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SOME PROPERTIES ABOUT COMPLEX DIFFERENCE EQUATIONS OF MALMQUIST TYPE

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Abstract. This article presents versions of the Malmquist type equation. We study the growth of transcendental meromorphic solutions of some complex $(qz+c)$ difference equations and find lower bounds for Nevanlinna lower order for meromorphic solutions of such equations. We also obtain a $(qz+c)$ difference version of Tumura-Clunie theorem, which improves the results of Zheng and Chen[25].

1. Introduction

Let $f(z)$ be a meromorphic function in the whole complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [14][18][20]. We also use notations $\rho(f)$, $\lambda(f)(\overline{\lambda}(f))$ and $\lambda(\frac{1}{f})(\overline{\lambda}(\frac{1}{f}))$ for the order, the exponent of convergence of zeros(distinct zeros) and the exponent of convergence of poles (distinct poles) of $f(z)$ respectively.

The celebrated Malmquist theorem states that a complex differential equation of the form

$$y' = R(z,y), \tag{1.1}$$

where the right-hand side is rational in both arguments, and which admits a transcendental meromorphic solution y in the complex plane, reduces into a Riccati differential equation

$$y' = a(z) + b(z)y + c(z)y^2 \tag{1.2}$$

with rational coefficients. For more details concerning the equations (1.1) and (1.2), as well as for generalizations of the Malmquist theorem, see[18].

Recently, meromorphic solutions of complex difference equations have gained increasing interest, due to the apparent role of the existence of such solutions of finite order for the integrability of difference and $q-$difference equations. For example, Halburd and Korhonen[12] showed that the existence of sufficiently many meromorphic

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solutions of finite order is enough to single out the second difference Painlevé equation. A number of papers(see e.g.[1, 2, 3, 4, 6, 9, 12, 13, 16, 17, 19, 26, 27]) focused on the growth and the existence of meromorphic solutions of difference equations and $q-$difference equations.

In 2001, Heittokangas et al.[16] considered the essential growth problem for transcendental meromorphic solutions of complex difference equations, which is to find lower bounds for their characteristic functions, and obtained the following results.

THEOREM 1. *Let* $c_1,\ldots,c_n\in\mathbb{C}\setminus\{0\}$ *and let* $m\geqslant 2$. *Suppose that* y *is a transcendental meromorphic solution of difference equation*

$$\sum_{i=1}^{n} a_i(z)y(z+c_i)=\sum_{i=0}^{m} b_i(z)y(z)^i \tag{1.3}$$

with rational coefficients $a_i(z)$, $b_i(z)$. *Denote* $C=\max\{|c_1|,\ldots,|c_n|\}$.

(1) If y *is entire or has finitely many poles, then there exist constants* $K>0$ *and* $r_0>0$ *such that* $\log M(r,y)\geqslant Km^{\frac{r}{C}}$ *holds for all* $r\geqslant r_0$.

(2) If y *has infinitely many poles, then there exist constants* $K>0$ *and* $r_0>0$ *such that* $n(r,y)\geqslant Km^{\frac{r}{c}}$ *holds for all* $r\geqslant r_0$.

(3) Thus, all transcendental meromorphic solutions of (1.3) have infinitely lower order.

THEOREM 2. *Let* $c_1,\ldots,c_n\in\mathbb{C}\setminus\{0\}$ *and* y *be a transcendental meromorphic solution of difference equation*

$$\sum_{i=1}^{n} d_i(z)y(z+c_i)=\frac{a_0(z)+a_1(z)y(z)+\cdots+a_p(z)y(z)^p}{b_0(z)+b_1(z)y(z)+\cdots+b_t(z)y(z)^t}, \tag{1.4}$$

where all coefficients in (1.4) are of growth $o(T(r,y))$ *without an exceptional set as* $r\to\infty$, *and* $d_i^{\prime}s$ *are non-vanishing. If* $d=\max\{p,t\}>n$, *then for any* ε $(0<\varepsilon<\frac{d-n}{d+n})$, *there exists an* r_0 *such that* $T(r,y)\geqslant K(\frac{d}{n}(\frac{1-\varepsilon}{1+\varepsilon}))^{\frac{r}{C}}$ *for all* $r\geqslant r_0$, *where* $C=\max\{|c_1|,\ldots,|c_n|\}$ *and* $K>0$ *is a constant.*

THEOREM 3. *Suppose that all coefficients in (1.4) are of growth* $S(r,y)$ *and that all other assumptions of theorem B hold. Then* $\mu(y)=\infty$.

