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PROGRESSIVE CONTRACTIONS, MEASURES OF NON–COMPACTNESS AND QUADRATIC INTEGRAL EQUATIONS

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Abstract. Classical fixed point theorems often begin with the assumption that we have a mapping P of a non-empty, closed, bounded, convex set G in a Banach space into itself. Then a number of conditions are added which will ensure that there is at least one fixed point in the set G. These fixed point theorems have been very effective with many problems in applied mathematics, particularly for integral equations containing a term

$$\int_0^t A(t-s)v(t,s,x(s))ds,$$

because such terms frequently map sets of bounded continuous functions into compact sets. But there is a large and important class of integral equations from applied mathematics containing such a term with a coefficient function $f(t,x)$ which destroys all compactness. Investigators have then turned to Darbo's fixed point theorem and measures of non-compactness to get a (possibly non-unique) fixed point. In this paper:

a) We offer an elementary alternative to measures of non-compactness and Darbo's theorem by using progressive contractions. This method yields a unique fixed point (unlike Darbo's theorem) which, in turn, by default yields asymptotic stability as introduced in [1].

b) We lift the growth requirements in both x and t seen using Darbo's theorem.

c) We offer a technique for finding the mapping set G.

1. Introduction

By way of a quadratic integral equation we offer an elementary alternative to Darbo's fixed point theorem and measures of non-compactness, lift the growth requirements in both x and t found in the Darbo method, and develop a method for finding a fixed point mapping set. Our method is progressive contractions and it yields a unique global fixed point. The uniqueness then automatically yields asymptotic stability [3] as defined by Banas and Rzepka [1] to deal with the non-uniqueness from Darbo's theorem. Briefly, a solution x is asymptotically stable if for any $\varepsilon > 0$ there is a $T > 0$ such that for any other solution y then $t > T$ implies that $|x(t) - y(t)| < \varepsilon$.

Integral equations of the form

$$x(t) = g(t,x(t)) + \int_0^t A(t-s)v(t,s,x(s))ds$$

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have been studied very effectively by means of a number of classical fixed point theorems because of compactness of the integral operator [4]. They are of the exact form for Krasnoselskii's fixed point theorem for the sum of two operators when g is a contraction, v is continuous, and A satisfies mild conditions [10, p. 31].

There is a large class of integral equations from diverse areas of applied mathematics of that form which can be handled in the same way illustrated here. But there are calculations which are very important to understand and so we will focus on an exact form of A and deal with the equation

$$x(t) = g(t,x(t)) + f(t,x(t)) \int_0^t (t-s)^{\beta-1} v(t,s,x(s))ds, \tag{1}$$

where $0 < \beta < 1$ and the functions f and v satisfy a type of Lipschitz condition with respect to x. That poses a growth condition in x which we are able to remove as shown in Section 3. This equation was studied in a very informative paper by Darwish and Henderson [7]. The challenge here is that the coefficient function f destroys the compactness of the integral part of the map. We refer to a paper by Darwish [6] for a description of a wide set of real-world problems governed by this equation.

With the failure of the integral term to define a compact map, investigators have turned to Darbo's fixed point theorem and measures of non-compactness to obtain a non-unique global solution. It is most informative to study the conditions on f and v and the subsequent calculations found in Darwish and Henderson [7, pp. 76-82].

The idea here is to examine Darbo's fixed point theorem which we found in our three motivating papers [6, p. 48], [7, p. 78], and [1, p. 2] which differ in several ways, but an in-depth study is found in [8]. In spite of differences there is the common theme that there is a Banach space $(\mathscr{B}, \|\cdot\|)$ containing a non-empty, closed, bounded, convex set G with a continuous mapping $P : G \to G$ which is a contraction with respect to a measure of non-compactness.

In our effort to offer an elementary alternative we focus on all except the last. That is, suppose that we take from this theorem only the condition that the natural mapping defined by (1) does map such a set G into itself. By asking that g be a contraction and that f and v satisfy Lipschitz type conditions in the set G itself, can we conclude that there is a unique global solution of (1) residing in G? Moreover, Darbo's theorem does not give uniqueness which can be extremely important in real-world problems since G may have been constructed to include only points favorable to the problem at hand, while some points outside G could promote a disaster.

The method we employ is called progressive contractions and it is critical to note that it only applies to Volterra operators which are non-anticipative maps extensively discussed in [5, p. 84]. At the risk of belaboring the obvious we feel it is crucial to explain "non-anticipative". It proceeds as follows. For a given set of functions G which map the interval $[0,E] \to \Re$, if $\phi \in G$ then the natural mapping defined by (1) is

$$(P\phi)(t) = g(t,\phi(t)) + f(t,\phi(t)) \int_0^t (t-s)^{\beta-1} v(t,s,\phi(s))ds, \quad t \in [0,E].$$

A fixed point of P is a point $\phi \in G$ with $(P\phi)(t) = \phi(t)$ for $0 \leqslant t \leqslant E$. In other words,

$$\phi(t) = g(t,\phi(t)) + f(t,\phi(t)) \int_0^t (t-s)^{\beta-1} v(t,s,\phi(s))ds$$

is an identity in t. In particular, if $t_1 > 0$ then

$$\phi(t_1) = g(t_1,\phi(t_1)) + f(t_1,\phi(t_1)) \int_0^{t_1} (t_1-s)^{\beta-1} v(t_1,s,\phi(s))ds,$$

which shows us most clearly that $\phi(t_1)$ depends only on the values of ϕ for $0 \leqslant t \leqslant t_1$. More to the point, no matter what happens to the fixed point for $t > t_1$, the value of the fixed point at t_1 is not changed.

