M\|u\|,\ \xi \in (2,\infty) \right\}. \tag{37}$$

Next, we define the (p,q)-integral operator $A : E \to E$ as follows:

$$Au(t) := \int_0^1 G(t, pqs) f(s, u(s)) d_{p,q}s, \tag{38}$$

where, $G(t,s)$ stands for the Green function (29). With a straightforward calculation based on Lemma 2.13, it follows that the integral operator A is completely continuous on the Banach space E.
Since our investigation on positive solutions of the (p,q)-BVP (1)-(3) will be managed on the cone K, so it is necessary to prove that the (p,q)-integral operator A leaves the cone K invariant. Thus, we have the following lemma.

LEMMA 2.14. *The cone K is invariant by the (p,q)-integral operator A defined by (38), that is $AK \subset K$.*

Proof. Lemma 2.13 is the cornerstone of our proof. As a result of positivity of the Green function $G(t, pqs)$, positivity of the operator A on cone K is clear. Furthermore, for each $u \in K$ and for each $\xi \in (2,\infty)$ it follows that

$$\begin{aligned}
\min_{\frac{1}{\xi} \leqslant t \leqslant \frac{\xi-1}{\xi}} Au(t) &= \min_{\frac{1}{\xi} \leqslant t \leqslant \frac{\xi-1}{\xi}} \int_0^1 G(t, pqs) f(s, u(s)) d_{p,q}s \\
&\geqslant M \int_0^1 \max_{0 \leqslant s \leqslant 1} G(pqs, pqs) f(s, u(s)) d_{p,q}s \\
&\geqslant M \max_{0 \leqslant t \leqslant 1} \int_0^1 G(t, pqs) f(s, u(s)) d_{p,q}s \\
&= M\|Au\|.
\end{aligned}$$

Thus $AK \subset K$. □

2.3. Positivity of solutions

This is the last part of the technical background and we are interested in a brief and effective discussion about positivity of solutions of boundary value problems with a mathematical perspective. In fact, we are going to show relationship between positivity of solutions and positivity of the Green functions. To this aim, we first consider the family of boundary value problems having positive solutions. Generally, positive solutions of these boundary value problems can be represented as integral equations with related Green functions in their kernel. So, let us consider the abstract boundary value problem

$$\mathscr{L}u = \mathscr{F}, \ a \leqslant t \leqslant b, \qquad U(a,b) = 0, \tag{39}$$

where $\mathscr{L}$ and $\mathscr{F}$ stand for an n-th order differential operator and the nonlinearity of differential equation (39) (that may include $u^{(k)}$, $k = 1,2,...,n-1$), respectively. Also,

$$U(a,b) := \sum_{k=0}^{n-1} \left[\alpha_k u^{(k)}(a) + \beta_k u^{(k)}(b) \right], \quad \alpha_k, \beta_k \in \mathbb{R}, \ n \in \mathbb{Z}^+. \tag{40}$$

Discussing about positivity of solutions of the boundary value problem (39), depends on our ability to transform (39) to the integral equation

$$u(t) := \int_a^b K(s,t) \mathscr{F} ds, \tag{41}$$

in which $K(s,t)$ as an integral kernel is a positive function. In practice, the integral kernel $K(s,t)$ is the *possible* Green function of the homogenous boundary value problem $\mathscr{L}u = 0$ subject to the boundary conditions (40). This leads us to this fact that positivity of solutions of this boundary value problem is directly related to the positivity of its Green function and positivity of the nonlinearity $\mathscr{F}$ follows basically from positivity of the Green function.