Zheng and Chen[25] considered a similar growth problem for transcendental meromorphic solutions of complex q-difference equations instead of difference equations, where the usual shift $f(z+c)$ will be replaced by the q-shift $f(qz)$, and obtain the following results.

THEOREM 4. *Suppose that* f *is a transcendental meromorphic solution of equation*

$$\sum_{j=1}^{n} a_j(z)f(q^jz)=\sum_{i=0}^{d} b_i(z)f(z)^i \tag{1.5}$$

where $q \in \mathbb{C}$, $|q| > 1$, $d \geqslant 2$ *and the coefficients* $a_j(z)$, $b_i(z)$ *are rational functions.*

(1) If f *is entire or has finitely many poles, then there exist constants* $K > 0$ *and* $r_0 > 0$ *such that for all* $r \geqslant r_0$, $\log M(r,f) \geqslant Kd^{\frac{\log r}{n \log |q|}}$.

(2) If y *has infinitely many poles, then there exist constants* $K > 0$ *and* $r_0 > 0$ *such that for all* $r \geqslant r_0$, $n(r,f) \geqslant Kd^{\frac{\log r}{n \log |q|}}$.

(3) Thus, the lower order of f *satisfies* $\mu(f) \geqslant \frac{\log d}{n \log |q|}$.

THEOREM 5. *Suppose that* f *is a transcendental meromorphic solution of equation*

$$\sum_{j=1}^{n} a_j(z) f(q^j z) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \tag{1.6}$$

where $q \in \mathbb{C}$, $|q| > 1$, *all coefficients* $a_j(z)$ *are rational functions and* P, Q *are relatively prime polynomials in* f *over the field of rational functions satisfying* $p = \deg_f P$, $t = \deg_f Q$, $d = p - t \geqslant 2$. *If* f *has infinitely many poles, then for sufficiently large* r, $n(r,f) \geqslant Kd^{\frac{\log r}{n \log |q|}}$ *holds of some constant* $K > 0$. *Thus, the lower order of* f, *which has infinitely many poles, satisfies* $\mu(f) \geqslant \frac{\log d}{n \log |q|}$.

THEOREM 6. *Suppose that* f *is a transcendental meromorphic solution of equation*

$$\frac{\sum_{\lambda \in I} d_\lambda(z) f(qz)^{i_{\lambda,1}} f(q^2 z)^{i_{\lambda,2}} \cdots f(q^n z)^{i_{\lambda,n}}}{\sum_{\mu \in I} e_\mu(z) f(qz)^{j_{\mu,1}} f(q^2 z)^{j_{\mu,2}} \cdots f(q^n z)^{j_{\mu,n}}} = \frac{a_0(z) + a_1(z) f(z) + \cdots + a_p(z) f(z)^p}{b_0(z) + b_1(z) f(z) + \cdots + b_t(z) f(z)^t} \tag{1.7}$$

where $I = \{(i_{\lambda,1}, i_{\lambda,2}, \ldots, i_{\lambda,n})\}$, $J = \{(j_{\mu,1}, j_{\mu,2}, \ldots, j_{\mu,n})\}$ *are two finite index sets,*

$$\max_{\lambda,\mu} \{i_{\lambda,1} + i_{\lambda,2} + \cdots + i_{\lambda,n}, j_{\mu,1} + j_{\mu,2} + \cdots + j_{\mu,n}\} = \sigma,$$

$q \in \mathbb{C}$, $|q| > 1$ *and all coefficients of (1.7) are of growth* $S(r,f)$. *If* $d = \max\{p,t\} > 2n\sigma$, *then for sufficiently large* r, $T(r,f) \geqslant K(\frac{d}{2n\sigma})^{\frac{\log r}{n \log |q|}}$, *where* $K (> 0)$ *is a constant. Thus, the lower order of* f *satisfies* $\mu(f) \geqslant \frac{\log d - \log 2n\sigma}{n \log |q|}$.

From the ideas of theorem 1 to theorem 6, It is natural to ask from theorem 1 to theorem 6 whether the shift $f(z+c)$ or $f(qz)$ can be replaced by $f(qz+c)$ in above theorems. The answer is positive. In 2011, Wang obtained a more general case(see theorem 1 in [24]), which more improve the theorem 4 and theorem 5. Now, also base on this idea, we also prove the following result which improve the theorem 6.