2. The assumptions and a fixed point

In our work here we begin with the Banach space $(\mathscr{B}, \|\cdot\|_E)$ of continuous functions $\phi : [0,E] \to \Re$ where E is an arbitrary positive number. After finishing our work we will then consider the sequence of fixed points on the intervals $[0,1], [0,2], ..., [0,n]$ and use a limiting technique to prove that there is a solution on $[0,\infty)$ without any resort to Zorn's lemma as is so often the case [9, p. 42].

The assumption that the natural mapping $P : G \to G$ enables us to consider the pair $(G, \|\cdot\|_E)$ as a complete metric space of continuous functions in that closed, bounded, convex, non-empty set G. This is the entire space in which we work. It is complete because it is a closed subset of the Banach space $\mathscr{B}$.

This work is inspired in large measure by a paper by Darwish and Henderson [7]. In that paper the authors assume global Lipschitz conditions controlling the growth rate of a different sort than ours here. Moreover, since ours need only hold in G they can be much less restrictive than if they were global. There are many examples to illustrate this but suffice it to note that $f(t,x) = x^2$ does satisfy a Lipschitz condition with constant $2D$ if the bound on G is D. In Section 3 we reduce the growth condition even more.

We will be considering spaces $G_1, .., G_n$ being G restricted to the intervals, respectively, $[0,T_1], ..., [0,T_n]$ with $T_n = E$. If P_i is P restricted to G_i then $P_i : G_i \to G_i$ and $(G_i, \|\cdot\|)$ is again a complete metric space.

Here is the merit of the present work. Each P_i will become a very simple contraction mapping with unique fixed point ϕ_i. The ϕ_i will piece together to get a unique solution on $[0,E]$. We will then continue as in the first paragraph of this section to get the global unique solution.

Next, notice that with v continuous, if x is continuous then

$$\lim_{t \downarrow 0} \int_0^t (t-s)^{\beta-1} v(t,s,x(s))ds = 0.$$

Hence, if g is a contraction with solution ϕ then

$$(P\phi)(0) = \phi(0) = g(0,\phi(0)),$$

and this is a unique solution $\phi(0)$. If the solution of (1) is not unique then all others, say ϕ_j, satisfy $\phi_j(0) = \phi(0)$.

Now it is imperative that G be chosen so that some function ψ contained in the interior of G satisfies

$$\phi(0) = \psi(0) \tag{2}$$

and this is always assumed.

Here are the formal assumptions on f, g, v. We assume that there exist $\alpha \in (0,1)$, $J > 0$ and a function $L : [0,\infty) \to [0,\infty)$ so that $x, y \in G, 0 \leqslant t \leqslant E$ implies:

$$|f(t,x) - f(t,y)| \leqslant L(t)\,|x-y|, \tag{3}$$

$$|v(t,s,x) - v(t,s,y)| \leqslant J|x-y|, \tag{4}$$

$$|g(t,x) - g(t,y)| \leqslant \alpha|x-y|, \tag{5}$$

so that there is a unique $x(0)$ with

$$g(0,x(0)) = x(0). \tag{6}$$

We also assume that there is a function ψ in the interior of G with

$$\psi(0) = x(0). \tag{7}$$

Note that we can find $M > 0$ with

$$M = \max_{0 \leqslant s \leqslant t \leqslant 2E, |x| \leqslant r} \{|g(t,x)|, |f(t,x)|, |v(t,s,x)|\},$$

where r is a bound of the set G.

Firstly, select $T_1 \in (0,E]$ to satisfy

$$c_0 := \alpha + \frac{M(J+L^*)}{\beta} T_1^{\beta} < 1, \tag{8}$$

with

$$L^* = \sup_{t \in [0,2E]} L(t). \tag{9}$$

Then, let $T \in (0,\delta)$ be such that

$$c := \alpha + \gamma + \frac{M(J+L^*)}{\beta} T^{\beta} < 1, \tag{10}$$

where $\gamma \in (0, 1-\alpha)$ and δ are as in Theorem 1.

We divide the interval $[T_1, E]$ into $n-1$ equal segments of length less than or equal to T. Thus, we have

$$0 = T_0 < T_1 < \ldots < T_n = E.$$

Now we are starting the process called progressive contractions, a technique we devised some time ago and which has been illustrated in several papers (see, [2]).

Let G_1 be the complete metric space of functions ϕ in G restricted to the interval $[0,T_1]$ and let $(G_1^*, \|\cdot\|)$ be the complete metric space of all continuous functions ϕ in G_1 satisfying $\phi(0)=x(0)$ with $x(0)$ given in (6).

To see that this is a complete metric space, let x_n be a Cauchy sequence in G_1^* and note that it is also a Cauchy sequence in G_1 which is certainly complete since it is a closed subset of a Banach space. Thus this sequence has a limit in G_1 which is a continuous function and so it is also in G_1^*.

Next, notice that if P_1 is P restricted to $0 \leqslant t \leqslant T_1$ then $P: G \to G \implies P_1 : G_1 \to G_1$ because P is a Volterra operator. But if any function ϕ in G_1 satisfies $\phi(0)=x(0)$ then $(P\phi)(0)=x(0)$ and so $P_1 : G_1^* \to G_1^*$.

The metric is that induced by the supremum norm which we will always denote by $\|\cdot\|$ letting the context define the interval over which the supremum is taken. We are all set to show that P_1 is a contraction mapping on the complete metric space with a unique fixed point. That will be Step 1.

