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ASYMPTOTICALLY SELF–SIMILAR GLOBAL SOLUTIONS FOR HARDY–HÉNON PARABOLIC SYSTEMS

BYRAME BEN SLIMENE

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Abstract. In this paper we study the nonlinear parabolic system $\partial_t u = \Delta u + a|x|^{-\gamma}|v|^{p-1}v,\ \partial_t v = \Delta v + b|x|^{-\rho}|u|^{q-1}u,\ t>0,\ x\in\mathbb{R}^N\backslash\{0\},\ N\geqslant 1,\ a,b\in\mathbb{R},\ 0\leqslant\gamma<\min(N,2),\ 0<\rho<\min(N,2),\ p,q>1$. Under conditions on the parameters p,q,γ and ρ we show the existence and uniqueness of global solutions for initial values small with respect of some norms. In particular, we show the existence of self-similar solutions with initial value $\Phi=(\varphi_1,\varphi_2)$, where φ_1, φ_2 are homogeneous initial data. We also prove that some global solutions are asymptotic for large time to self-similar solutions.

1. Introduction

In this paper we consider global in time solutions of the following nonlinear parabolic system

$$(S)\begin{cases}\partial_t u = \Delta u + a|.|^{-\gamma}|v|^{p-1}v,\\ \partial_t v = \Delta v + b|.|^{-\rho}|u|^{q-1}u,\end{cases}$$

with initial data

$$u(0,x)=\varphi_1(x),\ \ v(0,x)=\varphi_2(x), \tag{1.1}$$

where $u=u(t,x)\in\mathbb{R}$, $v=v(t,x)\in\mathbb{R}$, $t>0$, $x\in\mathbb{R}^N$, $a,\ b\in\mathbb{R}$, $0\leqslant\gamma<\min(N,2)$, $0<\rho<\min(N,2)$, $p,\ q>1$.

In what follows, we denote $\|.\|_{L^r(\mathbb{R}^N)}$ by $\|.\|_r$. For $f,g:I\to\mathbb{R}$, we denote when there exists $sup_{t\in I}\left[f(t),g(t)\right]=\max\left[sup_{t\in I}f(t),sup_{t\in I}g(t)\right]$. For all $t>0$, $e^{t\Delta}$ denotes the heat semi-group, that is

$$\left(e^{t\Delta}f\right)(x)=\int_{\mathbb{R}^N}G(t,x-y)f(y)dy,$$

where

$$G(t,x)=(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}},\ t>0,\ x\in\mathbb{R}^N,$$

and $f\in L^r(\mathbb{R}^N)$, $r\in[1,\infty)$ or $f\in C_0(\mathbb{R}^N)$. For $f\in\mathscr{S}'(\mathbb{R}^N)$, $e^{t\Delta}f$ is defined by duality.

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A mild solution of the system (S)-(1.1) is a solution of the integral system

$$\begin{cases} u(t) = e^{t\Delta}\varphi_1 + a\int_0^t e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\big)d\sigma, \\ v(t) = e^{t\Delta}\varphi_2 + b\int_0^t e^{(t-\sigma)\Delta}\big(|.|^{-\rho}|u(\sigma)|^{q-1}u(\sigma)\big)d\sigma. \end{cases} \tag{1.2}$$

We investigate the existence of global solutions, including self-similar solutions for the semilinear system (1.2). Moreover, we are concerned with estimating the decaying rate in time of some global solutions and their asymptotic behavior.

Using the key estimate established by Proposition 2.1 in [1] we can adapt the method in Fujita and Kato [9, 10] and recently used in [1, 3, 4, 5, 6, 7, 8, 13, 14, 15, 16].

This method is based on a contraction mapping argument on the associated integral system (1.2). Precisely we transform the problem of existence and uniqueness of global solutions into a problem of a fixed point for a function defined in a suitable Banach space equipped with a norm chosen so that we obtain directly the global character of the solution.

In this paper we seek conditions for the following parameters p, q, γ and ρ such that we have the global existence of some class of solutions, including self-similar solutions and the nonlinear asymptotic self-similar behavior of these solutions. For this we define k, α_1, α_2, β_1 and β_2 by

$$k = \frac{(2-\gamma)q + (2-\rho)}{(2-\rho)p + (2-\gamma)}, \tag{1.3}$$

$$\alpha_1 = \frac{1}{2(pq-1)}[(2-\rho)p + (2-\gamma)], \tag{1.4}$$

$$\alpha_2 = \frac{1}{2(pq-1)}[(2-\gamma)q + (2-\rho)], \tag{1.5}$$

$$\beta_1 = \alpha_1 - \frac{N}{2r_1} = \frac{1}{2(pq-1)}[(2-\rho)p + (2-\gamma)] - \frac{N}{2r_1}, \quad r_1 > 1, \tag{1.6}$$

$$\beta_2 = \alpha_2 - \frac{N}{2r_2} = \frac{1}{2(pq-1)}[(2-\gamma)q + (2-\rho)] - \frac{N}{2r_2}, \quad r_2 > 1. \tag{1.7}$$

Note that α_1 and α_2 verify the following system

$$\begin{cases} 2-\gamma + 2\alpha_1 = 2\alpha_2 p, \\ 2-\rho + 2\alpha_2 = 2\alpha_1 q \end{cases} \tag{1.8}$$

and that

$$kp > 1, \quad q > k \text{ and } \frac{\alpha_2}{\alpha_1} = k. \tag{1.9}$$

Let us summarize the results of this paper. First of all if we suppose that the following conditions

$$2\alpha_1 < \min\left(N, \frac{p}{q}(N-\rho)\frac{(2-\gamma)q + (2-\rho)}{[2 + (2-\rho)p - \gamma pq]_+}\right), \tag{1.10}$$

and

$$2\alpha_2 < \min\left(N, \frac{q}{p}(N-\gamma)\frac{(2-\rho)p+(2-\gamma)}{[2+(2-\gamma)q-\rho pq]_+}\right), \tag{1.11}$$

are satisfied, then we prove the global existence of solutions for some initial data $\Phi = (\varphi_1, \varphi_2)$ small with respect to the norm $\mathscr{N}$ defined by

$$\mathscr{N}(\Phi) := \sup_{t>0}\left[t^{\beta_1}\|e^{t\Delta}\varphi_1\|_{r_1}, t^{\beta_2}\|e^{t\Delta}\varphi_2\|_{r_2}\right], \tag{1.12}$$

where β_1 and β_2 are given by (1.6) and (1.7), r_1 and r_2 are defined in Lemma 1 below. See Theorem 1 below. We also prove, for φ_1 homogeneous of degree $-2\alpha_1$ and φ_2 homogeneous of degree $-2\alpha_2$, where α_1 and α_2 are given by (1.4) and (1.5), that the initial data $\Phi = (\varphi_1, \varphi_2)$ gives rise to a global self-similar solution. See Theorem 2 below. Next we show as in [1], that solutions with initial data Ψ which behaves asymptotically like Φ in some appropriate sense as $|x| \to \infty$, are asymptotically self-similar in the L^∞-norm. See Theorem 3 below. The norm $\mathscr{N}$ given in (1.12) is weak enough so that initial data $\Phi = (\varphi_1, \varphi_2)$ with homogeneous components have finite norm. We prove finally stronger uniqueness results in Lebesgue spaces for initial values small with respect of some norm. See Theorem 4 below.

Yamauchi in [18] studied the parabolic system (S). In [18, Theorem 2.1, p. 339] it is shown that for some nonnegative initial values under the conditions $\gamma < \min(N,2)$, $\rho < \min(N,2)$, $pq-1>0$ and $\max(\alpha_1,\alpha_2) \geqslant \mathbb{N}/2$, that no nonnegative nontrivial solutions exist.

The case $\gamma = \rho = 0$ has been already covered in [16]. In the case where $p = q$ and $\gamma = \rho > 0$, the parabolic system (S) behaves like a parabolic equation with singularity in the nonlinearity. For more reading about Hardy-Hénon equations see [1, 11, 12, 17].

The rest of the paper is organized as follows. In Section 2, we state the main results. In Section 3, we give the proofs of the main theorems. Finally, in Section 4, we give stronger uniqueness results. Throughout this paper C will be a positive constant which may have different values at different places. We denote sometimes $u(t)$ by $u(t,.)$.

2. Main results

We now state the main results of the paper. Let $e^{t\Delta}$ be the linear heat semi-group defined by

$$(e^{t\Delta}\varphi)(x) = (G(t,.)*\varphi)(x),$$

where G is the heat kernel

$$G(t,x) = (4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}, \quad t>0, \quad x\in\mathbb{R}^N.$$

We recall the smoothing effect of the heat semi-group

$$\|e^{t\Delta}f\|_{s_2} \leqslant (4\pi t)^{-\frac{N}{2}\left(\frac{1}{s_1}-\frac{1}{s_2}\right)}\|f\|_{s_1}, \tag{2.1}$$

for $1 \leqslant s_1 \leqslant s_2 \leqslant \infty$, $t > 0$ and $f \in L^{s_1}(\mathbb{R}^N)$. We recall also the following key estimate from [1]

$$\|e^{t\Delta}(|.|^{-\gamma} f)\|_{q_2} \leqslant C(N,\gamma,q_1,q_2) t^{-\frac{N}{2}\left(\frac{1}{q_1}-\frac{1}{q_2}\right)-\frac{\gamma}{2}} \|f\|_{q_1}, \tag{2.2}$$

for $0 \leqslant \gamma < N$, q_1 and q_2 such that $0 \leqslant 1/q_2 < \gamma/N + 1/q_1 < 1$, $t > 0$ and $f \in L^{q_1}(\mathbb{R}^N)$. We note that if $q_2 = \infty$, then $e^{t\Delta}(|.|^{-\gamma} f) \in C_0(\mathbb{R}^N)$.

We begin with the following technical lemma.