THEOREM 7. *Let* $c_1, \ldots, c_n \in \mathbb{C} \setminus \{0\}$ *and* f *be a transcendental meromorphic solution of difference equation*

$$\frac{\sum_{\lambda \in I} d_\lambda(z) f(qz + c_1)^{i_{\lambda,1}} f(q^2 z + c_2)^{i_{\lambda,2}} \cdots f(q^n z + c_n)^{i_{\lambda,n}}}{\sum_{\mu \in I} e_\mu(z) f(qz + c_1)^{j_{\mu,1}} f(q^2 z + c_2)^{j_{\mu,2}} \cdots f(q^n z + c_n)^{j_{\mu,n}}}$$

$$= \frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_t(z)f(z)^t}, \tag{1.8}$$

where $I=\{(i_{\lambda,1},i_{\lambda,2},...,i_{\lambda,n})\}$, $J=\{(j_{\mu,1},j_{\mu,2},...,j_{\mu,n})\}$ *are two finite index sets,*

$$\max_{\lambda,\mu}\{i_{\lambda,1}+i_{\lambda,2}+\cdots+i_{\lambda,n}, j_{\mu,1}+j_{\mu,2}+\cdots+j_{\mu,n}\}=\sigma,$$

$q\in\mathbb{C}$, $|q|>1$ *and all coefficients of (1.7) are of growth* $S(r,f)$. *If* $d=\max\{p,t\}>2n\sigma$, *then for sufficiently large* r, $T(r,f)\geqslant K(\frac{d}{2n\sigma})^{\frac{\log r}{(n+1)\log|q|}}$, *where* $K(>0)$ *is a constant. Thus, the lower order of* f *satisfies* $\mu(f)\geqslant\frac{\log d-\log 2n\sigma}{(n+1)\log|q|}$.

We also use the shift $f(qz+c)$ to replace the results of $f(qz)$ or $f(z+c)$ in Zheng and Chen[25] and in Laine[19]. We also obtain the following results.

THEOREM 8. *Let* $c_1,...,c_n\in\mathbb{C}\setminus\{0\}$ *and* $f(z)$ *be a transcendental meromorphic solution of difference equation*

$$\sum_{j=1}^{n}a_j(z)f(q^jz+c_j)=R(z,f(z))=\frac{P(z,f(z))}{Q(z,f(z))}, \tag{1.9}$$

where $q\in\mathbb{C}$, $|q|>1$, *all coefficients* $a_j(z)$ *are rational functions and* P, Q *are relatively prime polynomials in* f *over the field of small functions relative to* f. *Moreover, we assume that* $t=\deg_f Q>0$, $n=\max\{p,t\}=\max\{\deg_f P,\deg_f Q\}$ *and that, without restricting generality,* Q *is a monic polynomial. If there exist* $\alpha\in[0,n)$ *such that for all sufficiently large* r

$$\overline{N}(r,\sum_{j=1}^{n}a_j(z)f(q^jz+c_j))\leqslant\alpha\overline{N}(|q|^nr+c,f(z))+S(r,f), \tag{1.10}$$

then either the order $\rho(f)>0$, *or* $Q(z,f(z))\equiv(f(z)+h(z))^t$, *where* $h(z)$ *is a small meromorphic function and* $c=\max\{|c_1|,|c_2|,...,|c_n|\}$.

THEOREM 9. *Let* $c_1,...,c_n\in\mathbb{C}\setminus\{0\}$ *and* f *be a transcendental meromorphic solution of the equation*

$$\sum_{\{J\}}\alpha_J(z)(\Pi_{j\in J}f(q^jz+c_j))=f(p(z)),$$

where $p(z)$ *is a polynomial of degree* $k\geqslant 2$. *Moreover, we assume that the coefficients* $\alpha_J(z)$ *are small functions relative to* f *and that* $n\geqslant k$. *Then*

$$T(r,f)=O((\log r)^{\alpha+\varepsilon}),$$

where $\alpha=\frac{\log n}{\log k}$, $\{J\}$ *the collection of non-empty subsets of* $\{1,...,n\}$.

2. Preliminary lemmas

In order to prove our results, we need the following lemmas.

LEMMA 1. (Valiron-Mohon'ko[18]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f(z)) = \frac{\sum_{i=0}^{m} a_i(z)f(z)^i}{\sum_{j=0}^{n} b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies

$$T(r,R(z,f(z))) = dT(r,f) + O(\psi(r)),$$

where $d = \max\{m,n\}$ and $\psi(r) = \max_{i,j}\{T(r,a_i),T(r,b_j)\}$.

LEMMA 2. *Given distinct complex numbers $c_1,...,c_n$, a meromorphic function f, and small functions $\alpha_J(z)$ relative to f, we have*

$$T(r,\Sigma_{\{J\}}\alpha_J(z)(\Pi_{j\in J}f(z+c_j))) \leqslant \sum_{k=1}^{n} T(r,f(z+c_k)) + S(r,f)$$

where $\{J\}$ the collection of non-empty subsets of $\{1,...,n\}$.