THEOREM 1. *Let $E>0$ and let G be a closed, bounded, convex, non-empty set in the Banach space $(\mathscr{B}, \|\cdot\|)$ of continuous functions $\phi : [0,E] \to \Re$ with the supremum norm. Let P be the natural mapping $G \to \mathscr{B}$ defined by (1) and assume that $P(G) \subseteq G$. Moreover, suppose that the above notation (8)-(10) and conditions (3)–(7) hold and that there exist $\delta, \gamma > 0$ such that*

$$L(u+h)\int_0^u (u+h-s)^{\beta-1} \sup_{x\in G} |v(u+h,s,x(s))|\, ds \leqslant \gamma < 1-\alpha, \tag{11}$$

for all $h \in [0,\delta]$ and any $u \in [T_1,E]$ with T_1 defined in (8). Then P has a unique fixed point in G. Moreover, as this holds for any $E>0$ these fixed points can be parlayed into a unique fixed point on $[0,\infty)$ which is asymptotically stable.

Proof. **Step 1**: We will show that the mapping P_1 defined above is a contraction with unique fixed point $\phi_1 \in G_1^*$ and $\phi_1(0)=x(0)$. Let $x,y\in G_1$ and L^* defined in (9), and notice that for $t\in[0,T_1]$ we have

$$\begin{aligned}
&|(P_1x)(t)-(P_1y)(t)| \leqslant |g(t,x(t))-g(t,y(t))| \\
&+\left[f(t,x(t))\int_0^t (t-s)^{\beta-1}v(t,s,x(s))ds - f(t,x(t))\int_0^t (t-s)^{\beta-1}v(t,s,y(s))ds\right] \\
&+|f(t,x(t))-f(t,y(t))|\int_0^t (t-s)^{\beta-1}|v(t,s,y(s))|ds \\
\leqslant&\alpha\|x-y\| + |f(t,x(t))|J\|x-y\|\int_0^t (t-s)^{\beta-1}ds + L^*\|x-y\|M\int_0^t (t-s)^{\beta-1}ds \\
\leqslant&\alpha\|x-y\| + MJ\|x-y\|\int_0^t (t-s)^{\beta-1}ds + L^*M\|x-y\|\int_0^t (t-s)^{\beta-1}ds \\
=&\|x-y\|\left[\alpha + M(J+L^*)\frac{t^\beta}{\beta}\right] \leqslant \|x-y\|\left[\alpha+\frac{M(J+L^*)}{\beta}T_1^\beta\right] = c_0\|x-y\|,
\end{aligned}$$

and so

$$\|(P_1x)-(P_1y)\| \leqslant c_0\|x-y\|$$

as required. We conclude that there exists a unique solution x_1 of the equation (1) defined on the interval $[0,T_1]$.

Remark. Notice that at this point we have not used (11) and we have established that there is a solution x_1 defined on a sufficiently small interval $[0,T_1]$, i.e., $x_1 \in G_1$. It remains to show that it can be continued on the (arbitrary) interval $[0,E]$, thus to $[0,\infty)$, and that is where (11) is used.

Step 2: Clearly if $T_1 = E$ then there is nothing else to prove, so now we assume that $0 < T_1 < E$. Having obtained the (unique) solution x_1 on the interval $[0,T_1]$, we proceed to prove existence of a (unique) solution to (1) on $[0,T_1+T = T_2]$. Our first task is to transfer equation (1) by taking

$$\begin{aligned} x(t+T_1) =& g(t+T_1,x(t+T_1)) \\ &+ f(t+T_1,x(t+T_1))\int_0^{t+T_1}(t+T_1-s)^{\beta-1}v(t+T_1,s,x(s))ds, \\ & t\in[0,E-T_1]. \end{aligned}$$

Setting

$$z(t) := x(t+T_1),\ t\in[0,E-T_1], \tag{12}$$

in view of x_1 being the unique solution obtained on the interval $[0,T_1]$, the transferred equation is written as

$$\begin{aligned} z(t) =& g(t+T_1,z(t)) + f(t+T_1,z(t))\int_0^{T_1}(t+T_1-s)^{\beta-1}v(t+T_1,s,x_1(s))ds \\ &+ f(t+T_1,z(t))\int_{T_1}^{t+T_1}(t+T_1-s)^{\beta-1}v(t+T_1,s,z(s-T_1))ds, \end{aligned}$$

for $t\in[0,E-T_1]$. By using $s = u+T_1$ in the last integral, we see that equation (1) is transferred to the equation

$$\begin{aligned} z(t) =& g(t+T_1,z(t)) + f(t+T_1,z(t))\int_0^{T_1}(t+T_1-s)^{\beta-1}v(t+T_1,s,x_1(s))ds \\ &+ f(t+T_1,z(t))\int_0^{t}(t-u)^{\beta-1}v(t+T_1,u+T_1,z(u))du,\ t\in[0,E-T_1]. \end{aligned} \tag{13}$$

We consider the set G_2^* consisting of continuous functions in G restricted on $[0,T_2]$ and coinciding with x_1 on $[0,T_1]$, and we define the operator $P_2 : G_2^* \to \mathscr{B}$ given by

$$(P_2\phi)(t) := g(t,\phi(t)) + f(t,\phi(t))\int_0^t (t-s)^{\beta-1}v(t,s,\phi(s))ds, \quad t\in[0,T_2].$$