LEMMA 1. (Technical lemma) *Let N be a positive integer. Let $p,\ q > 1$. Let $0 \leqslant \gamma < \min(N,2)$ and $0 < \rho < \min(N,2)$. Let k be given by (1.3). Let α_1, α_2 be defined by (1.4) and (1.5). Suppose that (1.10) and (1.11) are satisfied. Let β_1, β_2 be given by (1.6) and (1.7). Then there exist $r_1 > 1$ and $r_2 > 1$ satisfying*

$$r_1 = k r_2, \tag{2.3}$$

such that

(i) $\beta_1 > 0$, $\beta_2 > 0$ *and* $\beta_2 = k\beta_1$,

(ii) $\frac{1}{r_1} < \frac{\gamma}{N} + \frac{p}{r_2} < 1$ *and* $\frac{1}{r_2} < \frac{\rho}{N} + \frac{q}{r_1} < 1$,

(iii) $\beta_2 p < 1$ *and* $\beta_1 q < 1$,

(iv) $\frac{N}{2r_1}\left(-1 + \frac{r_1}{r_2} p\right) < \frac{2-\gamma}{2}$ *and* $\frac{N}{2r_2}\left(-1 + \frac{r_2}{r_1} q\right) < \frac{2-\rho}{2}$,

(v) $\frac{1}{r_1} < \frac{2\alpha_1}{N} < \frac{\gamma}{N} + \frac{p}{r_2}$ *and* $\frac{1}{r_2} < \frac{2\alpha_2}{N} < \frac{\rho}{N} + \frac{q}{r_1}$,

(vi) $-\frac{N}{2}\left(\frac{p}{r_2} - \frac{1}{r_1}\right) - \frac{\gamma}{2} - \beta_2 p + 1 + \beta_1 = 0$ *and* $-\frac{N}{2}\left(\frac{q}{r_1} - \frac{1}{r_2}\right) - \frac{\rho}{2} - \beta_1 q + 1 + \beta_2 = 0.$

We prove this lemma in the appendix.

THEOREM 1. (Global existence and continuous dependence) *Let N be a positive integer. Let $p,\ q > 1$. Let $0 \leqslant \gamma < \min(N,2)$ and $0 < \rho < \min(N,2)$. Let α_1, α_2 be defined by (1.4) and (1.5). Suppose that (1.10) and (1.11) are satisfied. Let β_1, β_2 be given by (1.6) and (1.7). Let r_1 and r_2 be as in Lemma 1. Let $M > 0$ be such that*

$$\nu = \max(M^{p-1}\nu_1, M^{q-1}\nu_2) < 1, \tag{2.4}$$

where ν_1 and ν_2 are two positive constants given by (3.8) and (3.9) below. Choose $R > 0$ such that

$$R + M\nu \leqslant M. \tag{2.5}$$

Let $\Phi = (\varphi_1, \varphi_2)$ be an element of $\mathscr{S}'(\mathbb{R}^N) \times \mathscr{S}'(\mathbb{R}^N)$ such that

$$\mathscr{N}(\Phi) := \sup_{t>0}\left[t^{\beta_1}\|e^{t\Delta}\varphi_1\|_{r_1}, t^{\beta_2}\|e^{t\Delta}\varphi_2\|_{r_2}\right] \leqslant R. \tag{2.6}$$

Then there exists a unique global solution $U=(u,v)\in C((0,\infty);L^{r_1}(\mathbb{R}^N)\times L^{r_2}(\mathbb{R}^N))$ *of the integral system* (1.2) *such that*

$$\sup_{t>0}\left[t^{\beta_1}\|u(t)\|_{r_1},t^{\beta_2}\|v(t)\|_{r_2}\right]\leqslant M. \tag{2.7}$$

Furthermore,

(a) $\lim_{t\searrow 0}u(t)=\varphi_1$ *and* $\lim_{t\searrow 0}v(t)=\varphi_2$ *in the sense of tempered distributions,*

(b) $u(t)-e^{t\Delta}\varphi_1\in C\left([0,\infty),L^{\tau_1}\left(\mathbb{R}^N\right)\right)$ *for* τ_1 *satisfying* $\frac{2\alpha_1}{N}<\frac{1}{\tau_1}<\frac{\gamma}{N}+\frac{p}{r_2}$,

(c) $v(t)-e^{t\Delta}\varphi_2\in C\left([0,\infty),L^{\tau_2}\left(\mathbb{R}^N\right)\right)$ *for* τ_2 *satisfying* $\frac{2\alpha_2}{N}<\frac{1}{\tau_2}<\frac{\rho}{N}+\frac{q}{r_1}$,

(d) $\sup_{t>0}t^{\alpha_1-\frac{N}{2r}}\|u(t)\|_r<\infty,\quad \forall r\in[r_1,\infty]$ *and* $u\in C\left((0,\infty),L^r\left(\mathbb{R}^N\right)\cap C_0\left(\mathbb{R}^N\right)\right)$,

(e) $\sup_{t>0}t^{\alpha_2-\frac{N}{2r}}\|v(t)\|_r<\infty,\quad \forall r\in[r_2,\infty]$ *and* $v\in C\left((0,\infty),L^r\left(\mathbb{R}^N\right)\cap C_0\left(\mathbb{R}^N\right)\right)$.

In addition, if $\Phi=(\varphi_1,\varphi_2)$ *and* $\Psi=(\psi_1,\psi_2)$ *satisfy* (2.6) *and if* $U_1=(u_1,v_1)$ *and* $U_2=(u_2,v_2)$ *respectively are the solutions of the system* (1.2) *with initial values* Φ *and* Ψ*, then*

$$\sup_{t>0}\left[t^{\beta_1}\|u_1(t)-u_2(t)\|_{r_1},t^{\beta_2}\|v_1(t)-v_2(t)\|_{r_2}\right]\leqslant(1-\nu)^{-1}\mathcal{N}(\Phi-\Psi). \tag{2.8}$$

Furthermore, if the initial data Φ *and* Ψ *are such that*

$$\mathcal{N}_\delta(\Phi-\Psi)=\sup_{t>0}\left[t^{\beta_1+\delta}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1},t^{\beta_2+\delta}\|e^{t\Delta}(\varphi_2-\psi_2)\|_{r_2}\right]<\infty, \tag{2.9}$$

for some $0<\delta<\delta_0$*, where*

$$\delta_0=\min\left\{1-\beta_1 q,1-\beta_2 p\right\}. \tag{2.10}$$

Then

$$\sup_{t>0}\left[t^{\beta_1+\delta}\|u_1(t)-u_2(t)\|_{r_1},t^{\beta_2+\delta}\|v_1(t)-v_2(t)\|_{r_2}\right]\leqslant(1-\nu')^{-1}\mathcal{N}_\delta(\Phi-\Psi), \tag{2.11}$$

where the positive constant M *is chosen small enough so that* $0<\nu'<1$*, where* ν' *is given by the relations* (3.16)-(3.18) *below.*

Finally, if we suppose also that $\Phi=(\varphi_1,\varphi_2)\in L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$ *such that*

$$\mathcal{N}'(\Phi):=\max\left[\|\varphi_1\|_{\frac{N}{2\alpha_1}},\|\varphi_2\|_{\frac{N}{2\alpha_2}}\right]<R, \tag{2.12}$$

then the solution $U=(u,v)$ *of the integral system* (1.2) *satisfies also* $U\in C\left([0,\infty),L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\right)\times C\left([0,\infty),L^{N/2\alpha_2}\left(\mathbb{R}^N\right)\right)$ *and*

$$\sup_{t\geqslant 0}\left[\|u(t)\|_{\frac{N}{2\alpha_1}},\|v(t)\|_{\frac{N}{2\alpha_2}}\right]\leqslant M. \tag{2.13}$$

Where M *and* R *are sufficiently small.*

Now we give the following result which proves the existence of self-similar solutions.

THEOREM 2. (Self-similar solutions) *Let N be a positive integer. Let $p,\ q>1$. Let $0 \leqslant \gamma < \min(N,2)$ and $0<\rho<\min(N,2)$. Let $\alpha_1,\ \alpha_2$ be defined by* (1.4) *and* (1.5)*. Suppose that* (1.10) *and* (1.11) *are satisfied. Let $\varphi_1(x)=\omega_1(x)|x|^{-2\alpha_1}$, $\varphi_2(x)=\omega_2(x)|x|^{-2\alpha_2}$, where $\omega_1,\omega_2\in L^\infty(\mathbb{R}^N)$ are homogeneous of degree 0 and $\|\omega_1\|_\infty,\|\omega_2\|_\infty$ are sufficiently small. Denote $\Phi=(\varphi_1,\varphi_2)$, then there exists a global self-similar solution $U_S=(u_S,v_S)$ of* (1.2) *with initial data Φ. Moreover $U_S(t)\to\Phi$ in $\mathscr{S}'\left(\mathbb{R}^N\right)$, as $t\to 0$.*

We turn now to the asymptotic behavior.

THEOREM 3. (Asymptotic behavior) *Let N be a positive integer. Let $p,\ q>1$. Let $0\leqslant\gamma<\min(N,2)$ and $0<\rho<\min(N,2)$. Let $\alpha_1,\ \alpha_2$ be defined by* (1.4) *and* (1.5)*. Suppose that* (1.10) *and* (1.11) *are satisfied. Let $\beta_1,\ \beta_2$ be given by* (1.6) *and* (1.7)*. Let r_1 and r_2 be as in Lemma* 1*. Define $\beta_1(q)$ and $\beta_2(q)$ by*

$$\beta_1(q)=\alpha_1-\frac{N}{2q},\quad \beta_2(q)=\alpha_2-\frac{N}{2q},\quad q>1. \tag{2.14}$$

Let Φ be given by

$$\Phi(x)=\big(\varphi_1(x),\varphi_2(x)\big):=\big(\omega_1(x)|x|^{-2\alpha_1},\omega_2(x)|x|^{-2\alpha_2}\big)$$

with $\omega_1,\ \omega_2$ homogeneous of degree 0, $\omega_1,\omega_2\in L^\infty\left(\mathbb{R}^N\right)$ and $\|\omega_1\|_\infty,\ \|\omega_1\|_\infty$ are sufficiently small. Let

$$U_{\mathscr{S}}(t,x)=\left(t^{-\alpha_1}u_{\mathscr{S}}\left(1,\frac{x}{\sqrt{t}}\right),t^{-\alpha_2}v_{\mathscr{S}}\left(1,\frac{x}{\sqrt{t}}\right)\right)$$

be the self-similar solution of (1.2) *given by Theorem* 2*.*

Let $\Psi=(\psi_1,\psi_2)\in C_0\left(\mathbb{R}^N\right)\times C_0\left(\mathbb{R}^N\right)$ be such that

$$|\psi_1(x)|\leqslant\frac{c}{(1+|x|^2)^{\alpha_1}},\ \forall x\in\mathbb{R}^N,\ \psi_1(x)=\omega_1(x)|x|^{-2\alpha_1},\ |x|\geqslant A,$$

$$|\psi_2(x)|\leqslant\frac{c}{(1+|x|^2)^{\alpha_2}},\ \forall x\in\mathbb{R}^N,\ \psi_2(x)=\omega_2(x)|x|^{-2\alpha_2},\ |x|\geqslant A,$$

for some constant $A>0$, where c is a small positive constant. (We take $\|\omega_1\|_\infty$, $\|\omega_2\|_\infty$ and c sufficiently small so that (2.6) *is satisfied by Φ and Ψ).*

Let $U=(u,v)$ be the global solution of (1.2) *with initial data Ψ constructed by Theorem* 1*. Then there exists $\delta>0$ sufficiently small such that*

$$\|u(t)-u_{\mathscr{S}}(t)\|_{q_1}\leqslant C_\delta t^{-\beta_1(q_1)-\delta},\ \forall\, t>0, \tag{2.15}$$

$$\|v(t)-v_{\mathscr{S}}(t)\|_{q_2}\leqslant C_\delta t^{-\beta_2(q_2)-\delta},\ \forall\, t>0, \tag{2.16}$$

for all $q_1 \in [r_1, \infty]$, $q_2 \in [r_2, \infty]$. *Also, we have*

$$\|t^{\alpha_1} u(t, .\sqrt{t}) - u_{\mathscr{S}}(1, .)\|_{q_1} \leqslant C_\delta t^{-\delta},\ \forall\, t > 0, \tag{2.17}$$

$$\|t^{\alpha_2} v(t, .\sqrt{t}) - v_{\mathscr{S}}(1, .)\|_{q_2} \leqslant C_\delta t^{-\delta},\ \forall\, t > 0, \tag{2.18}$$

for all $q_1 \in [r_1, \infty]$, $q_2 \in [r_2, \infty]$.