LEMMA 3. *Let $g:(0,+\infty)\to R$, $h:(0,+\infty)\to R$ be monotone increasing functions such that $g(r)\leqslant h(r)$ outside of an exceptional set of finite linear measure. Then for any $\alpha>1$, there exist $r_0>0$ such that $g(r)\leqslant h(\alpha r)$ for all $r>r_0$.*

The following lemma 4 is a variant of the famous Tumura-Clunie theorem([7, 22]).

LEMMA 4. (Tumura-Clunie theorem [23]) *Let f be a meromorphic function, and let ϕ be given by $\phi = f^n + a_{n-1}f^{n-1} + \cdots + a_0$, where $a_0,a_1,...,a_{n-1}$ are small meromorphic functions relative to f. Then either*

$$\phi = (f+\frac{a_{n-1}}{n})^n$$

or

$$T(r,f) \leqslant \overline{N}(r,\frac{1}{\phi}) + \overline{N}(r,f) + S(r,f).$$

LEMMA 5. [19] *Let f be a non-constant meromorphic function, and let $P(z,f)$, $Q(z,f)$ be two polynomials in f with meromorphic coefficients small relative to f. If P and Q have no common factors of positive degree in f over the field of small functions relative to f, then*

$$\overline{N}(r,\frac{1}{Q(z,f)}) \leqslant \overline{N}(r,\frac{P(z,f)}{Q(z,f)}) + S(r,f).$$

The following lemma 6 is a special case of [15, lemma 4].

LEMMA 6. *If* $T : R^+ \to R^+$ *is an increasing function such that* $\limsup_{r\to\infty} \frac{\log T(r)}{\log r} = 0$*, then the set* $E = \{r : T(C_1 r) \geqslant C_2 T(r)\}$ *has logarithmic density 0 for all* $C_1 > 1$ *and* $C_2 > 1$*.*

The following result follows immediately by lemma 6.

LEMMA 7. *If* f *be a non-constant meromorphic function,* $\beta > 1$*,* $\alpha < 1$ *are given constants, and let* $F \subset R^+$ *be the set of all* r *such that* $\overline{N}(r, f) \leqslant \alpha\overline{N}(\beta r, f)$*. If the logarithmic density of* F *is non-zero, that is* $logdensF > 0$*, then the exponent of convergence of distinct poles* $\overline{\lambda}(\frac{1}{f})$ *is non-zero. Thus,* $\rho(f)$ *is non-zero.*

LEMMA 8. ([5, theorem 2.1]) *Let* f *be a meromorphic function of finite order* ρ *and* c *is a non-zero complex constant. Then, for each* $\varepsilon > 0$*, we have*

$$T(r, f(z+c)) = T(r, f) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

It is evident that $S(r, f(z+c)) = S(r, f)$ from lemma 2.8.

LEMMA 9. ([5, theorem 2.2]) *Let* f *be a meromorphic function with finite exponent of convergence of poles* $\lambda(\frac{1}{f})$ *and* c *is a non-zero complex constant. Then, for each* $\varepsilon > 0$*, we have*

$$N(r, f(z+c)) = N(r, f) + O(r^{\lambda(\frac{1}{f})-1+\varepsilon}) + O(\log r).$$

Obviously, by lemma 8 and lemma 9, if $\rho(f) = 0$ and f is a transcendental meromorphic function then $T(r, f(z+c)) = T(r, f) + O(1)$, $N(r, f(z+c)) = N(r, f) + O(1)$.

LEMMA 10. [25] *Let* f *be a meromorphic function*

$$M(r, f(qz)) = M(|q|r, f), \qquad N(r, f(qz)) = N(|q|r, f) + O(1)$$

and

$$T(r, f(qz)) = T(|q|r, f) + O(1)$$

hold for any meromorphic function f *and any non-zero constant* q*.*

LEMMA 11. [10] *Let* $P(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$*,* $a_k \neq 0$*, be a non-constant polynomial of degree* k *and let* f *be a transcendental meromorphic function. Given* $0 < \delta < |a_k|$*, denote* $\lambda := |a_k| + \delta$ *and* $u := |a_k| - \delta$*. Then, given* $\varepsilon > 0$ *and* $a \in \mathbb{C} \cup \{\infty\}$*, we have*

$$kn(\mu r^k, a, f) \leqslant n(r, a, f \circ p) \leqslant kn(\lambda r^k, a, f),$$