Note that by our hypothesis that $P(G)\subseteq G$ we have that $P_2(G_2^*)\subseteq G$. Furthermore, in view of the definitions of P_2 and x_1 we may see that

$$P_2\phi(t) := x_1(t), \quad t\in[0,T_1], \ \phi\in G_2^*, \tag{14}$$

and that implies that $P_2(G_2^*) \subseteq G_2^*$. For $t \in [T_1, T_2]$ by putting $z(t) := \phi(t + T_1)$, $t \in [0, T_2 - T_1 = T]$ we have that

$$(P_2 z)(t) := g(t+T_1, z(t)) + f(t+T_1, z(t)) \int_0^{T_1} (t+T_1-s)^{\beta-1} v(t+T_1, s, x_1(s)) ds$$
$$+ f(t+T_1, z(t)) \int_0^t (t-u)^{\beta-1} v(t+T_1, u+T_1, z(u)) du, \quad t \in [0, T]. \tag{15}$$

Clearly, if a function z satisfies (13) on $[0, T]$ then by (12) the function $x(t) := z(t - T_1)$, $t \in [T_1, T_2]$ satisfies (1) on $[T_1, T_2]$, and so, in order to obtain a (unique) solution to (1) on $[0, T_2]$ all we have to do is to prove existence of a (unique) fixed point of the operator P_2 on G_2^*. In fact, as for any $z_1, z_2 \in G_2^*$ it holds

$$(P_2 z_1)(t) - (P_2 z_2)(t) = (P_2 x_1)(t) - (P_2 x_1)(t) = 0, \; t \in [0, T_1], \tag{16}$$

we may restrict ourselves to showing that P_2 given in (15) is a contraction.

Firstly we note that for any $z_1, z_2 \in G_2^*$ it holds

$$|g(t+T_1, z_1(t)) - g(t+T_1, z_2(t))| \leqslant \alpha |z_1(t) - z_2(t)|, \quad t \in [0, T_2],$$

so

$$|g(t+T_1, z_1(t)) - g(t+T_1, z_2(t))| \leqslant \alpha \|z_1 - z_2\|, \quad t \in [0, T]. \tag{17}$$

As M is a bound of $|f|$ and $|v|$ on the compact set $\{0 \leqslant s \leqslant t \leqslant E\} \times [-r, r]$ and L^* is as defined in (9), we have for $t \in [0, T]$, $z_1, z_2 \in G_2^*$

$$\begin{aligned}
&\left| f(t+T_1, z_1(t)) \int_0^t (t-u)^{\beta-1} v(t+T_1, u+T_1, z_1(u)) du \right. \\
&\left. - f(t+T_1, z_2(t)) \int_0^t (t-u)^{\beta-1} v(t+T_1, u+T_1, z_2(u)) du \right| \\
&\leqslant |f(t+T_1, z_1(t)) - f(t+T_1, z_2(t))| \int_0^t (t-u)^{\beta-1} |v(t+T_1, u+T_1, z_1(u))| du \\
&\quad + |f(t+T_1, z_2(t))| \int_0^t (t-u)^{\beta-1} \times \\
&\quad \times |v(t+T_1, u+T_1, z_1(u)) - v(t+T_1, u+T_1, z_2(u))| du \\
&\leqslant L(t+T_1) |z_1(t) - z_2(t)| \int_0^t (t-u)^{\beta-1} |v(t+T_1, u+T_1, z_1(u))| du \\
&\quad + |f(t+T_1, z_2(t))| \int_0^t (t-u)^{\beta-1} J |z_1(u) - z_2(u)| du \\
&\leqslant \|z_1 - z_2\| (L^* + J) M \int_0^t (t-u)^{\beta-1} du \\
&= \|z_1 - z_2\| \frac{(L^* + J) M}{\beta} t^\beta,
\end{aligned}$$

and so,

$$\begin{aligned}&\left|f(t+T_1,z_1(t))\int_0^t (t-u)^{\beta-1}v(t+T_1,u+T_1,z_1(u))du\right.\\&\left.-f(t+T_1,z_2(t))\int_0^t (t-u)^{\beta-1}v(t+T_1,u+T_1,z_2(u))du\right|\\&\leqslant \|z_1-z_2\|\,\frac{(L^*+J)M}{\beta}T^{\beta},\quad t\in[0,T],z_1,z_2\in G_2^*.\end{aligned}\tag{18}$$

Now let $\delta,\gamma>0$ be such that (11) is satisfied. In particular, in view of $T\in(0,\delta)$ for $u=T_1$ we have

$$L(T_1+t)\int_0^{T_1}(T_1+t-s)^{\beta-1}\sup_{x\in G}|v(T_1+t,s,x(s))|\,ds\leqslant\gamma,\ t\in[0,T],\tag{19}$$

thus, by (17), (18) and (19) we have for $t\in[0,T]$, $z_1,z_2\in G_2^*$

$$\begin{aligned}&|(P_1z_1)(t)-(P_1z_2)(t)|\\=&\ |g(T_1+t,z_1(t))-g(T_1+t,z_2(t))|\\&+|f(t+T,z_1(t))-f(t+T,z_2(t))|\int_0^{T_1}(t+T_1-s)^{\beta-1}|v(t+T_1,s,x_1(s))|\,ds\\&+\left|f(t+T_1,z_1(t))\int_0^t(t-u)^{\beta-1}v(t+T_1,u+T_1,z_1(u))du\right.\\&\left.-f(t+T_1,z_2(t))\int_0^t(t-u)^{\beta-1}v(t+T_1,u+T_1,z_2(u))du\right|\\\leqslant&\ \alpha\|z_1-z_2\|\\&+\|z_1-z_2\|L(T_1+t)\int_0^{T_1}(T_1+t-s)^{\beta-1}\sup_{x\in G}|v(T_1+t,s,x(s))|\,ds\\&+\|z_1-z_2\|\,\frac{(L^*+J)M}{\beta}t^{\beta}\\\leqslant&\ \alpha\|z_1-z_2\|+\gamma\|z_1-z_2\|+\frac{M(L^*+J)}{\beta}T^{\beta}\|z_1-z_2\|\\=&\ \|z_1-z_2\|\left[\alpha+\gamma+\frac{M(L^*+J)}{\beta}T^{\beta}\right]\end{aligned}$$