To close this section we give the conditions on p, q, γ, ρ which guarantee that the relations (1.10) and (1.11) are satisfied.

PROPOSITION 1. *Let* N *be a positive integer. Let the real numbers* $p,\ q > 1$. *Suppose that*

$$\max[p, q] + 1 < \frac{N}{2}(pq - 1).$$

Then there exist $\gamma_0, \rho_0 > 0$ *such that for all* $0 \leqslant \gamma < \gamma_0$, $0 < \rho < \rho_0$, (1.10) *and* (1.11) *are satisfied.*

PROPOSITION 2. *Let* N *be a positive integer. Fix* $0 < \gamma < \min(2, N)$ *and* $0 < \rho < \min(2, N)$. *Let* $p, q > 1$ *such that*

$$p \geqslant \max\left(\frac{2-\gamma}{N} + \frac{2-\rho}{N} + 1, \frac{2-\gamma}{\rho} + \frac{2}{\rho}\right),$$

and

$$q \geqslant \max\left(\frac{2-\rho}{N} + \frac{2-\gamma}{N} + 1, \frac{2-\rho}{\gamma} + \frac{2}{\gamma}\right).$$

Then (1.10) *and* (1.11) *are satisfied.*

The proof of those two propositions is given in the next section.

3. Proof of main results

We look for global solutions of the system (1.2) via a fixed point argument. Let us denote $U = (u, v)$, $\Phi = (\varphi_1, \varphi_2)$ and

$$\mathscr{F}_\Phi(U) = (F_\Phi(U), G_\Phi(U)), \tag{3.1}$$

where

$$F_\Phi(U)(t) = e^{t\Delta}\varphi_1 + a\int_0^t e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\big)d\sigma, \tag{3.2}$$

$$G_\Phi(U)(t) = e^{t\Delta}\varphi_2 + b\int_0^t e^{(t-\sigma)\Delta}\big(|.|^{-\rho}|u(\sigma)|^{q-1}u(\sigma)\big)d\sigma, \tag{3.3}$$

with φ_1 and φ_2 being two tempered distributions, $a, b \in \mathbb{R}$, $0 \leqslant \gamma < \min(N, 2)$, $0 < \rho < \min(N, 2)$, $p, q > 1$.

Proof of Theorem 1. Let X be the set of continuous functions

$$\begin{array}{rcl} U:(0,\infty) & \to & L^{r_1}\left(\mathbb{R}^N\right)\times L^{r_2}\left(\mathbb{R}^N\right),\\ t & \mapsto & (u(t),v(t)) \end{array}$$

such that

$$\|U\|_X := \sup_{t>0}\left[t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|v(t)\|_{r_2}\right] < \infty,$$

where r_1, r_2 are two positive real numbers satisfying conditions in Lemma 1 and β_1, β_2 are respectively given by (1.6) and (1.7). Let $M > 0$ and define the closed ball in the Banach space X by

$$X_M = \left\{U \in X, \|U\|_X \leqslant M\right\}.$$

X_M, endowed with the metric $d(U_1,U_2) = \|U_1 - U_2\|_X$, is a complete metric space. Consider the mapping $\mathscr{F}_\Phi$ defined by (3.1), where $\Phi = (\varphi_1,\varphi_2) \in \mathscr{S}'(\mathbb{R}^n)\times\mathscr{S}'(\mathbb{R}^n)$ satisfies (2.6). We will show that $\mathscr{F}_\Phi = (F_\Phi, G_\Phi)$ is a strict contraction mapping on X_M. Let $\Phi = (\varphi_1,\varphi_2)$ and $\Psi = (\psi_1,\psi_2)$ belong to $\mathscr{S}'(\mathbb{R}^n)\times\mathscr{S}'(\mathbb{R}^n)$ satisfying (2.6). Let $U_1 = (u_1,v_1)$ and $U_2 = (u_2,v_2)$ be two elements of X_M. Then we have

$$\begin{aligned} & t^{\beta_1}\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_1}\\ \leqslant & t^{\beta_1}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1} + |a|t^{\beta_1}\int_0^t \|e^{(t-\sigma)\Delta}|.|^{-\gamma}[|v_1(\sigma)|^{p-1}v_1(\sigma) - |v_2(\sigma)|^{p-1}v_2(\sigma)]\|_{r_1}\,d\sigma. \end{aligned}$$

It follows, by the key estimate (2.2) with $(q_1,q_2) = (\frac{r_2}{p}, r_1)$ that

$$\begin{aligned} & t^{\beta_1}\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_1}\\ \leqslant & t^{\beta_1}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1}\\ & + |a|t^{\beta_1}\int_0^t C(t-\sigma)^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{r_1})-\frac{\gamma}{2}}\left\||v_1(\sigma)|^{p-1}v_1(\sigma) - |v_2(\sigma)|^{p-1}v_2(\sigma)\right\|_{\frac{r_2}{p}}\,d\sigma. \end{aligned} \tag{3.4}$$

Using the fact that, for $r > p > 1$,

$$\||f|^{p-1}f - |g|^{p-1}g\|_{r/p} \leqslant p\left(\|f\|_r^{p-1} + \|g\|_r^{p-1}\right)\|f-g\|_r,$$

we obtain by (3.4) and the fact that U_1 and U_2 belong to X_M, that

$$\begin{aligned} & t^{\beta_1}\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_1}\\ \leqslant & t^{\beta_1}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1} + 2p|a|Ct^{\beta_1}\times\left[\int_0^t (t-\sigma)^{-\frac{N}{2r_1}\left(-1+\frac{r_1}{r_2}p\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}M^{p-1}\,d\sigma\right]\|U_1-U_2\|_X. \end{aligned}$$

It follows that

$$\begin{aligned} & t^{\beta_1}\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_1}\\ \leqslant & t^{\beta_1}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1} + 2|a|CpM^{p-1}t^{\beta_1}\times\left[\int_0^t (t-\sigma)^{-\frac{N}{2r_1}\left(-1+\frac{r_1}{r_2}p\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}\,d\sigma\right]\|U_1-U_2\|_X \end{aligned}$$

$$\leqslant t^{\beta_1}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1}+2|a|CpM^{p-1}t^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{r_1}\right)-\frac{\gamma}{2}-\beta_2 p+1+\beta_1}$$
$$\times\left[\int_0^1(1-\sigma)^{-\frac{N}{2r_1}\left(-1+\frac{r_1}{r_2}p\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma\right]\|U_1-U_2\|_X. \tag{3.5}$$

Similarly using estimate (2.2) with $(q_1,q_2)=(r_1/q,r_2)$, we obtain an analogous estimate of $t^{\beta_2}\|G_\Phi(U_1)(t)-G_\Psi(U_2)(t)\|_{r_2}$. Thus

$$t^{\beta_2}\|G_\Phi(U_1)(t)-G_\Psi(U_2)(t)\|_{r_2}$$
$$\leqslant t^{\beta_2}\|e^{t\Delta}(\varphi_2-\psi_2)\|_{r_2}+2|b|CqM^{q-1}t^{\beta_2}\times\left[\int_0^t(t-\sigma)^{-\frac{N}{2r_2}\left(-1+\frac{r_2}{r_1}q\right)-\frac{\rho}{2}}\sigma^{-\beta_1 q}d\sigma\right]\|U_1-U_2\|_X$$
$$\leqslant t^{\beta_2}\|e^{t\Delta}(\varphi_2-\psi_2)\|_{r_2}+2|b|CqM^{q-1}t^{-\frac{N}{2}\left(\frac{q}{r_1}-\frac{1}{r_2}\right)-\frac{\rho}{2}-\beta_1 q+1+\beta_2}$$
$$\times\left[\int_0^1(1-\sigma)^{-\frac{N}{2r_2}\left(-1+\frac{r_2}{r_1}q\right)-\frac{\rho}{2}}\sigma^{-\beta_1 q}d\sigma\right]\|U_1-U_2\|_X. \tag{3.6}$$

Now, due to part (vi) of Lemma 1, inequalities (3.5) and (3.6) we obtain

$$\|\mathscr{F}_\Phi(U_1)-\mathscr{F}_\Psi(U_2)\|_X\leqslant\mathscr{N}(\Phi-\Psi)+\nu\|U_1-U_2\|_X, \tag{3.7}$$

where

$$\nu=\max(M^{p-1}\nu_1,M^{q-1}\nu_2),$$

with

$$\nu_1=2|a|Cp\int_0^1(1-\sigma)^{-\frac{N}{2r_1}\left(-1+\frac{r_1}{r_2}p\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma, \tag{3.8}$$
$$\nu_2=2|b|Cq\int_0^1(1-\sigma)^{-\frac{N}{2r_2}\left(-1+\frac{r_2}{r_1}q\right)-\frac{\rho}{2}}\sigma^{-\beta_1 q}. \tag{3.9}$$

Finally, from parts (iii)-(iv) of Lemma 1, we see that both quantities ν_1 and ν_2 are finite. Setting $\Psi=0$ and $U_2=0$, the inequality (3.7) becomes

$$\|\mathscr{F}_\Phi(U_1)\|_X\leqslant\mathscr{N}(\Phi)+\nu\|U_1\|_X. \tag{3.10}$$

If we choose M and R such that (2.5) and (2.6) are satisfied, then by (3.10) $\mathscr{F}_\Phi$ maps X_M into itself. Letting $\Phi=\Psi$, we observe that (3.7) becomes

$$\|\mathscr{F}_\Phi(U_1)-\mathscr{F}_\Phi(U_2)\|_X\leqslant\nu\|U_1-U_2\|_X.$$

Hence inequality (2.4) gives that $\mathscr{F}_\Phi$ is a strict contraction mapping from X_M into itself. So $\mathscr{F}_\Phi$ has a unique fixed point $U=(u,v)$ in X_M which is solution of (1.2). This achieves the proof of the existence of a unique global solution of (1.2) in X_M.