$$N(\mu r^k, a, f) + O(\log r) \leqslant N(r, a, f \circ p) \leqslant N(\lambda r^k, a, f) + O(\log r),$$

$$(1-\varepsilon)T(\mu r^k, f) \leqslant T(r, f \circ p) \leqslant (1+\varepsilon)T(\lambda r^k, f),$$

for all r *large enough.*

LEMMA 12. [11] *Let* $\psi : [r_0, +\infty) \to (0, +\infty)$ *be positive and bounded in every finite interval, and suppose that* $\psi(\mu r^m) \leqslant A\psi(r) + B$ *holds for all* r *large enough, where* $\mu > 0$, $m > 1$, $A > 1$ *and* B *are real constants. Then*

$$\psi(r) = O((\log r)^{\alpha}),$$

where $\alpha = \frac{\log A}{\log m}$.

3. Proof of theorem 7

Proof. Given $0 < \varepsilon < \frac{d-2n\sigma}{d+2n\sigma}$, noting that $|q| > 1$, we have by (1.8), lemma 1 and lemma 2 that

$$\begin{aligned} d(1-\varepsilon)T(r,f) &\leqslant dT(r,f) + S(r,f) \leqslant 2\sigma \textstyle\sum_{j=1}^{n} T(r, f(q^j(z + \frac{c_j}{q^j}))) + S(r,f) \\ &\leqslant 2\sigma \textstyle\sum_{j=1}^{n} T(r + \frac{|c_j|}{|q|^j}, f(q^j z)) + S(r,f) \\ &\leqslant 2\sigma \textstyle\sum_{j=1}^{n} T(r + \frac{c}{|q|}, f(q^j z))) + S(r,f) \\ &= 2\sigma(1+\varepsilon) \textstyle\sum_{j=1}^{n} T(|q|^j r + |q|^{j-1} c, f), \end{aligned} \tag{3.1}$$

outside of a possible exceptional set of finite linear measure, where $c = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. By lemma 3 and (3.1), it follows that for any given $\alpha > 1$, there exists an $r_0 > 0$ such that

$$d(1-\varepsilon)T(r,f) \leqslant 2n\sigma(1+\varepsilon)T(\alpha|q|^n r + \alpha|q|^{n-1}c, f)$$

holds for all $r \geqslant r_0$. Hence

$$T(\alpha|q|^n r + \alpha|q|^{n-1}c, f) \geqslant \frac{d(1-\varepsilon)}{2n\sigma(1+\varepsilon)} T(r,f), r \geqslant r_0. \tag{3.2}$$

Inductively, for any $k \in \mathbb{N}$, we have by (3.1) and (3.2) that

$$T(\alpha^k |q|^{nk} r + \sum_{\mu=1}^{k} \alpha^{\mu} |q|^{\mu n-1} c, f) \geqslant (\frac{d(1-\varepsilon)}{2n\sigma(1+\varepsilon)})^k T(r,f), r \geqslant r_0. \tag{3.3}$$

For sufficiently large s, using the same method as in the proof of theorem 6(see [25]), we obtain by (3.3) that

$$T(s,f) \geqslant (\frac{d(1-\varepsilon)}{2n\sigma(1+\varepsilon)})^{\frac{\log s}{\log \alpha |q|^{(n+1)}} - \frac{n}{n+1}} T(r_0, f). \tag{3.4}$$

Letting $\varepsilon \to 0$ and $\alpha \to 1$, we have by (3.4) that

$$T(s,f) \geqslant K(\frac{d}{2n\sigma})^{\frac{\log s}{\log |q|^{n+1}}},$$

where K is a constant. Thus, we get $\mu(f) \geqslant \frac{\log d - \log 2n\sigma}{(n+1)\log |q|}$.

The proof of theorem 7 is complete. □

4. Proof of theorem 8

Proof. Suppose the second alternative of the assertion do not hold. Then by lemmas 4 and 5, we get

$$T(r,f) \leqslant \overline{N}(r,\tfrac{1}{Q}) + \overline{N}(r,f) + S(r,f) \leqslant \overline{N}(r,\tfrac{P(z,f)}{Q(z,f)}) + \overline{N}(r,f) + S(r,f). \tag{4.1}$$

By (1.9), (1.10) and (4.1), we obtain that

$$\begin{aligned} T(r,f) - \overline{N}(r,f) &\leqslant \overline{N}(r,\tfrac{P(z,f)}{Q(z,f)}) + S(r,f) = \overline{N}(r,\textstyle\sum_{j=1}^{n} a_j(z)f(q^jz+c_j)) + S(r,f) \\ &\leqslant \alpha\overline{N}(|q|^n r + c, f) + S(r,f). \end{aligned} \tag{4.2}$$

where $c = \max\{|c_1|,|c_2|,\ldots,|c_n|\}$. Assume contrary to the assertion, that is $\rho(f) = 0$. By lemma 8, lemma 10, we obtain the $T(r,f(q^jz+c_j)) = T(r,|q|^j r,f) + O(1)$. So lemma 6 implies that for any constant $C > 1$,

$$T(r,f(q^jz+c_j)) = T(|q|^j r,f) + O(1) < CT(r,f),\ j = 1,\ldots,n \tag{4.3}$$

on a set of logarithmic density 1.

