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POSITIVE SOLUTIONS TO A NONLINEAR SIXTH ORDER BOUNDARY VALUE PROBLEM

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Abstract. We consider a sixth order two point boundary value problem. Upper and lower estimates for positive solutions of the problem are proved. Sufficient conditions for the existence and nonexistence of positive solutions for the problem are obtained. An example is included to illustrate the results.

1. Introduction

In this paper, we consider the boundary value problem

$$u^{(6)}(t)+g(t)f(u(t))=0, \quad 0 \leqslant t \leqslant 1, \tag{1.1}$$

$$\left.\begin{aligned} u(0)=u'(0)=u''(0)=0, \\ u'(1)=u'''(1)=u^{(5)}(1)=0. \end{aligned}\right\} \tag{1.2}$$

Our interest here is in the existence and nonexistence of positive solutions of the problem (1.1)–(1.2). By a *positive solution*, we mean a solution $u(t)$ to the boundary value problem such that $u(t)>0$ for $0<t<1$. Throughout the paper, we assume that

(H) The functions $f:[0,\infty)\to[0,\infty)$ and $g:[0,1]\to[0,\infty)$ are continuous, and $g(t)\not\equiv 0$ on $[0,1]$.

Sixth order boundary value problems arise from the study of elasticity. For example, according to Agarwal, Kovacs, and O'Regan [2], the deformation of the equilibrium state of an elastic circular ring segment with its two ends simply supported can be described by the sixth-order boundary value problem

$$u^{(6)}+2u^{(4)}+u''=f(t,u), \quad 0<t<1,$$

$$u(0)=u(1)=u''(0)=u''(1)=u^{(4)}(0)=u^{(4)}(1)=0.$$

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Sixth order boundary value problems have attracted some attention recently. For example, in 2006, Gyulov, Morosanu, and Tersian [12] studied the existence and multiplicity of nontrivial solutions of the following boundary value problem

$$u^{(6)} + Au^{(4)} + Bu'' + Cu = f(t,u), \quad 0 < t < L,$$

$$u(0) = u(L) = u''(0) = u''(L) = u^{(4)}(0) = u^{(4)}(L) = 0,$$

where $A, B,$ and C are real constants. For some other results on sixth order boundary value problems, we refer the readers to [1, 3, 5, 6, 7, 8, 9, 10, 11, 13, 15, 16, 17, 18, 20].

The problem (1.1)–(1.2) is closely related to the (3,3) conjugate boundary value problem. For example, if $u(t)$ solves the conjugate problem

$$u^{(6)}(t) + g(t) = 0, \quad 0 \leqslant t \leqslant 2, \tag{1.3}$$

$$\left.\begin{aligned} u(0) = u'(0) = u''(0) = 0, \\ u(2) = u'(2) = u''(2) = 0, \end{aligned}\right\} \tag{1.4}$$

and $u(t)$ is symmetric on $[0,2]$ in the sense that

$$u(t) = u(2-t), \quad 0 \leqslant t \leqslant 2,$$

then $u(t)$ solves the problem

$$u^{(6)}(t) + g(t) = 0, \quad 0 \leqslant t \leqslant 1, \tag{1.5}$$

$$\left.\begin{aligned} u(0) = u'(0) = u''(0) = 0, \\ u'(1) = u'''(1) = u^{(5)}(1) = 0. \end{aligned}\right\} \tag{1.6}$$

The interested reader is referred to Al Twaty and Eloe [4] for a recent paper on conjugate problems.

We will use the following fixed point theorem, which is due to Krasnosel'skii, to prove our existence results.

THEOREM 1. ([14]) *Let X be a Banach space over the reals, and let $P \subset X$ be a cone in X. Let $\leqslant$ be the partial order on X determined by P. Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let*

$$L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$$

be a completely continuous operator such that, either

(K1) $Lu \not\geqslant u$ if $u \in P \cap \partial\Omega_1$, and $Lu \not\leqslant u$ if $u \in P \cap \partial\Omega_2$; or

(K2) $Lu \not\leqslant u$ if $u \in P \cap \partial\Omega_1$, and $Lu \not\geqslant u$ if $u \in P \cap \partial\Omega_2$.