3. Existence, multiplicity and nonexistence via fixed point theory

For the sake of convenience, we introduce the following notation:

$$\Phi(h) := \max\{f(t,u) | \ (t,u) \in [0,1] \times [0,h]\}, \tag{42}$$

$$\Psi(h) := \min\left\{ f(t,u) \middle| \ (t,u) \in \left[\frac{1}{\xi}, \frac{\xi-1}{\xi}\right] \times [0,h] \right\}, \tag{43}$$

$$\mathscr{W}_1 := \frac{1}{\Theta}, \quad \mathscr{W}_2 := \frac{\mathscr{W}_1}{M}. \tag{44}$$

Since $\xi \in (2,\infty)$, it is obvious that $0 < M < 1$, and consequently $0 < \mathscr{W}_1 < \mathscr{W}_2$. Now we prove an existence result for the positive solutions of the (p,q)-BVP (1)-(3) as follows.

THEOREM 3.1. *Assume there exist two positive constants a and b, such that $\Phi(a) \leqslant a\mathscr{W}_1$ and $\Psi(b) \geqslant b\mathscr{W}_2$. Then, the (p,q)-BVP (1)-(3) has at least one solution $u^* \in K$, where*

$$\min\{a,b\} \leqslant \|u^*\| \leqslant \max\{a,b\}.$$

Proof. Since $0 < \mathscr{W}_1 < \mathscr{W}_2$, then relying on Φ and Ψ defined by (42) and (43), respectively, it is easy to check that necessarily $a \neq b$. Now, let

$$\Omega_1 := \{u \in E | \ \|u\| < a\}, \quad \Omega_2 := \{u \in E | \ \|u\| < b\}. \tag{45}$$

To reach the desired proof, we have to prove the following assertions:

(i) $\|Au\| \leqslant \|u\|$, for $u \in K \cap \partial\Omega_1$;

(ii) $\|Au\| \geqslant \|u\|$, for $u \in K \cap \partial\Omega_2$.

In what follows, we justify the assertions (i) and (ii) as below.

- To justify (i), assume $u \in K \cap \partial\Omega_1$. So, $0 \leqslant u(t) \leqslant \|u\| = a$ and consequently,

$$f(t,u) \leqslant \Phi(a) \leqslant a\mathscr{W}_1, \quad (t,u) \in [0,1] \times [0,a]. \tag{46}$$

Accordingly, it follows that

$$\begin{aligned} Au(t) &= \int_0^1 G(t,pqs) f(s,u(s)) d_{p,q}s \\ &\leqslant a\mathscr{W}_1 \int_0^1 G(t,pqs) d_{p,q}s \\ &\leqslant a\mathscr{W}_1 \int_0^1 \max_{0 \leqslant t, pqs \leqslant 1} G(t,pqs) d_{p,q}s \\ &= a\mathscr{W}_1 \mathscr{W}_1^{-1} = \|u\|. \end{aligned}$$

Thus, we reach the following

$$\|Au\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_1. \tag{47}$$

- To justify (ii), assume $u \in K \cap \partial\Omega_2$. So, $0 \leqslant u(t) \leqslant \|u\| = b$ and consequently,

$$f(t,u) \geqslant \Psi(b) \geqslant b\mathscr{W}_2, \quad (t,u) \in \left[\frac{1}{\xi}, \frac{\xi-1}{\xi}\right] \times [0,b]. \tag{48}$$

Accordingly, one has

$$\begin{aligned} Au(t) &= \int_0^1 G(t,pqs) f(s,u(s)) d_{p,q}s \\ &\geqslant b\mathscr{W}_2 \int_0^1 G(t,pqs) d_{p,q}s \\ &\geqslant b\mathscr{W}_2 \int_0^1 \min_{\frac{1}{\xi} \leqslant t \leqslant \frac{\xi-1}{\xi}} G(t,pqs) d_{p,q}s \\ &\geqslant b\mathscr{W}_2 M\Theta = \|u\|. \end{aligned}$$

This yields the following

$$\|Au\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_2. \tag{49}$$

Using (47), (49) and Theorem 2.11, we come to the conclusion that the integral operator A has a fixed point u^* in K that indicates a positive solution of the (p,q)-BVP (1)-(3). $\square$

Actually, by a natural way, the existence result of Theorem 3.1 can be extended to obtain arbitrary number of positive solutions for the (p,q)-BVP (1)-(3).