We see that if a set is of finite linear measure, then the set is of logarithmic density 0. Thus, combing (4.3), we obtain that for $j = 1,\ldots,n$,

$$S(r,f(q^jz+c_j)) = o(T(r,f(z))) \tag{4.4}$$

on a set of logarithmic density 1. Now (4.2) applies to $f(q^j+c_j)(j=1,\ldots,n)$, then by (4.4) we have

$$T(r,f(q^jz+c_j)) - \overline{N}(r,f(q^jz+c_j)) \leqslant \alpha\overline{N}(|q|^n r + c, f(q^jz+c_j)) + o(T(r,f)), \tag{4.5}$$

on a set of logarithmic density 1. Applying lemma 1 on both sides of (1.9), we conclude by (1.9), lemma 9 and lemma 10 that

$$\begin{aligned} nT(r,f) &= T(r,\textstyle\sum_{j=1}^{n} a_j(z)f(q^jz+c_j)) + o(T(r,f)) \\ &= T(r,\textstyle\sum_{j=1}^{n} a_j(z)f(q^jz+c_j)) - \overline{N}(r,\textstyle\sum_{j=1}^{n} a_j(z)f(q^jz+c_j)) \\ &\quad + \overline{N}(r,\textstyle\sum_{j=1}^{n} a_j(z)f(q^jz+c_j)) + o(T(r,f)) \\ &\leqslant m(r,\textstyle\sum_{j=1}^{n} a_j(z)f(q^jz+c_j)) + N_1(r,\textstyle\sum_{j=1}^{n} a_j(z)f(q^jz+c_j)) \\ &\quad + \alpha\overline{N}(|q|^n r + c, f) + o(T(r,f)) \\ &\leqslant \textstyle\sum_{j=1}^{n}(m(r,f(q^jz+c_j)) + N_1(r,f(q^jz+c_j))) + \alpha\overline{N}(|q|^n r + c, f) + o(T(r,f)) \\ &= \textstyle\sum_{j=1}^{n}(T(r,f(q^jz+c_j) - \overline{N}(r,f(q^jz+c_j))) + \alpha\overline{N}(|q|^n r + c, f) + o(T(r,f)), \end{aligned} \tag{4.6}$$

on a set of logarithmic density 1, where $N_1(r,f) = N(r,f) - \overline{N}(r,f)$. By (4.5),(4.6) and lemma 9, we have

$$\begin{aligned} nT(r,f) &\leqslant \textstyle\sum_{j=1}^{n} \alpha\overline{N}(|q|^n r + c, f(q^jz+c_j)) + \alpha\overline{N}(|q|^n r + c, f) + o(T(r,f)) \\ &\leqslant n\alpha\overline{N}(|q|^{2n} r + c, f) + \alpha\overline{N}(|q|^n r + c, f) + o(T(r,f)) \\ &= (n+1)\alpha\overline{N}(|q|^{2n} r + c, f) + o(T(r,f)) \\ &= (n+1)\alpha\overline{N}(|q|^{2n} r, f) + o(T(r,f)), \end{aligned} \tag{4.7}$$

on a set of logarithmic density 1 and r sufficiently large. Therefore, we have obtained by (4.7) that

$$T(r,f)-\overline{N}(r,f)\leqslant \frac{n+1}{n}\alpha\overline{N}(|q|^{2n}r,f)-\overline{N}(r,f)+o(T(r,f)) \tag{4.8}$$

on a set of logarithmic density 1.