Then L has a fixed point in $P \cap (\overline{\Omega}_2 - \Omega_1)$.

Throughout we let $X = C[0,1]$ be equipped with the supremum norm

$$\|v\| = \max_{t\in[0,1]} |v(t)| \quad \text{for all } v \in X.$$

Clearly, X is a Banach space.

This paper is organized as follows. In Section 2, we obtain some *priori* estimates to positive solutions to the problem (1.1)–(1.2). In Section 3, we apply Theorem 1 to establish some existence results for positive solutions of the problem (1.1)–(1.2). In Section 4, we present some nonexistence results. An example is given at the end of the section to illustrate our results.

2. Green's function

In this section, we give Green's function for the problem (1.1)–(1.2) and prove some upper and lower estimates for positive solutions of the problem.

We define the function $G : [0,1]\times[0,1] \to \mathbb{R}$ by

$$G(t,s) = \left(\frac{t^3}{2} - \frac{t^4}{8}\right)\frac{(1-s)^4}{24} + \left(\frac{-t^3}{12} + \frac{t^4}{16}\right)\frac{(1-s)^2}{2} + \frac{t^3}{48} - \frac{5t^4}{192} + \frac{t^5}{120} - \frac{(t-s)^5}{120}H(t-s).$$

Here, $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside function

$$H(t) = \begin{cases} 1, \ t \geqslant 0, \\ 0, \ t < 0. \end{cases}$$

Then, $G(t,s)$ is Green's function associated with the problem (1.1)–(1.2). That is, the problem (1.1)–(1.2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)g(s)f(u(s))ds, \quad 0 \leqslant t \leqslant 1. \tag{2.1}$$

It is easily seen that

$$G(1,s) = \frac{s^3}{960}(20 - 25s + 8s^2), \quad 0 \leqslant s \leqslant 1.$$

It is obvious that $G(1,s) \geqslant 0$ for $0 \leqslant s \leqslant 1$.

We define the functions $a : [0,1] \to [0,1]$ and $b : [0,1] \to [0,1]$ by

$$b(t) = 4t^2 - 4t^3 + t^4, \quad 0 \leqslant t \leqslant 1,$$

$$a(t) = \frac{t^3}{3}(20 - 25t + 8t^2), \quad 0 \leqslant t \leqslant 1.$$

These functions will be used to give the upper and lower estimates for Green's function $G(t,s)$. We leave it to the reader to show that both $a(t)$ and $b(t)$ are increasing on the interval $[0,1]$, and

$$a(0) = b(0) = 0, \quad a(1) = b(1) = 1.$$

Also, since

$$1-b(t)=(1-t)^2(1+2t-t^2)\geqslant 0, \quad 0\leqslant t\leqslant 1,$$

$$b(t)-a(t)=\frac{4t^2}{3}(1-t)^2(3-2t)\geqslant 0, \quad 0\leqslant t\leqslant 1,$$

and

$$a(t)=\frac{t^3}{3}(3+(1-t)(17-8t))\geqslant 0, \quad 0\leqslant t\leqslant 1,$$

we have

$$0\leqslant a(t)\leqslant b(t)\leqslant 1, \quad 0\leqslant t\leqslant 1.$$

The next lemma gives an upper bound for Green's function $G(t,s)$.

LEMMA 1. *We have, for all* $t,s\in[0,1]$,

$$G(t,s)\leqslant b(t)G(1,s).$$

Proof. Our strategy of proof is to decompose the difference $b(t)G(1,s)-G(t,s)$ into a number of positive terms. We take two cases. If $0\leqslant s\leqslant t\leqslant 1$, then

$$b(t)G(1,s)-G(t,s) = \frac{s^4(1-t)^2}{240}\left[s(2s^2-9s+8)+(5-2s)(2-s-t)(t-s)\right]\geqslant 0.$$

In the second case, that is, if $0\leqslant t\leqslant s\leqslant 1$, then

$$\begin{aligned} b(t)G(1,s)-G(t,s) &= \frac{t^2}{240}\big[s^3(2s^2-9s+8)(1-s)^2+2(s-t)s^3(1-s)^2(5-2s)\\ &\quad +2(s-t)s^2(1-s)(3-2s)+2(s-t)^3+(s-t)^2s(4-3s)\\ &\quad +(s-t)^2s^2(1-s)(3+3s-2s^2)\big]\\ &\geqslant 0. \end{aligned}$$

The proof of the lemma is now complete. □

The next lemma provides a lower bound for Green's function $G(t,s)$.