REMARK 3.2. Let us consider a finite sequence $\{a_k\}_{k=1}^{n+1}$ with $a_k < a_{k+1}$, $k = 1,2,...,n$, such that either

(i) $\Phi(a_{2k-1}) \leqslant a_{2k-1}\mathscr{W}_1$, $k = 1,2,...,[(n+2)/2]$, and
$\Psi(a_{2k}) \geqslant a_{2k}\mathscr{W}_2$, $k = 1,2,...,[(n+1)/2]$, or

(ii) $\Phi(a_{2k}) \leqslant a_{2k}\mathscr{W}_1$, $k = 1,2,...,[(n+1)/2]$, and
$\Psi(a_{2k-1}) \geqslant a_{2k-1}\mathscr{W}_2$, $k = 1,2,...,[(n+2)/2]$.

Then, similar to Theorem 3.1 one can prove that the (p,q)-BVP (1)-(3) has at least n positive solutions u_k^*, $k = 1,2,...,n$ in K such that

$$a_k \leqslant \|u_k^*\| \leqslant a_{k+1}, \quad k = 1,2,...,n.$$

For instance if we focus on the case (i), since $\Phi, \Psi \in C([0,\infty),[0,\infty))$, also since $0 < \mathscr{W}_1 < \mathscr{W}_2$, so for each pair (a_k, a_{k+1}), $k = 1,2,..,n$, there exists pair (b_k, c_k), $k = 1,2,...,n$ with

$$a_k < b_k < c_k < a_{k+1}, \quad k = 1,2,...,n,$$

such that

$$\begin{aligned} \Phi(b_{2k-1}) \leqslant b_{2k-1}\mathscr{W}_1, \quad \Psi(c_{2k-1}) \geqslant c_{2k-1}\mathscr{W}_2, \qquad k = 1,2,..,[(n+2)/2], \\ \Phi(c_{2k}) \leqslant c_{2k}\mathscr{W}_1, \qquad \Psi(b_{2k}) \geqslant b_{2k}\mathscr{W}_2, \qquad k = 1,2,..,[(n+1)/2]. \end{aligned}$$

Now, Theorem 3.1 implies that for each pair (b_k, c_k), $k = 1,2,...,n$ the (p,q)-BVP (1)-(3) has at least one positive solution u_k^* in K.

We complete solvability criteria with a nonexistence criterion for positive solutions of the (p,q)-BVP (1)-(3) as follows.

THEOREM 3.3. *Let* $0 < c < d < \infty$. *Assume that either*

(i) $\Phi(h) < h\mathscr{W}_1, \quad h \in [c,d]$, *or*

(ii) $\Psi(h) > h\mathscr{W}_2, \quad h \in [c,d]$.

Then, the (p,q)*-BVP (1)-(3) has no positive solution* u^* *in* K *satisfying* $c \leqslant \|u^*\| \leqslant d$.

Proof. Assume that the hypothesis (i) is satisfied. Suppose on the contrary that $u^* \in K$ be a positive solution of the boundary value problem (1)-(3) satisfying $c \leqslant \|u^*\| \leqslant d$. So, according to the hypothesis (i) it follows that

$$f(t,u) \leqslant \Phi(\|u^*\|) < \|u^*\|\mathscr{W}_1, \quad (t,u) \in [0,1] \times [0, \|u^*\|]. \tag{50}$$

Since u^* is a positive solution of the boundary value problem (1)-(3), hence $u^* = Au^*$ and consequently, we get

$$\begin{aligned}\|u^*\| &= \int_0^1 G(t,pqs)f(s,u^*(s))d_{p,q}s \\ &\leqslant \max_{0\leqslant t,pqs\leqslant 1}\int_0^1 G(t,pqs)f(s,u^*(s))d_{p,q}s \\ &< \|u^*\|\mathscr{W}_1\mathscr{W}_1^{-1} \\ &= \|u^*\|.\end{aligned}$$

This contradiction indicates that existence of positive solution of the boundary value problem (1)-(3) under the hypothesis (i) is impossible. Since, the proof under the hypothesis (ii) is similar, so we omit it here. □

In this position, we present some illustrative examples to justify the obtained theoretical existence and nonexistence criteria.