i.e.,

$$|(P_1z_1)(t)-(P_1z_2)(t)|\leqslant c\|z_1-z_2\|,\quad t\in[0,T],$$

and so, taking (16) into consideration we find

$$\|P_1z_1-P_1z_2\|\leqslant c\|z_1-z_2\|,$$

where $c<1$ is given in (10). It follows that P_2 is a contraction in G_2^* so it has a unique fixed point x_2 in G_2^* which is defined on $[0,T_2]$ and coincides with x_1 on $[0,T_1]$.

If $T_2 = E$ then the procedure is finalized. Otherwise, as T and δ are independent of T_i, $(i = 1,...,n-1)$, we may consider the space G_3^* of continuous functions on $[0, T_3 = T_2 + T]$ coinciding with x_2 on $[0, T_2]$ (with the sup norm) and the operator P_3 acting on G_3^* and defined in a way analogous to the definition of P_2 in (14) and (15), yet we employ the same argumentation as in Step 2 to prove existence of a unique solution to (1) on $[0, T_3 = T_2 + T]$, and so on until we reach $T_n = E$.

To obtain a unique fixed point on $[0,\infty)$ we let E successively be $1,2,\ldots$, and obtain unique fixed points ϕ_i on $[0,i]$, $i = 1,2,\ldots$, then extend each of those functions to the interval $[0,\infty)$ by letting $\Phi_i(t) = \phi_i(t)$ for $0 \leqslant t \leqslant i$ and let $\Phi_i(t) = \phi_i(i)$ for $t \geqslant i$. Now consider the sequence of functions $\{\Phi_i(t)\}$ which converges uniformly on compact sets to a continuous function $\Phi(t)$ which is a fixed point on $[0,\infty)$. It is unique because each piece is unique and is asymptotically stable. □

While condition (11) seems abstract and rather difficult to verify, the next lemma presents sufficient conditions so that (11) holds true.

LEMMA 1. *Let $E > 0$ be arbitrary, $r > 0$, and the functions $L : [0,\infty) \to [0,\infty)$, $v, v_1 : \{0 \leqslant s \leqslant t \leqslant 2E\} \times [-r, r] \to \Re^+$ be continuous with*

$$|v(t,s,x)| \leqslant v_1(t,s), \quad |x| \leqslant r, 0 \leqslant s \leqslant t \leqslant 2E. \tag{20}$$

If

$$\sup_{u>0} L(u) \int_0^u (u-s)^{\beta-1} v_1(u,s)ds := \gamma_0 < 1-\alpha, \tag{21}$$

then there are γ, $\delta > 0$ such that (11) is satisfied.

Proof. Firstly we prove that for an arbitrary $T \in (0,E]$ the function $V(u,t)$: $[T,E] \times [0,E] \to \mathbb{R}$ with

$$V(u,t) := L(u+t) \int_0^u (u+t-s)^{\beta-1} v_1(u+t,s)ds, \tag{22}$$

is continuous. Continuity at points (u,t) with $t \neq 0$ follows immediately by continuity of L on $[0,\infty)$ and of the integrand on the sets $[T,E] \times \{d \leqslant s \leqslant t \leqslant E\}$ for any $d \in (0,E]$. For the continuity of V at $(u,0)$, $u \in [T,E]$, in view of the continuity of the function L we may consider only continuity of the integral

$$V_1(u,t) := \int_0^u (u+t-s)^{\beta-1} v_1(u+t,s)ds, \quad (u,t) \in [T,E] \times [0,d],$$

at $(u,0)$. For $t \in [0,d]$, $u, u_1 \in [T,E]$ say with $u \leqslant u_1$ we have

$$\begin{aligned} |V_1(u,t) - V_1(u_1,0)| &= \left| \int_0^u (u+t-s)^{\beta-1} v_1(u+t,s)ds - \int_0^{u_1} (u_1-s)^{\beta-1} v_1(u_1,s)ds \right| \\ &\leqslant \int_0^u \left| (u+t-s)^{\beta-1} v_1(u+t,s) - (u_1-s)^{\beta-1} v_1(u_1,s) \right| ds \\ &\quad + \int_u^{u_1} (u_1-s)^{\beta-1} v_1(u_1,s)ds. \end{aligned} \tag{23}$$

Let N be a bound of $v_1(w,s)$ on $S:=[T_1,2E]\times[0,2E]$. We may easily see that

$$\int_u^{u_1}(u_1-s)^{\beta-1}v_1(u_1,s)ds \leqslant N\frac{(u_1-u)^\beta}{\beta}\to 0, \quad \text{for } u_1\to u. \tag{24}$$