We now prove the statements (a)-(c). Let τ_1 be a positive real number satisfying

$$\frac{2\alpha_1}{N}<\frac{1}{\tau_1}<\frac{\gamma}{N}+\frac{p}{r_2}, \tag{3.11}$$

then by (2.2) with $(q_1,q_2)=(r_2/p,\tau_1)$, we have

$$\begin{aligned}\|u(t)-e^{t\Delta}\varphi_1\|_{\tau_1} &\leqslant |a|\int_0^t \|e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\big)\|_{\tau_1}d\sigma \\ &\leqslant |a|\int_0^t C(t-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{\tau_1}\right)-\frac{\gamma}{2}}\|v(\sigma)\|_{r_2}^p d\sigma \\ &\leqslant |a|CM^p t^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{\tau_1}\right)-\beta_2 p+1-\frac{\gamma}{2}}\int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{\tau_1}\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma.\end{aligned}$$

Therefore

$$\|u(t)-e^{t\Delta}\varphi_1\|_{\tau_1}\leqslant C_1 t^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{\tau_1}\right)-\beta_2 p+\frac{2-\gamma}{2}}, \tag{3.12}$$

where

$$C_1=|a|CM^p\int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{\tau_1}\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma$$

is a positive constant. Owing to (3.11) and part (iii) of Lemma 1, the constant C_1 is finite. Similarly using (2.2) with $(q_1,q_2)=(r_1/q,\tau_2)$, we obtain for τ_2 satisfying

$$\frac{2\alpha_2}{N}<\frac{1}{\tau_2}<\frac{\rho}{N}+\frac{q}{r_1}, \tag{3.13}$$

the following inequality

$$\|v(t)-e^{t\Delta}\varphi_2\|_{\tau_2}\leqslant C_2 t^{-\frac{N}{2}\left(\frac{q}{r_1}-\frac{1}{\tau_2}\right)-\beta_1 q+\frac{2-\rho}{2}}, \tag{3.14}$$

where C_2 is a positive constant given by

$$C_2=|b|CM^q\int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{q}{r_1}-\frac{1}{\tau_2}\right)-\frac{\rho}{2}}\sigma^{-\beta_1 q}d\sigma,$$

which is finite by (3.13) and part (iii) of Lemma 1. Owing to the conditions (3.11) and (3.13), the right hand sides of (3.12) and (3.14) converges to zero as $t\searrow 0$. This proves statements (a)-(c) of Theorem 1.

Finally, the continuous dependence relation (2.8) of Theorem 1 follows by considering $\mathscr{F}_\Phi(U_1)=U_1$ and $F_\Psi(U_2)=U_2$ in the inequality (3.7).

Now, if in addition Φ and Ψ satisfy (2.9), then following the same steps as above but with the norm

$$\|U=(u,v)\|_{X,\delta}=\sup_{t>0}\left[t^{\beta_1+\delta}\|u(t)\|_{r_1},t^{\beta_2+\delta}\|v(t)\|_{r_2}\right],$$

we obtain by the key estimate (2.2) with $(q_1,q_2)=(r_2/p,r_1)$, the fact that U_1 and U_2 belong to X_M and the estimate $\|v_1(\sigma)-v_2(\sigma)\|_{r_2}\leqslant\sigma^{-\beta_2-\delta}\|U_1-U_2\|_{X,\delta}$

$$\begin{aligned}&t^{\beta_1+\delta}\|F_\Phi(U_1)(t)-F_\Psi(U_2)(t)\|_{r_1}\\ &\leqslant t^{\beta_1+\delta}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1}\\ &\quad+|a|t^{\beta_1+\delta}\times\int_0^t C(t-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{r_1}\right)-\frac{\gamma}{2}}\left\||v_1(\sigma)|^{p-1}v_1(\sigma)-|v_2(\sigma)|^{p-1}v_2(\sigma)\right\|_{\frac{r_2}{p}}d\sigma\end{aligned}$$

$$\begin{aligned}&\leqslant t^{\beta_1+\delta}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1}+2|a|CpM^{p-1}t^{\beta_1+\delta}\\&\qquad\times\left[\int_0^t(t-\sigma)^{-\frac{N}{2r_1}\left(-1+\frac{r_1}{r_2}p\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p-\delta}d\sigma\right]\|U_1-U_2\|_{X,\delta}\\&\leqslant t^{\beta_1+\delta}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1}+2|a|CpM^{p-1}t^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{r_1}\right)-\frac{\gamma}{2}-\beta_2 p+1+\beta_1}\\&\qquad\times\left[\int_0^1(1-\sigma)^{-\frac{N}{2r_1}\left(-1+\frac{r_1}{r_2}p\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p-\delta}d\sigma\right]\|U_1-U_2\|_{X,\delta}.\end{aligned}$$

We obtain also

$$\begin{aligned}&t^{\beta_2+\delta}\|G_\Phi(U_1)(t)-G_\Psi(U_2)(t)\|_{r_2}\\&\leqslant t^{\beta_2+\delta}\|e^{t\Delta}(\varphi_2-\psi_2)\|_{r_2}+2|b|CqM^{q-1}t^{-\frac{N}{2}\left(\frac{q}{r_1}-\frac{1}{r_2}\right)-\frac{\rho}{2}-\beta_1 q+1+\beta_2}\\&\qquad\times\left[\int_0^1(1-\sigma)^{-\frac{N}{2r_2}\left(-1+\frac{r_2}{r_1}q\right)-\frac{\rho}{2}}\sigma^{-\beta_1 q-\delta}d\sigma\right]\|U_1-U_2\|_{X,\delta}.\end{aligned}$$

Then

$$\|\mathscr{F}_\Phi(U_1)-\mathscr{F}_\Psi(U_2)\|_{X,\delta}\leqslant\mathscr{N}_\delta(\Phi-\Psi)+\nu'\|U_1-U_2\|_{X,\delta},\tag{3.15}$$

where

$$\nu'=\max(M^{p-1}\nu_1',M^{q-1}\nu_2'),\tag{3.16}$$

with

$$\nu_1'=2|a|Cp\int_0^1(1-\sigma)^{-\frac{N}{2r_1}\left(-1+\frac{r_1}{r_2}p\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p-\delta}d\sigma,\tag{3.17}$$

$$\nu_2'=2|b|Cq\int_0^1(1-\sigma)^{-\frac{N}{2r_2}\left(-1+\frac{r_2}{r_1}q\right)-\frac{\rho}{2}}\sigma^{-\beta_1 q-\delta}.\tag{3.18}$$

Since $\mathscr{F}_\Phi(U_1)=U_1$ and $\mathscr{F}_\Psi(U_2)=U_2$, then (3.15) becomes

$$\sup_{t>0}\left[t^{\beta_1+\delta}\|u_1(t)-u_2(t)\|_{r_1},t^{\beta_2+\delta}\|v_1(t)-v_2(t)\|_{r_2}\right]\leqslant(1-\nu')^{-1}\mathscr{N}_\delta(\Phi-\Psi).$$

Now, since $0<\delta<\delta_0$ with δ_0 given by (2.10), ν_1' and ν_2' are finite. Thus, (2.11) holds by choosing $\nu'<1$ (this choice is possible for M small enough), where ν' is given by (3.16)-(3.18).

We now prove statements (d)-(e) of Theorem 1 for $r=\infty$, we use some arguments of [15]. Let us consider two real numbers r and r' such that $r=kr'$ and

$$\begin{array}{ll}1<r_1<r\leqslant\infty & 1<r_2<r'\leqslant\infty,\\ 0<\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{r}\right)<\frac{2-\gamma}{2}, & 0<\frac{N}{2}\left(\frac{q}{r_1}-\frac{1}{r'}\right)<\frac{2-\rho}{2}.\end{array}\tag{3.19}$$

Remark that such a choice is possible owing to Lemma 4. Write now,

$$u(t)=e^{\frac{t}{2}\Delta}u(t/2)+a\int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\right)d\sigma.$$

Then by using the smoothing properties of the heat semigroup (2.1), the estimate (2.2) with $(q_1,q_2)=(r_2/p,r)$, (3.19) and the estimate (2.7), we obtain

$$\begin{aligned}
&t^{\alpha_1-\frac{N}{2r}}\|u(t)\|_r\\
&\leqslant C\sup_{t>0}\left[t^{\alpha_1-\frac{N}{2r_1}}\|u(t)\|_{r_1}\right]+|a|t^{\alpha_1-\frac{N}{2r}}\int_{\frac{t}{2}}^{t}\|e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\big)\|_r d\sigma\\
&\leqslant CM+Ct^{\alpha_1-\frac{N}{2r}}\int_{\frac{t}{2}}^{t}(t-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{r}\right)-\frac{\gamma}{2}}\|v(\sigma)\|_{r_2}^{p}d\sigma\\
&\leqslant CM+CM^p t^{\alpha_1-\frac{N}{2r}}\int_{\frac{t}{2}}^{t}(t-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{r}\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma\\
&\leqslant CM+CM^p\int_{\frac{1}{2}}^{1}(1-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{1}{r}\right)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma,
\end{aligned}$$

which leads to

$$\sup_{t>0}\left[t^{\alpha_1-\frac{N}{2r}}\|u(t)\|_r\right]\leqslant C(M)<\infty.$$

Analogously, we obtain the following estimate on the second component v:

$$\sup_{t>0}\left[t^{\alpha_2-\frac{N}{2r'}}\|v(t)\|_{r'}\right]\leqslant C(M)<\infty.$$

We iterate this procedure, for the next step we replace in (3.19) r_1 by r, r_2 by r' and we consider two real numbers s_2 and s_2' such that $s_2=ks_2'$ and

$$\begin{aligned}
&1<r<s_2\leqslant\infty && 1<r'<s_2'\leqslant\infty,\\
&0<\tfrac{N}{2}\left(\tfrac{p}{r'}-\tfrac{1}{s_2}\right)<\tfrac{2-\gamma}{2}, && 0<\tfrac{N}{2}\left(\tfrac{q}{r}-\tfrac{1}{s_2'}\right)<\tfrac{2-\rho}{2}.
\end{aligned}$$

We obtain

$$\sup_{t>0}\left[t^{\alpha_1-\frac{N}{2s_2}}\|u(t)\|_{s_2},t^{\alpha_2-\frac{N}{2s_2'}}\|v(t)\|_{s_2'}\right]\leqslant C(M)<\infty.$$

We therefore construct two sequences $(s_i)_i$ and $(s_i')_i$ with $s_0=r_1$, $s_0'=r_2$, $s_1=r$, $s_1'=r'$ and such that $s_i=ks_i'$, $\forall i=0,1,2,\ldots$ and

$$\begin{aligned}
&1<s_i<s_{i+1}\leqslant\infty, && 1<s_i'<s_{i+1}'\leqslant\infty,\\
&0<\tfrac{N}{2}\left(\tfrac{p}{s_i'}-\tfrac{1}{s_{i+1}}\right)<\tfrac{2-\gamma}{2}, && 0<\tfrac{N}{2}\left(\tfrac{q}{s_i}-\tfrac{1}{s_{i+1}'}\right)<\tfrac{2-\rho}{2}.
\end{aligned}$$

We prove that

$$\sup_{t>0}\left[t^{\alpha_1-\frac{N}{2s_i}}\|u(t)\|_{s_i},t^{\alpha_2-\frac{N}{2s_i'}}\|v(t)\|_{s_i'}\right]\leqslant C(M)<\infty,\quad \forall i=0,1,2,\ldots.$$

Now by Lemma 4, one can choose the sequences $(s_i)_i$ and $(s_i')_i$ such that they reach ∞ for some finite i. We finally obtain

$$\sup_{t>0}\left[t^{\alpha_1}\|u(t)\|_\infty,t^{\alpha_2}\|v(t)\|_\infty\right]\leqslant C(M)<\infty,$$

with $C(M) \searrow 0$ as $M \searrow 0$.