EXAMPLE 3.4. Let us consider the $\left(\frac{3}{4},\frac{1}{3}\right)$-BVP

$$\begin{cases} -D^2_{\frac{3}{4},\frac{1}{3}}u(t) = \dfrac{t(1+|u|)}{13}, & (t,u) \in [0,1]\times\mathbb{R}, \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{51}$$

Indeed, the following setting has been implemented on the (p,q)-BVP (1)-(3):

$$p := \frac{3}{4}, \quad q := \frac{1}{3}, \quad \alpha = \gamma := 1, \quad \beta = \delta := 0. \tag{52}$$

In this case for $\xi := 4$, a direct computation indicates that

$$\mathscr{W}_1 := \frac{1}{12}, \quad M := \frac{1}{4}, \quad \mathscr{W}_2 := \frac{1}{3}. \tag{53}$$

According to (42) and (43), it follows that

$$\Phi(h) := \max\left\{\frac{t(1+|u|)}{13}\,\middle|\,(t,u)\in[0,1]\times[0,h]\right\} = \frac{1+h}{13}, \tag{54}$$

$$\Psi(h) := \min\left\{\frac{t(1+|u|)}{13}\,\middle|\,(t,u)\in\left[\frac{1}{4},\frac{3}{4}\right]\times[0,h]\right\} = \frac{1}{52}. \tag{55}$$

Here we choose $a := 24$ and $b := \frac{1}{18}$ to reach the following

$$\Phi(a) := \frac{25}{13} < a\mathscr{W}_1 = 2, \quad \Psi(b) := \frac{1}{52} > b\mathscr{W}_2 = \frac{1}{54}.$$

Therefore, according to Theorem 3.1, the $(\frac{3}{4},\frac{1}{3})$-BVP (51) has at least one positive solution u^* in K, with

$$\frac{1}{18} < \|u^*\| < 24.$$

EXAMPLE 3.5. Consider the $\left(\frac{5}{6},\frac{4}{5}\right)$-BVP

$$-D^2_{\frac{5}{6},\frac{4}{5}}u(t)=\frac{(1+t)(0.1+|u|)}{10},\quad (t,u)\in[0,1]\times\mathbb{R}, \tag{56}$$

$$u(0)-D_{\frac{5}{6},\frac{4}{5}}u(0)=0, \tag{57}$$

$$u(1)+D_{\frac{5}{6},\frac{4}{5}}u(1)=0. \tag{58}$$

It is easy to check that this $\left(\frac{5}{6},\frac{4}{5}\right)$-BVP has been reduced from the (p,q)-BVP (1)-(3) under the setting

$$p:=\frac{5}{6},\quad q:=\frac{4}{5},\quad \alpha=\beta=\gamma=\delta:=1.$$

Now, taking $\xi:=4$ and a straightforward computation, it follows that

$$\mathscr{W}_1:=0.8939,\quad M:=0.625,\quad \mathscr{W}_2:=1.4302. \tag{59}$$

Also, according to (42) and (43), one has

$$\Phi(h):=\max\left\{\frac{(1+t)(0.1+|u|)}{10}\,\middle|\,(t,u)\in[0,1]\times[0,h]\right\}=\frac{0.1+h}{10}, \tag{60}$$

$$\Psi(h):=\min\left\{\frac{(1+t)(0.1+|u|)}{10}\,\middle|\,(t,u)\in\left[\frac{1}{4},\frac{3}{4}\right]\times[0,h]\right\}=\frac{1}{80}. \tag{61}$$

Here we choose $c:=0.07$, $d:=0.12$ and $h\in(c,d)$ with $h:=0.1$, to reach the following

$$\Phi(0.1):=0.04<0.1\times\mathscr{W}1=0.08939.$$

Thus, Theorem 3.3 ensures that the $\left(\frac{5}{6},\frac{4}{5}\right)$-BVP (56)-(58), has no positive solution in K.