We now proceed, inductively, to prove that

$$T(r,f)-\overline{N}(r,f)\leqslant \frac{n+m}{n}\alpha\overline{N}(|q|^{2mn}r,f)-m\overline{N}(r,f)+o(T(r,f)) \tag{4.9}$$

on a set of logarithmic density 1. Having already proved the case $m=1$ in (4.8), we continue to the inductive step. To this end, observe that the above reasoning also applies to the functions $f(q^jz+c_j), j=1,2,...,n$ instead of $f(z)$. Therefore, we may apply the inductive assertion to obtain by (4.6) that

$$\begin{aligned} nT(r,f) &\leqslant \textstyle\sum_{j=1}^{n}(T(r,f(q^jz+c_j))-\overline{N}(r,f(q^jz)))+\alpha\overline{N}(|q|^nr+c,f)+o(T(r,f))\\ &\leqslant \textstyle\sum_{j=1}^{n}(\frac{n+m}{n}\alpha\overline{N}(|q|^{2mn}r,f)-m\overline{N}(r,f))+\alpha\overline{N}(|q|^nr,f)+o(T(r,f))+O(1)\\ &\leqslant (n+m+1)\alpha\overline{N}(|q|^{(2m+1)n}r,f)-mn\overline{N}(r,f)+o(T(r,f))+O(1)\\ &\leqslant (n+m+1)\alpha\overline{N}(|q|^{2(m+1)n}r,f)-mn\overline{N}(r,f)+o(T(r,f) \end{aligned}$$

on a set of logarithmic density 1. Therefore, we conclude that

$$T(r,f)-\overline{N}(r,f)\leqslant \frac{n+m+1}{n}\alpha\overline{N}(|q|^{2(m+1)n}r,f)-(m+1)\overline{N}(r,f)+o(T(r,f))$$

on a set of logarithmic density 1, completing the induction (4.9).

Thus, noting that $T(r,f)-o(T(r,f))\geqslant 0$, we immediately see by (4.9) that

$$\overline{N}(r,f)\leqslant \frac{n+m}{n(m-1)}\alpha\overline{N}(|q|^{2mn}r,f)$$

on a set of logarithmic density 1. Setting $\alpha'=\frac{n+m}{n(m-1)}\alpha$, we obtain

$$\overline{N}(r,f)\leqslant \alpha'\overline{N}(|q|^{2mn}r,f) \tag{4.10}$$

on a set of logarithmic density 1. Since $\alpha\in[0,n)$, we see that for sufficiently large m,

$$\alpha'\leqslant \frac{n+m}{n(m-1)}\alpha=(\frac{1}{m-1}+\frac{1}{n}\frac{m}{m-1})\alpha<1. \tag{4.11}$$

So by lemma 7, (4.10), (4.11) and $|q|^{2mn}>1$, we get $\rho(f)>0$, a contradiction.

The proof of theorem 8 is complete. □

5. Proof of theorem 9

Proof. Combing now the last assertion of lemma 11 with lemma 12, we obtain that

$$(1-\varepsilon)T(\mu r^k, f) \leqslant T(r, f(p(z))) = T(r, \sum_J \alpha_J(z)(\Pi_{j\in J} f(q^j z + c_j)))$$
$$\leqslant \sum_{j=1}^{n} T(r, f(q^j z + c_j)) + S(r, f).$$

Defining $C := \max\{|c_1|, ..., |c_n|\}$, we immediately infer that

$$(1-\varepsilon)T(\mu r^k, f) \leqslant \sum_{j=1}^{n} T(r, f(q^j z + C)) + S(r, f) \leqslant \sum_{j=1}^{n} T(|q|^j r, f(z + \frac{c_j}{q^j})) + S(r, f)$$
$$\leqslant \sum_{j=1}^{n} T(|q|^j r + \frac{C}{|q|}, f(z)) + S(r, f).$$

Since $T(r+C, f) \leqslant T(|q|r, f)$ holds always for r large enough for given $|q| > 1$, we may assume r to be large enough to satisfy

$$(1-\varepsilon)T(\mu r^k, f) \leqslant n(1+\varepsilon)T(|q|^{j+1} r, f)$$

outside of a possible exceptional set of finite linear measure. By a standard reasoning to remove the exceptional set, we note that whenever $\sigma > 1$,

$$(1-\varepsilon)T(\mu r^k, f) \leqslant n(1+\varepsilon)T(\sigma |q|^{j+1} r, f)$$

holds for all r large enough. Set now $t := \sigma |q|^{j+1} r$, the last inequality can be written as

$$T(\frac{\mu t^k}{\sigma^k |q|^{k(j+1)}}, f) \leqslant \frac{n(1+\varepsilon)}{1-\varepsilon} T(t, f).$$

We now make use of lemma 12 to conclude that

$$\alpha = \frac{\log(\frac{n(1+\varepsilon)}{(1-\varepsilon)})}{\log k} = \frac{\log n}{\log k} + O(1),$$

which has the required form.