Finally, if in addition Φ satisfies (2.12), the fact that the solution $U=(u,v)$ of the integral system (1.2) with initial value Φ belongs to $C\left([0,\infty),L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\right)\times C\left([0,\infty),L^{N/2\alpha_2}\left(\mathbb{R}^N\right)\right)$ and the proof of the affirmation (2.13) are based on a contraction mapping argument in the set

$$Y_M=\Big\{U=(u,v)\in C\left([0,\infty),L^{\frac{N}{2\alpha_1}}\left(\mathbb{R}^N\right)\right)\times C\left([0,\infty),L^{\frac{N}{2\alpha_2}}\left(\mathbb{R}^N\right)\right)\cap C\left((0,\infty),L^{r_1}\left(\mathbb{R}^N\right)\right)\times C\left((0,\infty),L^{r_2}\left(\mathbb{R}^N\right)\right);\ \max\left[\sup_{t\geqslant 0}[\|u(t)\|_{\frac{N}{2\alpha_1}},\|v(t)\|_{\frac{N}{2\alpha_2}}],\sup_{t>0}[t^{\beta_1}\|u(t)\|_{r_1},t^{\beta_2}\|v(t)\|_{r_2}]\right]\leqslant M\Big\}.$$

Endowed with the metric

$$d(U_1,U_2):=d\big((u_1,v_1),(u_2,v_2)\big)=\max\Big[\sup_{t\geqslant 0}[\|u_1(t)-u_2(t)\|_{\frac{N}{2\alpha_1}},\|v_1(t)-v_2(t)\|_{\frac{N}{2\alpha_2}}],\ \sup_{t>0}[t^{\beta_1}\|u_1(t)-u_2(t)\|_{r_1},t^{\beta_2}\|v_1(t)-v_2(t)\|_{r_2}]\Big],$$

Y_M is a nonempty complete metric space. Consider the mapping $\mathscr{F}_\Phi$ defined by (3.2)-(3.3), where $\Phi=(\varphi_1,\varphi_2)\in L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$ satisfies (2.12). We will show that $\mathscr{F}_\Phi=(F_\Phi,G_\Phi)$ is a strict contraction mapping on Y_M. Let $\Phi=(\varphi_1,\varphi_2)$ and $\Psi=(\psi_1,\psi_2)$ belong to $L^{N/2\alpha_1}(\mathbb{R}^N)\times L^{N/2\alpha_2}(\mathbb{R}^N)$ satisfying (2.12). Let $U_1=(u_1,v_1)$ and $U_2=(u_2,v_2)$ be two elements of Y_M. Then we have

$$\|F_\Phi(U_1)(t)-F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}}\leqslant\|e^{t\Delta}(\varphi_1-\psi_1)\|_{\frac{N}{2\alpha_1}}+|a|\int_0^t\left\|e^{(t-\sigma)\Delta}|.|^{-\gamma}\left[|v_1(\sigma)|^{p-1}v_1(\sigma)-|v_2(\sigma)|^{p-1}v_2(\sigma)\right]\right\|_{\frac{N}{2\alpha_1}}d\sigma.$$

It follows, by the key estimate (2.2) with $(q_1,q_2)=(r_2/p,N/2\alpha_1)$ that

$$\|F_\Phi(U_1)(t)-F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}}\leqslant\|\varphi_1-\psi_1\|_{\frac{N}{2\alpha_1}}+|a|\int_0^t C(t-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{2\alpha_1}{N}\right)-\frac{\gamma}{2}}\left\||v_1(\sigma)|^{p-1}v_1(\sigma)-|v_2(\sigma)|^{p-1}v_2(\sigma)\right\|_{\frac{r_2}{p}}d\sigma, \tag{3.20}$$

we obtain by (3.20) and the fact that U_1 and U_2 belong to Y_M, that

$$\|F_\Phi(U_1)(t)-F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}}\leqslant\|\varphi_1-\psi_1\|_{\frac{N}{2\alpha_1}}+|a|C\times\left[\int_0^t(t-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{2\alpha_1}{N}\right)-\frac{\gamma}{2}}2p\sigma^{-\beta_2 p}M^{p-1}d\sigma\right]d(U_1,U_2),$$

it follows that

$$\|F_\Phi(U_1)(t)-F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}}$$

$$\leqslant \|\varphi_1 - \psi_1\|_{\frac{N}{2\alpha_1}} + 2|a|CpM^{p-1} \times \left[\int_0^t (t-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{2\alpha_1}{N}\right)-\frac{\gamma}{2}} \sigma^{-\beta_2 p} d\sigma\right] d(U_1,U_2)$$

$$\leqslant \|\varphi_1 - \psi_1\|_{\frac{N}{2\alpha_1}} + 2|a|CpM^{p-1} t^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{2\alpha_1}{N}\right)-\frac{\gamma}{2}-\beta_2 p+1}$$

$$\times \left[\int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{2\alpha_1}{N}\right)-\frac{\gamma}{2}} \sigma^{-\beta_2 p} d\sigma\right] d(U_1,U_2).$$

Owing to (1.7), we get

$$\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}}$$

$$\leqslant \|\varphi_1 - \psi_1\|_{\frac{N}{2\alpha_1}} + 2|a|CpM^{p-1} t^{\alpha_1 - p\alpha_2 + \frac{2-\gamma}{2}} \times \left[\int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{2\alpha_1}{N}\right)-\frac{\gamma}{2}} \sigma^{-\beta_2 p} d\sigma\right] d(U_1,U_2).$$

Since α_1, α_2 satisfy (1.4) and (1.5), using the fact that $r_1 > N/2\alpha_1$ and due to part (iv) of Lemma 1, it follows that

$$\alpha_1 - p\alpha_2 + \frac{2-\gamma}{2} = 0, \quad \frac{N}{2}\left(\frac{p}{r_2} - \frac{2\alpha_1}{N}\right) + \frac{\gamma}{2} < \frac{N}{2}\left(\frac{p}{r_2} - \frac{1}{r_1}\right) + \frac{\gamma}{2} < 1.$$

Using also the fact that $\beta_2 p < 1$, we get

$$\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}} \leqslant \mathscr{N}'(\Phi - \Psi) + M^{p-1}\nu_1'' d(U_1,U_2), \tag{3.21}$$

with ν_1'' is a finite positive constant defined by

$$\nu_1'' = 2|a|Cp \times \left[\int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{p}{r_2}-\frac{2\alpha_1}{N}\right)-\frac{\gamma}{2}} \sigma^{-\beta_2 p} d\sigma\right].$$

Similarly, we get

$$\|G_\Phi(U_1)(t) - G_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_2}} \leqslant \mathscr{N}'(\Phi - \Psi) + M^{q-1}\nu_2'' d(U_1,U_2), \tag{3.22}$$

with ν_2'' is a finite positive constant defined by

$$\nu_2'' = 2|b|Cq \times \left[\int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{q}{r_1}-\frac{2\alpha_2}{N}\right)-\frac{\rho}{2}} \sigma^{-\beta_1 q} d\sigma\right].$$

Owing to (3.21) and (3.22) we get

$$\sup_{t\geqslant 0}\left[\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}}, \|G_\Phi(U_1)(t) - G_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_2}}\right]$$
$$\leqslant \mathscr{N}'(\Phi - \Psi) + \nu'' d(U_1,U_2), \tag{3.23}$$

where

$$\nu'' = \max(M^{p-1}\nu_1'', M^{q-1}\nu_2'').$$

We can conclude now from (3.7) and (3.23) and from the estimate $\mathscr{N}(\Phi-\Psi) \leqslant \mathscr{N}'(\Phi-\Psi)$, that

$$d(\mathscr{F}_{\Phi}(U_1),\mathscr{F}_{\Psi}(U_2)) \leqslant \mathscr{N}'(\Phi-\Psi)+\max(\nu,\nu'')d(U_1,U_2). \tag{3.24}$$

It is clear that if $U \in Y_M$, then $\mathscr{F}_{\Phi}(U) \in C\left([0,\infty),L^{\frac{N}{2\alpha_1}}\left(\mathbb{R}^N\right)\right) \times C\left([0,\infty),L^{\frac{N}{2\alpha_2}}\left(\mathbb{R}^N\right)\right)$ $\cap C\left((0,\infty),L^{r_1}\left(\mathbb{R}^N\right)\right) \times C\left((0,\infty),L^{r_2}\left(\mathbb{R}^N\right)\right)$. Hence, by choosing M and R such that

$$R+M\max(\nu,\nu'') \leqslant M, \tag{3.25}$$

it follows that $\mathscr{F}_{\Phi}$ is a strict contraction from Y_M into itself. So $\mathscr{F}_{\varphi}$ has a unique fixed point in Y_M which is solution of (1.2).