We find

$$\begin{aligned}&\int_0^u \left|(u+t-s)^{\beta-1}v_1(u+t,s)-(u_1-s)^{\beta-1}v_1(u_1,s)\right|ds\\ \leqslant&\int_0^u (u+t-s)^{\beta-1}\left|v_1(u+t,s)-v_1(u_1,s)\right|ds\\ &+\int_0^u\left|(u+t-s)^{\beta-1}-(u_1-s)^{\beta-1}\right|v_1(u_1,s)ds,\end{aligned} \tag{25}$$

and note that by uniform continuity of v_1 on S, for $\varepsilon>0$ there exists a sufficiently small $\delta>0$ such that for $|t|,|u_1-u|\leqslant\delta$ we have

$$|v_1(u+t,s)-v_1(u_1,s)|\leqslant\varepsilon,$$

and so

$$\begin{aligned}\int_0^u (u+t-s)^{\beta-1}\left|v_1(u+t,s)-v_1(u_1,s)\right|ds &\leqslant \varepsilon\int_0^u (u+t-s)^{\beta-1}ds\\ &=\varepsilon\frac{(u+t)^\beta-t^\beta}{\beta}\leqslant\varepsilon\frac{(4E)^\beta}{\beta},\end{aligned}$$

from which we infer that

$$\int_0^u (u+t-s)^{\beta-1}\left|v_1(u+t,s)-v_1(u_1,s)\right|ds\to 0, \text{ as } t,u_1-u\to 0. \tag{26}$$

Finally, we find

$$\begin{aligned}\int_0^u\left|(u+t-s)^{\beta-1}-(u_1-s)^{\beta-1}\right|v_1(u_1,s)ds&\leqslant N\int_0^u\left|(u+t-s)^{\beta-1}-(u_1-s)^{\beta-1}\right|ds\\ &=N\left(\frac{(u+t)^\beta-t^\beta}{\beta}-\frac{{u_1}^\beta-(u_1-u)^\beta}{\beta}\right)\end{aligned}$$

so

$$\int_0^u\left|(u+t-s)^{\beta-1}-(u_1-s)^{\beta-1}\right|v_1(u_1,s)ds\to 0, \quad \text{as } t,u_1-u\to 0. \tag{27}$$

By (26) and (27) we see that the right-hand-side of (25) tends to zero for $t,|u_1-u|\to 0$, which, in view of (24) implies that the right-hand-side of (23) tends to zero when $t,|u_1-u|\to 0$, therefore, V_1 is continuous at $(u,0)$. By continuity of L we conclude that the function V defined by (22) is continuous.

Now by continuity we have that V is uniformly continuous on $[T_1,2E]\times[0,2E]$, so for

$$0<\varepsilon<\min\left\{\frac{1-\alpha-\gamma_0}{2},E\right\}$$

there exists a $\delta \in (0,E)$ such that

$$|t_1 - t_2| < \delta, t_1, t_2 \in [0,E], u \in [T,E] \Longrightarrow |V(u,t_1) - V(u,t_2)| < \varepsilon,$$

so, for $t_1 = 0$, $t_2 = t$ with $0 \leqslant t \leqslant \delta$ it holds

$$|V(u,0) - V(u,t)| < \varepsilon, \quad \text{for any } u \in [T,E].$$

Thus, in view of (21) we have for $0 \leqslant t \leqslant \delta$, $u \in [T,E]$

$$\begin{aligned}|V(u,t)| &< \varepsilon + |V(u,0)| \\ &= \varepsilon + L(u)\int_0^u (u-s)^{\beta-1} v_1(u,s)ds \\ &\leqslant \varepsilon + \sup_{u>0} L(u)\int_0^u (u-s)^{\beta-1} v_1(u,s)ds \\ &\leqslant \varepsilon + \gamma_0.\end{aligned}$$

It follows that, for $t \in [0,\delta]$, $u \in [T,E]$ it holds

$$\begin{aligned}&L(u+t)\int_0^u (u+t-s)^{\beta-1} \sup_{x\in G} |v(u+t,s,x(s))|\,ds \\ &\leqslant L(u+t)\int_0^u (u+t-s)^{\beta-1} v_1(u+t,s)ds \\ &= |V(u,t)| \leqslant \varepsilon + \gamma_0 \\ &\leqslant \frac{1-\alpha-\gamma_0}{2} + \gamma_0 = \frac{1-\alpha+\gamma_0}{2} := \gamma < 1-\alpha,\end{aligned}$$

that is, for any $u \in [T,E]$ we have

$$L(u+t)\int_0^u (u+t-s)^{\beta-1} \sup_{x\in G} |v(T+t,s,x(s))|\,ds \leqslant \gamma, t \in [0,\delta],$$

which is (11). $\square$

From the proof of Lemma 1 it is clear that it suffices that the supremum in the left-hand-side in the inequality (21) be taken over $(0,E]$. However, as we want this result to hold for arbitrary $E > 0$, in fact we have to require that the inequality hold for the supremum taken over the whole half-line $[0,\infty)$.

3. An example

Consider the equation (4.1) in [7, pp. 82-83], namely the equation

$$\begin{aligned}x(t) = &\frac{1}{2\pi}\arctan(t + x(t)) \\ &+ \frac{t + t^2 x(t)}{5\Gamma\left(\frac{1}{3}\right)}\int_0^t \frac{|x(s)|e^{-3t-s} + \frac{1}{1+5t^{7/3}}}{(t-s)^{2/3}}ds, \quad t \geqslant 0.\end{aligned} \tag{28}$$

In terms of equation (1), here we have

$$g(t,x) := \frac{1}{2\pi}\arctan(t+x), \quad f(t,x) := \frac{t+t^2x}{5\Gamma\left(\frac{1}{3}\right)}, \quad t \geqslant 0, x \in \mathfrak{R},$$

and

$$v(t,s,x) := |x(s)|e^{-3t-s} + \frac{1}{1+5t^{7/3}}, \quad (t,s) \in \{(t,s) : 0 \leqslant s \leqslant t\}, x \in \mathfrak{R}.$$

Firstly, we note that for any $t \geqslant 0$ and $x, y \in \mathfrak{R}$ we have

$$|g(t,x) - g(t,y)| \leqslant \frac{1}{2\pi}|x-y|,$$

that is g is a contraction and (5) is satisfied with $\alpha = \frac{1}{2\pi}$.