4. Lower bound estimation for positive eigenvalues

In this section we consider the (p,q)-eigenvalue problem

$$\begin{cases} -D^2_{p,q}u(t)=\left(\dfrac{\lambda}{p-q}\right)^2u(t),\quad 0\leqslant t\leqslant 1,\ \lambda\in\mathbb{R},\\ \\ \alpha u(0)-\beta D_{p,q}u(0)=0,\\ \gamma u(1)+\delta D_{p,q}u(1)=0.\end{cases}$$

The main goal of this section is to present a lower bound for positive eigenvalues of this eigenvalue problem. In this way we are interested in the use of Lyapunov inequalities. But, before beginning the lower bound estimation procedure, this is a worthy opportunity to give a brief historical note on the Lyapunov inequalities.

The history of the Lyapunov inequalities turns to the last decade of nineteenth century, where the Russian mathematician A. M. Lyapunov was studying the concept of stability of motion. It was proved by Lyapunov that, if $q \in C(\mathbb{R};\mathbb{R}_+)$ be an ω-periodic ($\omega \in \mathbb{R}^+$) function and furthermore if $u(t)$ be a nontrivial solution of the second order differential equation

$$u'' + q(t)u = 0, \quad -\infty < t < \infty, \tag{62}$$

then, the roots of the characteristic equation of (62) are complex and their modulus is equal to 1, provided that

$$\int_0^\omega q(t)dt \leqslant \frac{4}{\omega}. \tag{63}$$

Nowadays, it is known that in the light of the Floquet theory the inequality (63) can be considered as a stability criterion for second order differential equation (62), in the sense that all solutions of (62) are bounded as $t \to \pm\infty$. The counter inequality of (63) is called Lyapunov inequality in honor of the A. M. Lyapunov. For more details the interested followers are refered to the Chap. III, Th. II of [23]. Besides, reviewing the literature it was P. Hartman that considered the second order Dirichlet boundary value problem

$$u'' + q(t)u = 0, \qquad a < t < b, \tag{64}$$

$$u(a) = 0 = u(b), \tag{65}$$

and proved that, if $q \in C([a,b];\mathbb{R})$ and $u(t)$ be a nontrivial solution of the boundary value problem (64)-(65), then the non periodic Lyapunov inequality

$$\int_a^b q_+(t)dt \leqslant \frac{4}{b-a}, \tag{66}$$

holds, in which $q_+ := \frac{q(t)+|q(t)|}{2}$ indicates the nonnegative part of $q(t)$. He, also was the first one who presented applicability of the Lyapunov inequality (66) in establishing the upper bound of maximum number of zeros of the nontrivial solutions of the boundary value problem (64)-(65) and some oscillatory behavior of such solutions. See Chap. XI, Sec. 5 of [13], for more consultation on the topic.

On the other hand, it seems that it was S. Cheng that initiated study of discrete Lyapunov inequalities. It was proven by Cheng that if $p(k)$ is a nonnegative function defined on the set of consecutive integers $\{a, a+1, ..., b\}$, and if $x(k)$ is a nontrivial solution of the second order forward difference Dirichlet boundary value problem

$$\Delta^2 x(k-1) + p(k)x(k) = 0, \quad a \leqslant k \leqslant b, \tag{67}$$

$$x(a-1) = 0 = x(b+1), \tag{68}$$

then, the sharp Lyapunov inequality

$$\sum_{k=a}^{b} p(k) \geqslant \mu(b-a+1), \tag{69}$$

is satisfied, in which $\mu(N)$ stands for a strictly decreasing function on N. Based on the discrete Lyapunov inequality (69) some nonexistence criteria for nontrivial solutions of