LEMMA 2. *We have, for all* $t,s\in[0,1]$,

$$G(t,s)\geqslant a(t)G(1,s)\geqslant 0.$$

Proof. Again, we take two cases. If $0\leqslant s\leqslant t\leqslant 1$, then

$$\begin{aligned} G(t,s)-a(t)G(1,s) &= \frac{s^3(1-t)^2}{360}\big[2t^2(9-4t)(1-t)^2+(t-s)t(1-t)(9-16t^2+27t)\\ &\quad +(t-s)^2(3-8t^3+9t^2+6t)\big]\\ &\geqslant 0. \end{aligned}$$

If $0 \leqslant t \leqslant s \leqslant 1$, then

$$\begin{aligned} G(t,s) - a(t)G(1,s) = & \frac{t^3(1-s)^2}{360}\big[2s^2(9-4s)(1-s)^2 + (s-t)s(1-s)(9-16s^2+27s) \\ & + (s-t)^2(3-8s^3+9s^2+6s)\big] \\ \geqslant & \ 0. \end{aligned}$$

The proof of the lemma is now complete. □

LEMMA 3. *If $u \in C^6[0,1]$ is such that*

$$u^{(6)}(t) \leqslant 0, \quad 0 \leqslant t \leqslant 1, \tag{2.2}$$

and $u(t)$ satisfies the boundary conditions (1.2), then

$$b(t)u(1) \geqslant u(t) \geqslant a(t)u(1), \quad 0 \leqslant t \leqslant 1.$$

Proof. On one hand, we have

$$u(t) = -\int_0^1 G(t,s)u^{(6)}(s)ds \geqslant -a(t)\int_0^1 G(1,s)u^{(6)}(s)ds = a(t)u(1), \quad 0 \leqslant t \leqslant 1.$$

On the other hand, we have

$$u(t) = -\int_0^1 G(t,s)u^{(6)}(s)ds \leqslant -b(t)\int_0^1 G(1,s)u^{(6)}(s)ds = b(t)u(1), \quad 0 \leqslant t \leqslant 1.$$

The proof of the lemma is now complete. □

The following lemma follows immediately.

LEMMA 4. *If $u \in C^6[0,1]$ satisfies (2.2) and (1.2), then $\|u\| = u(1)$.*

We now summarize our findings in the following theorem.

THEOREM 2. *Suppose that* (H) *hold. If $u \in C^6[0,1]$ satisfies (2.2) and the boundary conditions (1.2), then $b(t)u(1) \geqslant u(t) \geqslant a(t)u(1)$ on $[0,1]$. In particular, if $u \in C^6[0,1]$ is a nonnegative solution to the boundary value problem (1.1)–(1.2), then $b(t)u(1) \geqslant u(t) \geqslant a(t)u(1)$ on $[0,1]$.*

Now we define

$$P = \{v \in X : a(t)\|v\| \leqslant v(t) \leqslant b(t)\|v\| \text{ on } [0,1]\}.$$

Clearly, P is a positive cone of the Banach space X. Define an operator $T : P \to X$ by

$$Tu(t) = \int_0^1 G(t,s)g(s)f(u(s))ds, \quad 0 \leqslant t \leqslant 1, \text{ for all } u \in X.$$

It is well known that $T : P \to X$ is a completely continuous operator. And, by the same arguments as those used to prove Theorem 2, we can show that $T(P) \subset P$ provided (H) holds. We also note that if $v \in P$, then

$$\|v\| = v(1).$$

Now the integral equation (2.1) is equivalent to the equality

$$Tu = u, \quad u \in P,$$

so in order to solve the problem (1.1)–(1.2), we only need to find a fixed point of T in P.

3. Existence of positive solutions

We begin by defining the constants

$$A = \int_0^1 G(1,s)g(s)a(s)\,ds \quad \text{and} \quad B = \int_0^1 G(1,s)g(s)b(s)\,ds.$$

Also, we define the constants

$$F_0 = \limsup_{x\to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x\to 0^+} \frac{f(x)}{x},$$
$$F_\infty = \limsup_{x\to +\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x\to +\infty} \frac{f(x)}{x}.$$

These constants will be used in the statements of our existence and nonexistence theorems. Our first existence result is the following.