Furthermore, we see that the function $\widehat{g}(x) := g(0,x)$, $x \in \mathfrak{R}$, is a contraction, hence the algebraic equation $g(0,x) = x$, $x \in \mathfrak{R}$, i.e., the equation

$$x_0 = \frac{1}{2\pi}\arctan(x_0)$$

has a unique solution x_0 with $x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We may easily see that $x_0 = 0$ and that (6) is verified.

In view of the calculations in [7, pp. 82-83], we see that taking $G := \{x \in \mathscr{B} : \|x\| \leqslant 1\}$, we have $P(G) \subset G$. As $x_0 = 0 \in (-1,1)$, it follows that the function $\psi(t) = 0$, $t \in [0,E]$ belongs to the interior of G, thus (2) and (7) are satisfied.

Concerning the function f we see that for $x_1, x_2 \in \mathfrak{R}$ and $t \geqslant 0$ it holds

$$|f(t,x_1) - f(t,x_2)| = \frac{t^2|x_1 - x_2|}{5\Gamma\left(\frac{1}{3}\right)} = L(t)|x_1 - x_2|,$$

i.e., (3) is satisfied with $L(t) := \frac{t^2}{5\Gamma\left(\frac{1}{3}\right)}$, $t \geqslant 0$.

Next, as

$$|v(t,s,x) - v(t,s,y)| = e^{-3t-s}\,||x| - |y|| \leqslant |x-y|$$

we have that v satisfies (4) with $J = 1$. Moreover, condition (20) is fulfilled with

$$v_1(t,s) := e^{-3t-s} + \frac{1}{1+5t^{7/3}},$$

yet v_1 is continuous. It remains to show that (11) is satisfied. As v_1, v, and L are continuous, by Lemma 1 in order to show that (11) is satisfied it suffices to show that (21) holds true, i.e.,

$$\sup_{t \geqslant 0} L(t) \int_0^t (t-s)^{\beta-1} v_1(t,s)\,ds < 1 - \alpha.$$

We have for $t > 0$

$$\begin{aligned} L(t)\int_0^t (t-s)^{\beta-1} v_1(t,s)ds &= \frac{t^2}{5\Gamma\left(\frac{1}{3}\right)}\int_0^t (t-s)^{\frac{1}{3}-1}\left(e^{-3t-s}+\frac{1}{1+5t^{7/3}}\right)ds \\ &\leqslant \frac{t^2}{5\Gamma\left(\frac{1}{3}\right)}\int_0^t (t-s)^{-\frac{2}{3}}\left(e^{-3t}+\frac{1}{1+5t^{7/3}}\right)ds \\ &= \left(e^{-3t}+\frac{1}{1+5t^{7/3}}\right)\frac{t^2 t^{\frac{1}{3}}}{5\frac{1}{3}\Gamma\left(\frac{1}{3}\right)} \\ &= \left(e^{-3t}+\frac{1}{1+5t^{7/3}}\right)\frac{t^{7/3}}{5\Gamma\left(\frac{4}{3}\right)}, \end{aligned}$$

or

$$L(t)\int_0^t (t-s)^{\beta-1} v_1(t,s)ds \leqslant \frac{1}{\Gamma\left(\frac{4}{3}\right)}\left[\frac{1}{5}t^{7/3}e^{-3t}+\frac{1}{5}\frac{t^{7/3}}{1+5t^{7/3}}\right], t>0.$$

Borrowing calculations from ([7], p. 83) we see that for the function $\phi(t) := \frac{1}{5}t^{7/3}e^{-3t}$ we have

$$\phi^* := \sup_{t\geqslant 0}\frac{1}{5}t^{7/3}e^{-3t} = \phi\left(\frac{7}{6}\right) = 0.0277897...,$$

thus, as $\Gamma\left(\frac{4}{3}\right) \simeq 0.8929796...$, we take

$$\begin{aligned} \sup_{t\geqslant 0} L(t)\int_0^t (t-s)^{\beta-1} v_1(t,s)ds &\leqslant \frac{1}{\Gamma\left(\frac{4}{3}\right)}\left[t^{7/3}e^{-3t}+\frac{1}{5\cdot 5}\right] \\ &\simeq \frac{1}{0.8929796...}(0.0277897...+0.04) \\ &\leqslant \frac{1}{0.89}(0.068) < 0.0765 \end{aligned}$$

while

$$1-\alpha = 1-\frac{1}{2\pi} \simeq 0.68169...,$$

so (21) is satisfied, and by Lemma 1 it follows that condition (11) is also satisfied. As all assumptions of Theorem 1 are fulfilled we may conclude that equation (28) (i.e., equation (4.1) in [7, pp. 82-83]), has a unique solution x which starts from 0, it is defined on the half-line $[0,\infty)$ and is bounded by 1.