the boundary value problem (67)-(68) presented. Detailed discussions can be found in [5].
Since then by now the literature has been growing extraordinarily in almost all branches of differential and difference equations. Also at the present time we can evaluate more comprehensive collection of qualitative behavior of desired differential/difference equations via Lyapunov inequalities in comparison with other inequalities. In this way, some of the most effective papers and monographs dealing with Lyapunov inequalities are suggested here as [2],[5],[6],[10],[13],[23],[27]. So, in view point of novelty of the topic we are interested in study of Lyapunov inequality of the (p,q)-BVP (1)-(3).
In this section, we are interested in a particular form of the nonlinearity $f(t,u(t))$ as follows:

$$f(t,u(t)) = k(t)u(t), \quad k(t) \in C([0,1];\mathbb{R}^+),\ k(t) \equiv 0,\ t \in \left[1,\frac{1}{p}\right], 0 < p < 1.$$

So, we are concerned with the following (p,q)-BVP

$$-D^2_{p,q}u(t) = k(t)u(t), \quad 0 \leqslant t \leqslant 1, \tag{70}$$

subject to the Dirichlet-Neumann boundary conditions

$$\alpha u(0) - \beta D_{p,q}u(0) = 0, \tag{71}$$
$$\gamma u(1) + \delta D_{p,q}u(1) = 0. \tag{72}$$

We are now ready to find Lyapunov inequality of the (p,q)-BVP (70)-(72).

THEOREM 4.1. *Assume $u(t)$ is a nontrivial solution of the (p,q)-BVP (70)-(72). Then, the Lyapunov inequality*

$$\int_0^1 k(s)d_{p,q}s \geqslant \frac{\alpha\gamma(p+q)^2(\gamma(\alpha+\beta)+\delta\alpha)}{(\alpha(\gamma+\delta)(p+q-1)+\beta\gamma)(\beta\gamma(p+q-1)+\alpha(\gamma+\delta))}, \tag{73}$$

is satisfied.

Proof. According to Lemma 2.13, (p,q)-BVP (70)-(72) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,pqs)k(s)u(s)d_{p,q}s, \tag{74}$$

such that the Green function $G(t,pqs)$ is defined by (29). Also, Lemma 2.14, $(C.3)$ indicates that

$$G(t,pqs)_{0\leqslant t,pqs\leqslant 1} \leqslant \frac{(\alpha(\gamma+\delta)(p+q-1)+\beta\gamma)(\beta\gamma(p+q-1)+\alpha(\gamma+\delta))}{\alpha\gamma(p+q)^2(\gamma(\alpha+\beta)+\delta\alpha)}. \tag{75}$$

Thus we get the following:

$$\begin{aligned} u(t) &= \int_0^1 G(t,pqs)k(s)u(s)d_{p,q}s \\ &\leqslant \frac{(\alpha(\gamma+\delta)(p+q-1)+\beta\gamma)(\beta\gamma(p+q-1)+\alpha(\gamma+\delta))}{\alpha\gamma(p+q)^2(\gamma(\alpha+\beta)+\delta\alpha)} \int_0^1 k(s)u(s)d_{p,q}s. \end{aligned}$$

Taking max norm on both sides of the recent inequality, it follows that

$$\|u\| \leqslant \frac{(\alpha(\gamma+\delta)(p+q-1)+\beta\gamma)(\beta\gamma(p+q-1)+\alpha(\gamma+\delta))}{\alpha\gamma(p+q)^2(\gamma(\alpha+\beta)+\delta\alpha)} \int_0^1 k(s)\|u(s)\|d_{p,q}s.$$

Nontrivial nature of the solution $u(t)$ leads us to the sharp Lyapunov inequality

$$\int_0^1 k(s)d_{p,q}s \geqslant \frac{\alpha\gamma(p+q)^2(\gamma(\alpha+\beta)+\delta\alpha)}{(\alpha(\gamma+\delta)(p+q-1)+\beta\gamma)(\beta\gamma(p+q-1)+\alpha(\gamma+\delta))}.$$