The proof of theorem 9 is complete. □

REFERENCES

[1] M. J. ABLOWITZ, R. G. HALBURD, B. HERBST, *On the extension of the Painlevé property to difference equations*, Nonlinearly, **13**, (2000), 889–905.

[2] D. BARNETT, R. G. HALBURD, R. J. KORHONEN, W. MORGAN, *Nevanlinna theory for the q−difference equations*, Proc. Roy. Soc. Edinburgh Sect. **137**, (2007), 457–474.

[3] W. BERGWEILER, K. ISHIZAKI, N. YANAGIHARA, *Meromorphic solutions of some functional equations*, Methods Appl. Anal. **5**, 3 (1998), 248–259.
[4] W. BERGWEILER, J. K. LANGLEY, *Zeros of differences of meromorphic solutions*, Math. Proc. Cambridge Philos. Soc. **142**, (2007), 133–147.
[5] Y. M. CHIANG, S. J. FENG, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. **16**, (2008), 105–129.
[6] Z. X. CHEN, K. H. SHON, *On zeros and fixed points of differences of meromorphic functions*, J. Math. Anal. Appl. **344**, 1 (2008), 373–383.
[7] J. CLUNIE, *On integral and meromorphic functions*, J. London Math. Soc. **37**, (1962) 17–27.
[8] V. I. GROMAK, I. LAINE, S. SHIMOMURA, *Painlevé Differential equations in the Complex Plane*, Walter de Gruyter, Berlin, 2002.
[9] G. GUDERSEN, J. HEITTOKANGAS, I. LAINE, J. RIEPPO, D. YANG, *Meromorphic solutions of generalized Schröder equation*, Aequationes Math. **63**, (2002), 110–135.
[10] R. GOLDSTEIN, *Some results on factorisation of meromorphic functions*, J. London Math. Soc. **4**, 2 (1971), 357–364.
[11] R. GOLDSTEIN, *On meromorphic solutions of certain functional equations*, Aequationes Math. **18**, (1978), 112–157.
[12] R. G. HALBURD, R. J. KORHONEN, *Existence of finite-order meromorphic solutions as a detector of integrability in difference equation*, Phys. D **218**, (2006), 191–203.
[13] R. G. HALBURD, R. J. KORHONEN, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31**, (2006), 463–478.
[14] W. K. HAYMAN, *Meromorphic Functions*, Clarendon Press, Oxford, (1964).
[15] W. K. HAYMAN, *On the characteristic of functions meromorphic in the plane and of theire integrals*, Proc. London math. Soc. **3**, (1963), 93–128.
[16] J. HEITTOKANGAS, R. J. KORHONEN, I. LAINE, J. RIEPPO, K. TOHGE, *Complex difference equations of Malmquist type*, Comput. Methods Funct Theory **1**, (2001), 27–39.
[17] K. ISHIZAKI, *Hypertranscendency of meromorphic solutions of a linear functional equation*, Aequationes Math. **56**, (1998), 271–283.
[18] I. LAINE, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, (1993).
[19] I. LAINE, J. RIEPPO, H. SILVENNOINEN, *Remarks on complex difference equations*, Comput. Methods Funct. Theory **5**, 1 (2005), 77–88.
[20] R. NEVANLINNA, *Analytic Function*, Springer-Verlag. Berlin/ Heidelberg/ New York, (1970).
[21] A. RAMANI, B. GRAMMATICOS, T. TAMIZHMANI, K. M. TAMIZHMANI, *The road to the discrete analogue of the Painlevé property: Nevanlinna meets singularity confinement*, Comput. Math. Appl. **45**, (2003), 1001–1012.
[22] Y. TUMURA, *On the extensions of Borel's theorem and Saxer-Csillag's theorem*, Proc. Phys. Math. Soc. Japan **19**, (1937), 29–35.
[23] G. WEISSENBORN, *On the theorem of Tumura and Clunie*, Bull. London Math. Soc. **18**, (1986), 371–373.
[24] J. WANG, *Growth and poles of meromorphic solutions of some complex difference equations*, J. Math. Anal. Appl. **379**, (2011), 367–377.
[25] X. M. ZHENG, Z. X. CHEN, *Some properties of meromorphic solutions of $q-$difference equations*, J. Math. Anal. Appl. **361**, (2010), 472–480.
[26] K. LIU, T. B. CAO, *Entire solutions of Fermat type $q-$difference differential equations*, Electron. J. Differential Equations **59**, (2013), 1–10.
[27] Z. B. HUANG, R. R. ZHANG, *Properties on $q-$difference Riccati equations*, Bull. Korean. Math. Soc. **55**, (2018), 1755–1771.

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