Remark finally when the initial data Φ belongs to $L^{N/2\alpha_1}\left(\mathbb{R}^N\right) \times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$ with respect to the norm $\mathscr{N}'$, that the condition (2.6) is satisfied, since $\mathscr{N}(\Phi) \leqslant \mathscr{N}'(\Phi)$. We note also that by the previous calculations, precisely (3.24), we have the following continuous dependence property: Let $\Phi=(\varphi_1,\varphi_2)$, $\Psi=(\psi_1,\psi_2) \in L^{N/2\alpha_1}\left(\mathbb{R}^N\right) \times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$ and let $U_\Phi=(u_\Phi,v_\Phi)$ and $U_\Psi=(u_\Psi,v_\Psi)$ be the solutions of (1.2) with initial values Φ and respectively Ψ, with $\sup_{t\geqslant 0}\left[\|u_\Phi(t)\|_{N/2\alpha_1},\|v_\Phi(t)\|_{N/2\alpha_2}\right]$ $\leqslant M$ and $\sup_{t\geqslant 0}\left[\|u_\Psi(t)\|_{N/2\alpha_1},\|v_\Psi(t)\|_{N/2\alpha_2}\right] \leqslant M$. Then

$$\begin{aligned}&\sup_{t\geqslant 0}\left[\|u_\Phi(t)-u_\Psi(t)\|_{\frac{N}{2\alpha_1}},\|v_\Phi(t)-v_\Psi(t)\|_{\frac{N}{2\alpha_2}}\right]\\ &\leqslant(1-K)^{-1}\times\max\left[\|\varphi_1-\psi_1\|_{\frac{N}{2\alpha_1}},\|\varphi_2-\psi_2\|_{\frac{N}{2\alpha_2}}\right],\end{aligned} \tag{3.26}$$

for some positive constant $K=\max(\nu,\nu'')$. This finishes the proof of Theorem 1. □

Let us define the scaling operator d_λ by

$$[d_\lambda \varphi](x)=\varphi(\lambda x).$$

It follows that

$$e^{t\Delta}d_\lambda = d_\lambda e^{\lambda^2 t\Delta}, \forall\lambda>0.$$

Proof of Theorem 2. We now construct self-similar solution with initial data Φ. We adapt the method used in [1]. Let us define Φ_λ, for $\lambda>0$, by

$$\Phi_\lambda(x):=\left(\lambda^{2\alpha_1}\varphi_1(\lambda x),\lambda^{2\alpha_2}\varphi_2(\lambda x)\right).$$

It is clear that Φ_λ satisfies

$$\Phi_\lambda(x)=\Phi(x), \forall\lambda>0.$$

Let U be the solution of the integral system (1.2) with initial data Φ constructed by Theorem 1 (remark that $\mathscr{N}(\Phi)<\infty$, since r_1 satisfies parts (i)-(ii) of Lemma 4 below and by homogeneity, also $\mathscr{N}(\Phi)$ is sufficiently small since $\|\omega_1\|_\infty$ and $\|\omega_2\|_\infty$

are sufficiently small). That is U belong to X_M. We want to prove that $U_\lambda = U$, $\forall\lambda > 0$, where $U_\lambda(t,x) := \big(u_\lambda(t,x), v_\lambda(t,x)\big)$, $\forall\lambda > 0$, with

$$u_\lambda(t,x) = \lambda^{2\alpha_1}u(\lambda^2 t, \lambda x),$$

and

$$v_\lambda(t,x) = \lambda^{2\alpha_2}v(\lambda^2 t, \lambda x).$$

To do this it suffice to prove that U_λ is also a solution of (1.2) with the same initial data $\Phi_\lambda = \Phi$ and that U_λ belong to X_M. On one hand due the homogeneity properties of the system (1.2), if $U = (u,v)$ solves this system, then the scaled function solves it also. In fact

$$\begin{aligned}
d_\lambda u(\lambda^2 t) &= d_\lambda e^{\lambda^2 t\Delta}\varphi_1 + a\int_0^{\lambda^2 t} d_\lambda e^{(\lambda^2 t-\sigma)\Delta}\big(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\big)d\sigma \\
&= e^{t\Delta}d_\lambda\varphi_1 + a\int_0^{\lambda^2 t} e^{\left(t-\frac{\sigma}{\lambda^2}\right)\Delta}\big(d_\lambda(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma))\big)d\sigma \\
&= e^{t\Delta}d_\lambda\varphi_1 + a\int_0^{\lambda^2 t} \lambda^{-\gamma}e^{\left(t-\frac{\sigma}{\lambda^2}\right)\Delta}\big(|.|^{-\gamma}|d_\lambda v(\sigma)|^{p-1}d_\lambda v(\sigma)\big)d\sigma \\
&= e^{t\Delta}d_\lambda\varphi_1 + a\int_0^{t} \lambda^{2-\gamma}e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|d_\lambda v\left(\lambda^2\sigma\right)|^{p-1}d_\lambda v\left(\lambda^2\sigma\right)\right)d\sigma.
\end{aligned}$$

Hence by (1.8), we get

$$\begin{aligned}
&\lambda^{2\alpha_1}d_\lambda u\left(\lambda^2 t\right) \\
&= e^{t\Delta}d_\lambda\left(\lambda^{2\alpha_1}\varphi_1\right) + a\int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}\lambda^{2-\gamma+2\alpha_1}|d_\lambda v(\lambda^2\sigma)|^{p-1}d_\lambda v\left(\lambda^2\sigma\right)\right)d\sigma \\
&= e^{t\Delta}d_\lambda\left(\lambda^{2\alpha_1}\varphi_1\right) + a\int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|\lambda^{2\alpha_2}d_\lambda v\left(\lambda^2\sigma\right)|^{p-1}\lambda^{2\alpha_2}d_\lambda v\left(\lambda^2\sigma\right)\right)d\sigma,
\end{aligned}$$

we conclude finally that

$$u_\lambda(t) = e^{t\Delta}\varphi_1 + a\int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|v_\lambda(\sigma)|^{p-1}v_\lambda(\sigma)\right)d\sigma. \tag{3.27}$$

Similarly we obtain

$$v_\lambda(t) = e^{t\Delta}\varphi_2 + b\int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\rho}|u_\lambda(\sigma)|^{p-1}u_\lambda(\sigma)\right)d\sigma. \tag{3.28}$$

The affirmation follows from (3.27)-(3.28). On the other hand we have

$$\|u_\lambda(t)\|_{r_1} = \lambda^{2\alpha_1}\left\|d_\lambda u(\lambda^2 t)\right\|_{r_1} = \lambda^{2\alpha_1}\lambda^{-\frac{N}{r_1}}\left\|u\left(\lambda^2 t\right)\right\|_{r_1} = \left(\lambda^2\right)^{\beta_1}\left\|u\left(\lambda^2 t\right)\right\|_{r_1}.$$

Hence

$$\sup_{t>0} t^{\beta_1}\|u_\lambda(t)\|_{r_1} = \sup_{\lambda^2 t>0}\left(\lambda^2 t\right)^{\beta_1}\left\|u\left(\lambda^2 t\right)\right\|_{r_1} = \sup_{t>0} t^{\beta_1}\|u(t)\|_{r_1},$$

similarly $\sup_{t>0} t^{\beta_2}\|v_\lambda(t)\|_{r_2} = \sup_{t>0} t^{\beta_2}\|v(t)\|_{r_2}$. It follows so that $\|U_\lambda\|_X = \|U\|_X$. Then by uniqueness in X_M, we have $U_\lambda = U$ and thus U is self-similar. Let us denote it by U_S. The fact that $U_S(t) \to \Phi$ in $\mathscr{S}'(\mathbb{R}^N)$, as $t \to 0$, follows by statement (c) in Theorem 1. $\square$

Proof of Theorem 3. The proof is similar to the one of Theorem 5.1 in [1], we simply indicate that

(i) $\sup_{t>0} t^{\beta_1+\delta}\left\|e^{t\Delta}(\varphi_1-\psi_1)\right\|_{r_1} < \infty$, for $0<\delta<\frac{N}{2}-\alpha_1$.

(ii) $\sup_{t>0} t^{\beta_2+\delta}\left\|e^{t\Delta}(\varphi_2-\psi_2)\right\|_{r_2} < \infty$, for $0<\delta<\frac{N}{2}-\alpha_2$.

By the formula (2.11), we have that

$$\sup_{t>0}\left[t^{\beta_1+\delta}\|u(t)-u_{\mathscr{S}}(t)\|_{r_1}, t^{\beta_2+\delta}\|v(t)-v_{\mathscr{S}}(t)\|_{r_2}\right] \leqslant C\mathscr{N}_\delta(\Phi-\Psi).$$

That is

$$\sup_{t>0}\left[t^{\beta_1+\delta}\|u(t)-u_{\mathscr{S}}(t)\|_{r_1}, t^{\beta_2+\delta}\|v(t)-v_{\mathscr{S}}(t)\|_{r_2}\right] \leqslant \mathscr{C},$$

for $\delta>0$ sufficiently small and $\mathscr{C}$ a finite positive constant. This gives (2.15)-(2.16) directly for $q_1=r_1$ and $q_2=r_2$.

We now turn to prove the asymptotic result in the L^∞-norm. Write

$$\begin{aligned}
&u(t)-u_{\mathscr{S}}(t)\\
=&e^{\frac{t}{2}\Delta}(u(t/2)-u_{\mathscr{S}}(t/2))+a\int_{\frac{t}{2}}^{t} e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\left(|v(\sigma)|^{p-1}v(\sigma)-|v_{\mathscr{S}}(\sigma)|^{p-1}v_{\mathscr{S}}(\sigma)\right)\right]d\sigma,\\
&v(t)-v_{\mathscr{S}}(t)\\
=&e^{\frac{t}{2}\Delta}(v(t/2)-v_{\mathscr{S}}(t/2))+b\int_{\frac{t}{2}}^{t} e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\left(|u(\sigma)|^{q-1}u(\sigma)-|u_{\mathscr{S}}(\sigma)|^{q-1}u_{\mathscr{S}}(\sigma)\right)\right]d\sigma.
\end{aligned}$$

Let $T>0$ be an arbitrary real number. By using the smoothing properties of the heat semi-group with $(s_1,s_2)=(r_1,\infty)$ and the estimate (2.2) with $(q_1,q_2)=(\infty,\infty)$, it follows that

$$\begin{aligned}
t^{\alpha_1+\delta}\|u(t)-u_{\mathscr{S}}(t)\|_\infty \leqslant\ & t^{\alpha_1+\delta}\left\|e^{\frac{t}{2}\Delta}(u(t/2)-u_{\mathscr{S}}(t/2))\right\|_\infty + |a|t^{\alpha_1+\delta}\times\\
&\int_{\frac{t}{2}}^{t}\left\|e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\left(|v(\sigma)|^{p-1}v(\sigma)-|v_{\mathscr{S}}(\sigma)|^{p-1}v_{\mathscr{S}}(\sigma)\right)\right]\right\|_\infty d\sigma\\
\leqslant\ & Ct^{\beta_1+\delta}\|u(t/2)-u_{\mathscr{S}}(t/2)\|_{r_1} + |a|Ct^{\alpha_1+\delta}\times\\
&\int_{\frac{t}{2}}^{t}(t-\sigma)^{-\frac{\gamma}{2}}\left(\|v(\sigma)\|_\infty^{p-1}+\|v_{\mathscr{S}}(\sigma)\|_\infty^{p-1}\right)\|v(\sigma)-v_{\mathscr{S}}(\sigma)\|_\infty d\sigma.
\end{aligned}$$