THEOREM 3. *If*

$$BF_0 < 1 < Af_\infty,$$

then the problem (1.1)–(1.2) *has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ such that $B(F_0 + \varepsilon) < 1$. Then there exists $H_1 > 0$ such that

$$f(x) \leqslant (F_0 + \varepsilon)x \ \text{ for } \ 0 < x \leqslant H_1.$$

For each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}
(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\
&\leqslant \int_0^1 G(1,s)g(s)(F_0+\varepsilon)u(s)\,ds \\
&\leqslant (F_0+\varepsilon)\|u\| \int_0^1 G(1,s)g(s)b(s)ds \\
&= (F_0+\varepsilon)\|u\|B \\
&< \|u\|,
\end{aligned}$$

which means $Tu \not\geqslant u$. If we let $\Omega_1 = \{u \in X \mid \|u\| < H_1\}$, then

$$Tu \not\geqslant u, \quad \text{for any } u \in P \cap \partial\Omega_1.$$

To construct Ω_2, we first choose $c \in (0, 1/4)$ and $\delta > 0$ such that

$$(f_\infty - \delta)\int_c^1 G(1,s)g(s)a(s)\,ds > 1.$$

Now, there exists $H_3 > 0$ such that $f(x) \geqslant (f_\infty - \delta)x$ for $x \geqslant H_3$. Let $H_2 = H_1 + H_3/a(c)$. If $u \in P$ with $\|u\| = H_2$, then for $c \leqslant t \leqslant 1$, we have

$$u(t) \geqslant a(t)\|u\| \geqslant a(c)H_2 > H_3.$$

So, if $u \in P$ with $\|u\| = H_2$, then

$$\begin{aligned}(Tu)(1) &\geqslant \int_c^1 G(1,s)g(s)f(u(s))\,ds \\ &\geqslant \int_c^1 G(1,s)g(s)(f_\infty - \delta)u(s)ds \\ &\geqslant (f_\infty - \delta)\|u\| \int_c^1 G(1,s)g(s)a(s)\,ds \\ &> \|u\|,\end{aligned}$$

which means $Tu \not\leqslant u$. So, if we let $\Omega_2 = \{u \in X : \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$, and

$$Tu \not\leqslant u, \quad \text{for any } u \in P \cap \partial\Omega_2.$$

Therefore, condition (K1) of Theorem 1 is satisfied, and so there exists a fixed point of T in P. This completes the proof of the theorem. □

Our next theorem is a companion result to the one above.

THEOREM 4. *If*

$$BF_\infty < 1 < Af_0,$$

then the problem (1.1)–(1.2) *has at least one positive solution.*

The proof of Theorem 4 is similar to that of Theorem 3 and is therefore left to the reader.

4. Nonexistence results and example

In this section, we give some sufficient conditions for the nonexistence of positive solutions.

THEOREM 5. *Suppose that* (H) *holds. If* $Bf(x) < x$ *for all* $x \in (0, +\infty)$, *then the problem* (1.1)–(1.2) *has no positive solutions.*

Proof. Assume, to the contrary, that $u(t)$ is a positive solution of problem (1.1)–(1.2). Then $u \in P$, $u(t) > 0$ for $0 < t < 1$, and

$$\begin{aligned} u(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(1,s)g(s)u(s)\,ds \\ &\leqslant B^{-1}\|u\|\int_0^1 G(1,s)g(s)b(s)ds \\ &= \|u\|, \end{aligned}$$

which is a contradiction. The proof of the theorem is now complete. □

In a similar fashion, we can prove the next theorem.

THEOREM 6. *Suppose that* (H) *holds. If* $Af(x) > x$ *for all* $x \in (0,+\infty)$, *then the problem* (1.1)–(1.2) *has no positive solutions.*

We conclude the section with an example.