This completes the proof. □

Having the Lyapunov inequality (73) in hand, we are now ready to study the (p,q)-eigenvalue problem

$$-D_{p,q}^2 u(t) = \left(\frac{\lambda}{p-q}\right)^2 u(t), \quad 0 \leqslant t \leqslant 1,\ \lambda \in \mathbb{R}, \tag{76}$$

subject to the Dirichlet-Neumann boundary conditions

$$\alpha u(0) - \beta D_{p,q}u(0) = 0, \tag{77}$$

$$\gamma u(1) + \delta D_{p,q}u(1) = 0. \tag{78}$$

Making use of the Lyapunov inequality (73) we are going to present a lower bound for positive eigenvalues of the (p,q)-eigenvalue problem (76)-(78) as follows.

LEMMA 4.2. *Assume $0 < q < p \leqslant 1$ and $p+q > 1$. If λ be an eigenvalue of the (p,q)-eigenvalue problem (76)-(78), then*

$$|\lambda| \geqslant (p^2-q^2).\sqrt{\frac{\alpha\gamma(\gamma(\alpha+\beta)+\delta\alpha)}{(\alpha(\gamma+\delta)(p+q-1)+\beta\gamma)(\beta\gamma(p+q-1)+\alpha(\gamma+\delta))}}. \tag{79}$$

Proof. The proof is immediate. It is enough to set $k(t) = \frac{\lambda^2}{(p-q)^2}$ and substitute it into the Lyapunov inequality (73). This completes the proof. □

According to the Lemma (4.2), if λ be a positive eigenvalue of the (p,q)-eigenvalue problem (76)-(78), then

$$\lambda \in \left[(p^2-q^2).\sqrt{\frac{\alpha\gamma(\gamma(\alpha+\beta)+\delta\alpha)}{(\alpha(\gamma+\delta)(p+q-1)+\beta\gamma)(\beta\gamma(p+q-1)+\alpha(\gamma+\delta))}},\infty\right). \tag{80}$$

In the light of the lower bound (80) for positive eigenvalues of the (p,q)-eigenvalue problem (76)-(78), we can numerically examine validity of this lower bound. To this aim, we first choose the simplest framework by taking the setting $\alpha,\gamma := 1$ and $\beta,\delta := 0$ in the boundary conditions. Furthermore, we take $p := 1$ and replace the parameter

q with $q^{\frac{1}{2}}$. These setting gives us the following Dirichlet $(1,q^{\frac{1}{2}})$-eigenvalue problem:

$$\begin{cases} -D^2_{1,q^{\frac{1}{2}}}u(t)=\left(\dfrac{\lambda}{1-q^{\frac{1}{2}}}\right)^2 u(t), \quad 0\leqslant t\leqslant 1,\ \lambda\in\mathbb{R}, \\ \\ u(0)=0=u(1). \end{cases} \tag{81}$$

In this case, using (23), we get that

$$u(t):=c_1\sin_{q^{\frac{1}{2}}}(\lambda t)+c_2\cos_{q^{\frac{1}{2}}}(\lambda t), \quad c_1,\ c_2\in\mathbb{R}. \tag{82}$$

Now implementing the boundary condition $u(0)=0$ and this fact that $\sin_{q^{\frac{1}{2}}}(0)=0$ and $\cos_{q^{\frac{1}{2}}}(0)=1$, we conclude that

$$u(t):=c\sin_{q^{\frac{1}{2}}}(\lambda t). \tag{83}$$

Next, we shall impose the boundary condition $u(1)=0$ to obtain the structure of the eigenvalue λ. So, we have

$$\sin_{q^{\frac{1}{2}}}(\lambda):=0. \tag{84}$$

On the other hand, using the lower bound estimation (80) we get the Lyapunov inequality of the eigenvalue problem (81) as below:

$$\lambda\in\left[\frac{1-q}{q^{\frac{1}{4}}},\infty\right). \tag{85}$$

In order to present a numerical comparison to identify validity of the lower bound (85) for positive eigenvalues of the $q^{\frac{1}{2}}$-eigenvalue problem (81), we first must present a numerical evaluation of the eigenvalue λ of eigenfunction $\sin_{q^{\frac{1}{2}}}(\lambda)$. To this aim, we shall use the references [11] and [29], where in particular cases zeros of the so called basic sine and cosine q-trigonometric functions S_q and C_q have discussed, respectively. The following identities demonstrate relationship between these functions and $\sin_{1,q}$ and $\cos_{1,q}$ defined by (21) and (22):

$$S_q(\eta;\lambda)=E_{q^2}(q\lambda^2)\sin_{q^{\frac{1}{2}}}(\lambda), \tag{86}$$

$$C_q(\eta;\lambda)=E_{q^2}(q\lambda^2)\cos_{q^{\frac{1}{2}}}(\lambda), \tag{87}$$

where

$$\eta:=\frac{q^{\frac{1}{4}}+q^{\frac{-1}{4}}}{2}, \quad E_q(x):=\sum_{j=0}^{\infty}q^{\frac{j(j-1)}{2}}\frac{x^j}{[j]_q!}.$$

As observed above, the eigenvalue of the $S_q(\eta;\lambda)$ and $\sin_{q^{\frac{1}{2}}}(\lambda)$ are same. So, we can use the eigenvalues $S_q(\eta;\lambda)$ instead of $\sin_{q^{\frac{1}{2}}}(\lambda)$. To this aim, we refer to the numerical assessments for the first positive eigenvalue λ_1 of the q-eigenvalue problem (81), that has presented in the [11], Sec. 2, where, using the asymptotic formula

$$\lambda_n = q^{\frac{1}{4}-n} - c_1(q) + o(1), \quad n \to \infty, \tag{88}$$

with

$$c_1(q) := \frac{1}{2} \frac{q^{\frac{1}{4}}}{1-q^{\frac{1}{2}}} \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2},$$

finite number of the real positive zeros of the $S_q(\eta;\lambda)$ and $C_q(\eta;\lambda)$ for the values $q := 0.1, 0.2, 0.25, 0.5, 0.75$ and 0.9, have obtained. So, for the sake of the comparability we must choose the same values for q and then we have to use (85) to find a lower bound for the positive eigenvalues of the eigenvalue problem (81) in each case. Consequently, we are expected to reach an outcome that clearly illustrates validity of the lower bound estimation (85) in all of the above mentioned cases for the quantum parameter q. To ease of comparison, the obtained results are presented in the following table that shows the validity of the theoretically evaluated lower bound for the positive eigenvalues of the q-eigenvalue problem (81):

Table 1: Lower Bound Validity Identification

q	λ_1: Smallest Positive Eigenvalue evaluated by (88)	Lower Bound via (85)
0.1	5.6234	1.6005
0.2	3.3437	1.1963
0.25	2.8284	0.9458
0.5	1.6818	0.5946
0.75	1.2408	0.2689
0.9	1.0823	0.1027

5. Concluding remarks

In this paper, study of the second order (p,q)-BVPs has initiated. The main body of our study can be divided to two steps that we summarize them as follows:

- ► In the first step, we consider (p,q)-BVPs with Dirichlet-Neumann boundary conditions and general nonlinearities. The main aim of this step is to present some sufficient conditions to reach existence, multiplicity and nonexistence of positive solutions for (p,q)-BVPs of the form (1)-(3). The main strategy in this way is to use the Krasnoselskii fixed point theorem on positive cones in Banach spaces.

- ► In the second step, we consider the (p,q)-eigenvalue problems of the form (76)-(78). In the light of the Lyapunov inequalities, a lower bound estimation for the positive eigenvalues has provided. In addition, a numerical evaluation to show validity of the obtained lower bound has presented.

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