We note that the limit of the solution x at infinity may be calculated. Indeed, from (28) we have

$$x(t)\left[1-\frac{t^2}{5\Gamma\left(\frac{1}{3}\right)\left(1+5t^{7/3}\right)}\int_0^t \frac{|x(s)|\, e^{-3t-s}\left(1+5t^{7/3}\right)+1}{(t-s)^{2/3}}ds\right] \tag{29}$$
$$= \frac{1}{2\pi}\arctan(t+x(t)) + \frac{t}{5\Gamma\left(\frac{1}{3}\right)\left(1+5t^{7/3}\right)}\int_0^t \frac{|x(s)|\, e^{-3t-s}\left(1+5t^{7/3}\right)+1}{(t-s)^{2/3}}ds.$$

It is a matter of simple calculations to verify that, for any $x \in G$, we have

$$\lim_{t\to\infty} \frac{t^2}{1+5t^{7/3}} \int_0^t \frac{|x(s)|e^{-3t-s}\left(1+5t^{7/3}\right)+1}{(t-s)^{2/3}}ds = \frac{3}{5}, \tag{30}$$

$$\lim_{t\to\infty} \frac{t}{1+5t^{7/3}} \int_0^t \frac{|x(s)|e^{-3t-s}\left(1+5t^{7/3}\right)+1}{(t-s)^{2/3}}ds = 0, \tag{31}$$

and so, as the limit of the function in the brackets as well as that of the right-hand-side in (29) are nonzero real numbers, we find

$$\lim_{t\to\infty} x(t)\left[1-\frac{1}{5\Gamma\left(\frac{1}{3}\right)}\frac{1}{\frac{1}{3}\cdot 5}\right] = \frac{1}{2\pi}\arctan(\infty)+0 = \frac{1}{4},$$

from which we conclude that

$$\lim_{t\to\infty} x(t) = \frac{1}{4\left(1-\frac{1}{25\Gamma\left(\frac{4}{3}\right)}\right)}.$$

In view of the above discussion, we have the following result.

THEOREM 2. *There is a unique solution of (28) on* $[0,\infty)$ *which is bounded by* 1. *It tends to a finite limit as* $t \to \infty$ *and it is asymptotically stable in the sense of Banas and Rzepka* [1].

Finally, we cite an equation with large nonlinearities. For convenience, we modify equation (28) by replacing x in f by x^2, yet allowing arbitrarily large exponents of x in the function v inside the integral, namely the equation

$$x(t) = \frac{1}{2\pi}\arctan(t+x(t)) + \frac{2t+t^2x^2(t)}{10\Gamma\left(\frac{1}{3}\right)} \int_0^t \frac{x^{2m}(s)e^{-3t-s}+\frac{2m}{1+5t^{7/3}}}{2m(t-s)^{2/3}}ds, \ t \geqslant 0.$$

One can see that x in the function f has been replaced by $x^2/2$ while $|x|$ in the function v has been replaced by $x^{2m}/(2m), m \in \mathbb{N}$. Clearly neither f nor v are Lipschitzian in x on the whole real line, however they are in the set $G := \{\phi \in \mathscr{B} : ||\phi|| \leqslant 1\}$. It is not difficult to verify that Theorem 1 still applies yielding that the equation has a unique solution x which starts from zero, is defined on $[0,\infty)$ and is bounded by 1, yet for $t \geqslant 1$ it is positive and bounded below by $\frac{1}{2\pi}\arctan(t)$. In fact, in view of (30) and (31), we may prove that for $t \to \infty$, the unique solution x tends to the (unique) root ℓ of the quadratic equation

$$\frac{1}{50\Gamma\left(\frac{4}{3}\right)}\ell^2 - \ell + \frac{1}{4} = 0,$$

with $\ell \in [0,1]$, i.e., we have

$$\lim_{t\to\infty} x(t) = \frac{50\Gamma\left(\frac{4}{3}\right)}{2}\left(1-\sqrt{1-\frac{1}{50\Gamma\left(\frac{4}{3}\right)}}\right).$$

REFERENCES

[1] J. BANAS AND B. RZEPKA, *An application of a measure of noncompactness in the study of asymptotic stability*, Appl. Math. Lett. **16** (2003) 1–6.

[2] T. A. BURTON, *Existence and uniqueness results by progressive contractions for integro-differential equations*, Nonlinear Dynamics and Systems Theory, **16** (4) (2016), 366–371.

[3] T. A. BURTON AND BO ZHANG, *Fixed points and stability of an integral equation:nonuniqueness*, Appl. Math. Lett., **17** (2004), 839–846.

[4] T. A. BURTON AND BO ZHANG, *A NASC for equicontinuous maps for integral equations*, Nonlinear Dynamics and Systems Theory, **17** (3) (2017), 247–265.

[5] C. CORDUNEANU, Integral Equations and Applications, Cambridge, 1991.

[6] MOHAMED ABDALLA DARWISH, *On a quadratic fractional integral equation with linear modification of the argument*, Canadian Applied Mathematics Quarterly, **16** (2008), 45–58.

[7] MOHAMED ABDALLA DARWISH AND JOHNNY HENDERSON, *Existence and asymptotic stability of solutions of a perturbed quadratic fractional integral equation*, Fractional Calculus and Applied Analysis, **12** (2009), 71–86.

[8] J. DUGUNDJI AND A. GRANAS, Fixed Point Theory, Monografie Matematyczne, PWN, Warsaw, 1982.

[9] JACK HALE, Theory of Functional Differential Equations, Springer, New York, 1977.

[10] D. R. SMART, Fixed Point Theorems, Cambridge, 1980.

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