EXAMPLE 1. Consider the sixth-order boundary-value problem

$$u^{(6)}(t) + g(t)f(u(t)) = 0, \quad 0 \leqslant t \leqslant 1, \tag{4.1}$$

$$\left.\begin{aligned} u(0) = u'(0) = u''(0) = 0, \\ u'(1) = u'''(1) = u^{(5)}(1) = 0, \end{aligned}\right\} \tag{4.2}$$

where

$$\begin{aligned} g(t) &= 100(1+t), \quad 0 \leqslant t \leqslant 1, \\ f(u) &= \lambda u\frac{1+3u}{1+u}, \quad u \geqslant 0. \end{aligned}$$

Here, $\lambda > 0$ is a parameter. We easily see that $F_0 = f_0 = \lambda$ and $F_\infty = f_\infty = 3\lambda$. Calculations indicate that

$$A = \frac{36865}{199584}, \quad B = \frac{6689}{33264}.$$

From Theorem 3 we see that if

$$1.8046 \approx \frac{1}{3A} < \lambda < \frac{1}{B} \approx 4.9729,$$

then problem (4.1)–(4.2) has at least one positive solution. From Theorems 5 and 6, we see that if

$$\lambda < \frac{1}{3B} \approx 1.6576 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 5.4139,$$

then the problem (4.1)–(4.2) has no positive solutions.

This example shows that our existence and nonexistence conditions are quite sharp and work very well.

5. Application to the conjugate problem

As a special case of Theorem 1 of [14] and Theorem 4.3 of [19] we have the following result on positive solutions of the $(3,3)$ conjugate problem.

THEOREM 7. *If $u \in C^6[0,1]$ is such that*

$$u^{(6)}(t) \leqslant 0, \quad 0 \leqslant t \leqslant 1$$

and satisfies the conjugate boundary conditions

$$u(0) = u'(0) = u''(0) = u''(1) = u'(1) = u(1) = 0,$$

then we have

$$a_1(t)\|u\| \leqslant u(t) \leqslant b_1(t)\|u\|, \quad 0 \leqslant t \leqslant 1,$$

where

$$a_1(t) = \begin{cases} \dfrac{5^5}{2^2 \cdot 3^3} t^3(1-t)^2, & 0 \leqslant t \leqslant 1/2, \\ \dfrac{5^5}{2^2 \cdot 3^3} t^2(1-t)^3, & 1/2 \leqslant t \leqslant 1, \end{cases}$$

and

$$b_1(t) = \begin{cases} \dfrac{5^5}{2^2 \cdot 3^3} t^2(1-t)^3, & 0 \leqslant t \leqslant 2/5, \\ 1, & 2/5 \leqslant t \leqslant 3/5, \\ \dfrac{5^5}{2^2 \cdot 3^3} t^3(1-t)^2, & 3/5 \leqslant t \leqslant 1. \end{cases}$$

As explained in Section 1, problem (1.1)-(1.2) is closely related to the (3,3) conjugate boundary value problem. If we apply Theorem 2, we get the following result on symmetric positive solutions of the $(3,3)$ conjugate problem.

THEOREM 8. *Suppose that $u \in C^6[0,1]$ is such that*

$$u^{(6)}(t) \leqslant 0, \quad 0 \leqslant t \leqslant 1,$$

and that $u(t)$ satisfies the conjugate boundary conditions

$$u(0) = u'(0) = u''(0) = u''(1) = u'(1) = u(1) = 0.$$

If $u(t)$ is symmetric in the sense that

$$u(t) = u(1-t), \quad 0 \leqslant t \leqslant 1,$$

then $u'(1/2) = u'''(1/2) = u^{(5)}(1/2) = 0$, $u(1/2) = \|u\|$, and

$$a_2(t)\|u\| \leqslant u(t) \leqslant b_2(t)\|u\|, \quad 0 \leqslant t \leqslant 1,$$

where

$$a_2(t) = \begin{cases} \dfrac{16t^3}{3}\left(16t^2 - 25t + 10\right), & 0 \leqslant t \leqslant 1/2, \\ \dfrac{16(1-t)^3}{3}\left(16t^2 - 7t + 1\right), & 1/2 \leqslant t \leqslant 1, \end{cases}$$

and

$$b_2(t) = 16t^2(1-t)^2, \quad 0 \leqslant t \leqslant 1.$$

The proof of Theorem 8 is straightforward and therefore left to the reader.

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