Using (2.11) to estimate the first term and the fact that $\|v_{\mathscr{S}}(t)\|_\infty \leqslant Ct^{-\alpha_2}$, $\|v(t)\|_\infty \leqslant Ct^{-\alpha_2}$ to estimate the last term, we get

$$\begin{aligned}&t^{\alpha_1+\delta}\|u(t)-u_{\mathscr{S}}(t)\|_\infty\\ \leqslant &C(\delta)+|a|C\times\left[\int_{\frac{1}{2}}^{1}(1-\sigma)^{-\frac{\gamma}{2}}\sigma^{-\alpha_2 p-\delta}d\sigma\right]\sup_{t\in(0,T]}\left(t^{\alpha_2+\delta}\|v(t)-v_{\mathscr{S}}(t)\|_\infty\right).\end{aligned}$$

Which leads to

$$t^{\alpha_1+\delta}\|u(t)-u_{\mathscr{S}}(t)\|_\infty \leqslant C(\delta)+C\sup_{t\in(0,T]}\left[t^{\alpha_1+\delta}\|u(t)-u_{\mathscr{S}}(t)\|_\infty, t^{\alpha_2+\delta}\|v(t)-v_{\mathscr{S}}(t)\|_\infty\right]. \tag{3.29}$$

Similarly we have

$$t^{\alpha_2+\delta}\|v(t)-v_{\mathscr{S}}(t)\|_\infty \leqslant C(\delta)+C\sup_{t\in(0,T]}\left[t^{\alpha_1+\delta}\|u(t)-u_{\mathscr{S}}(t)\|_\infty, t^{\alpha_2+\delta}\|v(t)-v_{\mathscr{S}}(t)\|_\infty\right]. \tag{3.30}$$

Using (3.29) and (3.30) we obtain

$$\sup_{t\in(0,T]}\left[t^{\alpha_1+\delta}\|u(t)-u_{\mathscr{S}}(t)\|_\infty, t^{\alpha_2+\delta}\|v(t)-v_{\mathscr{S}}(t)\|_\infty\right]\leqslant C'(\delta).$$

Since the constant $C'(\delta)$ does not depend on $T>0$, one can take the supremum over $(0,\infty)$. This proves (2.15)-(2.16) for $r_1=\infty$ and $r_2=\infty$. Using the interpolation inequality

$$\|u(t)-u_{\mathscr{S}}(t)\|_{q_1}\leqslant \|u(t)-u_{\mathscr{S}}(t)\|_{r_1}^{\mu_1}\|u(t)-u_{\mathscr{S}}(t)\|_\infty^{1-\mu_1},$$

where

$$\frac{1}{q_1}=\frac{\mu_1}{r_1}+\frac{1-\mu_1}{\infty}=\frac{\mu_1}{r_1}.$$

We get

$$\begin{aligned}\|u(t)-u_{\mathscr{S}}(t)\|_{q_1}&\leqslant \|u(t)-u_{\mathscr{S}}(t)\|_{r_1}^{\mu_1}\|u(t)-u_{\mathscr{S}}(t)\|_\infty^{1-\mu_1}\\ &\leqslant Ct^{\mu_1[-\beta_1(r_1)-\delta]+(1-\mu_1)[-\beta_1(\infty)-\delta]}=Ct^{-\beta_1(q_1)-\delta}.\end{aligned}$$

We have also

$$\|v(t)-v_{\mathscr{S}}(t)\|_{q_2}\leqslant Ct^{-\beta_2(q_2)-\delta}.$$

Hence we obtain the results (2.15)-(2.16) in general case. The estimate (2.17)-(2.18) follows by a simple dilation argument. We prove just the first estimate (2.17), the proof of the second estimate is similar. We have

$$\begin{aligned}\|u(t)-u_{\mathscr{S}}(t)\|_{q_1}&=\left\|u(t,.)-t^{-\alpha_1}u_{\mathscr{S}}\left(1,\frac{.}{\sqrt{t}}\right)\right\|_{q_1}=\left\|d_{\frac{1}{\sqrt{t}}}u(t,.\sqrt{t})-t^{-\alpha_1}d_{\frac{1}{\sqrt{t}}}u_{\mathscr{S}}(1,.)\right\|_{q_1}\\ &=\left\|d_{\frac{1}{\sqrt{t}}}\left[u\left(t,.\sqrt{t}\right)-t^{-\alpha_1}u_{\mathscr{S}}(1,.)\right]\right\|_{q_1}\end{aligned}$$

$$= \left(\frac{1}{\sqrt{t}}\right)^{-\frac{N}{q_1}} \left\| u\left(t, .\sqrt{t}\right) - t^{-\alpha_1} u_{\mathscr{S}}(1,.) \right\|_{q_1}.$$

Then by using inequality (2.15) and relation (2.14), we get (2.17). □

Proof of Proposition 1. If $\gamma = 0$ and $\rho = 0$, then (1.10) and (1.11) are verified. Since these are strict inequalities, they must hold for small $\gamma \geqslant 0$ and $\rho > 0$. This finishes the proof of the proposition. □

Proof of Proposition 2. Let α_1 and α_2 defined by (1.4) and (1.5), respectively. Under the conditions

$$q \geqslant \frac{2-\rho}{\gamma} + \frac{2}{\gamma},$$

and

$$p \geqslant \frac{2-\gamma}{\rho} + \frac{2}{\rho},$$

we have that conditions (1.10) and (1.11) are equivalent to the conditions $2\alpha_1 < N$ and $2\alpha_2 < N$. Now, since $q \geqslant \frac{2-\rho}{N} + \frac{2-\gamma}{N} + 1$, we see that $2\alpha_1 < N$ and since $p \geqslant \frac{2-\gamma}{N} + \frac{2-\rho}{N} + 1$, we obtain that $2\alpha_2 < N$. This finishes the proof of the proposition. □

4. Stronger uniqueness results

It has been proved in Theorem 1 that for small initial data $\Phi = (\varphi_1, \varphi_2) \in L^{N/2\alpha_1}\left(\mathbb{R}^N\right) \times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$ with respect of the norm $\mathscr{N}'$, there exists a solution $U_\Phi = (u_\Phi, v_\Phi)$ of the integral system (1.2) and uniqueness is guaranteed only among continuous functions $U : [0,\infty) \to L^{N/2\alpha_1}\left(\mathbb{R}^N\right) \times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$ which also verify $\sup_{t>0}\left[t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|v(t)\|_{r_2}\right]$ is sufficiently small. Our aim in this section is to prove that uniqueness is guaranteed for solutions which belong to $C\left([0,\infty), L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\right) \times C\left([0,\infty), L^{N/2\alpha_2}\left(\mathbb{R}^N\right)\right) \cap C\left((0,\infty), L^{r_1}\left(\mathbb{R}^N\right)\right) \times C\left((0,\infty), L^{r_2}\left(\mathbb{R}^N\right)\right)$, which improves the result of uniqueness in Lebesgue spaces given in Theorem 1. We will use arguments of type Brezis Cazenave [2]. We have obtained the following result.

THEOREM 4. *Let N be a positive integer. Let p, $q > 1$. Let $0 \leqslant \gamma < \min(N,2)$ and $0 < \rho < \min(N,2)$. Let α_1, α_2 be defined by (1.4) and (1.5). Suppose that (1.10) and (1.11) are satisfied. Let β_1, β_2 be given by (1.6) and (1.7). Let r_1 and r_2 be as in Lemma 1. Let M, $R > 0$ be such that (3.25) is satisfied. Let $\Phi = (\varphi_1, \varphi_2) \in L^{N/2\alpha_1}\left(\mathbb{R}^N\right) \times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$ satisfying (2.12). Let $U_\Phi = (u_\Phi, v_\Phi) \in Y_M$ be the solution of the integral system (1.2) with initial data Φ constructed by Theorem 1. Let $V = (v_1, v_2) \in C\left([0,\infty), L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\right) \times C\left([0,\infty), L^{N/2\alpha_2}\left(\mathbb{R}^N\right)\right) \cap C\left((0,\infty), L^{r_1}\left(\mathbb{R}^N\right)\right) \times C\left((0,\infty), L^{r_2}\left(\mathbb{R}^N\right)\right)$ be a solution of (1.2) with the same initial data Φ. Then*

$$V(t) = U_\Phi(t), \quad \forall t \in [0,\infty).$$

The proof of this theorem relies on the following two lemmas.

LEMMA 2. *Let N be a positive integer. Let $p,\ q>1$. Let $0\leqslant\gamma<\min(N,2)$ and $0<\rho<\min(N,2)$. Let $\alpha_1,\ \alpha_2$ be defined by (1.4) and (1.5). Suppose that (1.10) and (1.11) are satisfied. Let $\beta_1,\ \beta_2$ be given by (1.6) and (1.7). Let r_1 and r_2 be as in Lemma 1. Let $M,\ R>0$ be such that (3.25) is satisfied. Let $\Phi=(\varphi_1,\varphi_2)\in L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$ satisfying (2.12). Let $U_\Phi=(u_\Phi,v_\Phi)$ be the solution of the integral system (1.2) with initial data Φ constructed by Theorem 1. Then for all $T>0$, there exists a unique solution $U_{\Phi,T}=U_\Phi\in Y_{M,T}$ of (1.2) with initial data Φ, where*

$$
\begin{aligned}
Y_{M,T}=\Bigg\{&U=(u,v)\in C\left([0,T),L^{\frac{N}{2\alpha_1}}\left(\mathbb{R}^N\right)\right)\times C\left([0,T),L^{\frac{N}{2\alpha_2}}\left(\mathbb{R}^N\right)\right)\cap\\
&\qquad C\left((0,T),L^{r_1}\left(\mathbb{R}^N\right)\right)\times C\left((0,T),L^{r_2}\left(\mathbb{R}^N\right)\right);\\
&\max\left[\sup_{t\in[0,T)}\left[\|u(t)\|_{\frac{N}{2\alpha_1}},\|v(t)\|_{\frac{N}{2\alpha_2}}\right],\ \sup_{t\in(0,T)}\left[t^{\beta_1}\|u(t)\|_{r_1},t^{\beta_2}\|v(t)\|_{r_2}\right]\right]\leqslant M\Bigg\}.
\end{aligned}
$$

Proof. The existence of the unique solution $U_{\Phi,T}$ of (1.2) with initial data Φ follows by a fixed point argument in $Y_{M,T}$. Let $U_\Phi\in Y_M$ be the solution of (1.2) with initial data Φ. Owing to the fact that $U_\Phi\in Y_M\subset Y_{M,T}$ and by uniqueness in $Y_{M,T}$, we obtain $U_{\Phi,T}=U_\Phi$. □

LEMMA 3. *Let N be a positive integer. Let $p,\ q>1$. Let $0\leqslant\gamma<\min(N,2)$ and $0<\rho<\min(N,2)$. Let $\alpha_1,\ \alpha_2$ be defined by (1.4) and (1.5). Suppose that (1.10) and (1.11) are satisfied. Let $M,\ R>0$ be such that (3.25) is satisfied. Let $\Phi=(\varphi_1,\varphi_2)\in L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$ satisfying (2.12). Let $U_\Phi=(u_\Phi,v_\Phi)$ be the solution of the integral system (1.2) with initial data Φ constructed by Theorem 1. Let $(\Phi_\tau)=\left((\varphi_{1,\tau},\varphi_{2,\tau})\right)$ be a family of functions satisfying (2.12) such that*

$$
\Phi_\tau\underset{\tau\to0}{\longrightarrow}\Phi,\ \ in\ \ L^{\frac{N}{2\alpha_1}}\left(\mathbb{R}^N\right)\times L^{\frac{N}{2\alpha_2}}\left(\mathbb{R}^N\right).
$$

Then the family of solutions $(U_{\Phi_\tau})=\left((u_{\Phi_\tau},v_{\Phi_\tau})\right)$ of the integral system (1.2) verify

$$
U_{\Phi_\tau}(t)\underset{\tau\to0}{\longrightarrow}U_\Phi(t),\ \ in\ \ L^{\frac{N}{2\alpha_1}}\left(\mathbb{R}^N\right)\times L^{\frac{N}{2\alpha_2}}\left(\mathbb{R}^N\right),\forall t\in[0,\infty).
$$

Proof. By continuous dependance (3.26) in Y_M, it follows that

$$
\begin{aligned}
&\max\left[\|u_{\Phi_\tau}(t)-u_\Phi(t)\|_{\frac{N}{2\alpha_1}},\|v_{\Phi_\tau}(t)-v_\Phi(t)\|_{\frac{N}{2\alpha_2}}\right]\\
\leqslant&(1-K)^{-1}\times\max\left[\|\varphi_{1,\tau}-\varphi_1\|_{\frac{N}{2\alpha_1}},\|\varphi_{2,\tau}-\varphi_2\|_{\frac{N}{2\alpha_2}}\right],\ \ \forall t\in[0,\infty).
\end{aligned}
$$

By letting $\tau\to0$, we obtain the result. □

Proof of Theorem 4. Since $V=(v_1,v_2)\in C\left([0,\infty),L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\right)\times C\left([0,\infty),L^{N/2\alpha_2}\left(\mathbb{R}^N\right)\right)$, then there exists $\varepsilon_1>0$ such that

$$\mathcal{N}'(V(s))=\max\left[\|v_1(s)\|_{\frac{N}{2\alpha_1}},\|v_2(s)\|_{\frac{N}{2\alpha_2}}\right]<R,\ \ \forall s\in[0,\varepsilon_1]. \tag{4.1}$$

Let us define $V_\tau=(v_{1,\tau},v_{2,\tau})$ by $V_\tau(t)=V(t+\tau)$, $\forall\tau\in(0,\varepsilon_1/2]$, $\forall t\in[0,\varepsilon_1/2]$. We have from (4.1) and since $\left(t^{\beta_1}\|v_{1,\tau}(t)\|_{r_1},t^{\beta_2}\|v_{2,\tau}(t)\|_{r_2}\right)\to(0,0)$ as $t\to 0$, $\forall\tau\in(0,\varepsilon_1/2]$:

(a) $\max\left[\|v_{1,\tau}(0)\|_{\frac{N}{2\alpha_1}},\|v_{2,\tau}(0)\|_{\frac{N}{2\alpha_2}}\right]=\max\left[\|v_1(\tau)\|_{\frac{N}{2\alpha_1}},\|v_2(\tau)\|_{\frac{N}{2\alpha_2}}\right]<R$, $\forall\tau\in\left(0,\frac{\varepsilon_1}{2}\right]$,

(b) $\sup\limits_{t\in\left[0,\frac{\varepsilon_1}{2}\right]}\left[\|v_{1,\tau}(t)\|_{\frac{N}{2\alpha_1}},\|v_{2,\tau}(t)\|_{\frac{N}{2\alpha_2}}\right]<R\leqslant M$, $\forall\tau\in\left(0,\frac{\varepsilon_1}{2}\right]$,

(c) there exists $0<T_\tau\leqslant\varepsilon_1$ such that $\sup\limits_{t\in\left(0,\frac{T_\tau}{2}\right]}\left[t^{\beta_1}\|v_{1,\tau}(t)\|_{r_1},t^{\beta_2}\|v_{2,\tau}(t)\|_{r_2}\right]\leqslant M$, $\forall\tau\in\left(0,\frac{\varepsilon_1}{2}\right]$.

It follows then that $V_\tau\in Y_{M,T_\tau/2}$, using now Lemma 2 we deduce that $V_\tau(t)=U_{V_\tau(0)}(t)$, $\forall\tau\in(0,\varepsilon_1/2]$, $\forall t\in[0,T_\tau/2]$, where $U_{V_\tau(0)}$ is the solution of the integral system (1.2) with initial data $V_\tau(0)$ constructed by Theorem 1. Hence $V_\tau(t)=U_{V_\tau(0)}(t)$, $\forall\tau\in(0,\varepsilon_1/2]$, $\forall t\in[0,\infty)$. By Lemma 3, we obtain $V_\tau(t)\underset{\tau\to 0}{\longrightarrow}U_\Phi(t)$, in $L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$, $\forall t\in[0,\infty)$. On the other hand $V_\tau(t)=V(t+\tau)\underset{\tau\to 0}{\longrightarrow}V(t)$, in $L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$, $\forall t\in[0,\infty)$, (since V is continuous in $[0,\infty)$). Finally, we conclude by uniqueness of the limit that $V(t)=U_\Phi(t)$, $\forall t\in[0,\infty)$. $\square$

Consider now the integral equation

$$u(t)=e^{t\Delta}\varphi+a\int_0^t e^{(t-s)\Delta}\left(|.|^{-\gamma}|u(s)|^{p-1}u(s)\right)ds, \tag{4.2}$$

where $u=u(t,x)\in\mathbb{R}$, $t>0$, $x\in\mathbb{R}^N$, $a\in\mathbb{R}$, $0<\gamma<\min(N,2)$ and $p>1$. Set

$$q_c=\frac{N(p-1)}{2-\gamma}. \tag{4.3}$$

Suppose that

$$\frac{N(p-1)}{2-\gamma}>1,\ \ (\text{i.e. } q_c>1). \tag{4.4}$$

By choosing $\gamma=\rho$, $p=q$ and $r_1=r$ in Lemma 4, using the fact that $\frac{1}{q_c}-\frac{2}{Np}=\frac{2+(2-\gamma)p-\gamma p^2}{Np(p^2-1)}$ and the equivalence $q_c>1\Leftrightarrow$ (A.1), it follows that there exists $r>q_c$ satisfying

$$\frac{1}{q_c}-\frac{2}{Np}<\frac{1}{r}<\frac{N-\gamma}{Np}. \tag{4.5}$$

COROLLARY 1. *Let N be a positive integer. Suppose that $p > 1$. Let $0 < \gamma < \min(N,2)$. Let q_c be defined by (4.3). Suppose that (4.4) is satisfied. Let $r > q_c$ satisfying (4.5). Let $\varphi \in L^{q_c}(\mathbb{R}^N)$ sufficiently small. Then there exists a global solution of the integral equation (4.2), which is unique in the class of functions $u \in C\left([0,\infty), L^{q_c}\left(\mathbb{R}^N\right)\right) \cap C\left((0,\infty), L^r\left(\mathbb{R}^N\right)\right)$.*

Proof. Let N be a positive integer. Suppose that $p = q > 1$. Suppose that $\gamma = \rho$ with $0 < \gamma < \min(N,2)$. Let $\alpha_1 = \alpha_2$ defined by (1.4). Suppose that (A.1) is satisfied. Let β_1, β_2 be given by (1.6) and (1.7). Let $r_1 = r_2$ be as in Lemma 1. Let M, $R > 0$ be such that (3.25) is satisfied. Let $\Phi = (\varphi_1, \varphi_1) \in L^{N/2\alpha_1}\left(\mathbb{R}^N\right) \times L^{N/2\alpha_2}\left(\mathbb{R}^N\right)$ satisfying (2.12). Let $U_\Phi = (u_\Phi, u_\Phi) \in Y_M$ be the solution of the integral system (1.2) with initial data Φ constructed by Theorem 1. Let $V = (v_1, v_1) \in C\left([0,\infty), L^{N/2\alpha_1}\left(\mathbb{R}^N\right)\right) \times C\left([0,\infty), L^{N/2\alpha_2}\left(\mathbb{R}^N\right)\right) \cap C\left((0,\infty), L^{r_1}\left(\mathbb{R}^N\right)\right) \times C\left((0,\infty), L^{r_2}\left(\mathbb{R}^N\right)\right)$ be a solution of (1.2) with the same initial data Φ. Then by Theorem 4

$$V(t) = U_\Phi(t), \quad \forall t \in [0,\infty).$$

This finishes the proof. $\square$

REMARK 1. The previous corollary improves the class of uniqueness for the scalar Hardy-Hénon parabolic equations given by Theorem 1.1 (iii)-(b) in [1].

REMARK 2. Using the same steps as above we prove that for initial data $\Phi = (\varphi_1, \varphi_2) \in L^{q_1}\left(\mathbb{R}^N\right) \times L^{q_2}\left(\mathbb{R}^N\right)$ such that $N/2\alpha_1 < q_1 < r_1$ and $N/2\alpha_2 < q_2 < r_2$, there exists a local solution $U_\Phi = (u_\Phi, v_\Phi)$ of the integral system (1.2) and uniqueness is guaranteed in the class of solutions which belong to $C\left([0,T], L^{q_1}\left(\mathbb{R}^N\right)\right) \times C\left([0,T], L^{q_2}\left(\mathbb{R}^N\right)\right) \cap C\left((0,T], L^{r_1}\left(\mathbb{R}^N\right)\right) \times C\left((0,T], L^{r_2}\left(\mathbb{R}^N\right)\right)$, for any fixed $0 < T < T_{\max}$, where $T_{\max}$ is the maximal existence time. This improves the result of uniqueness in Lebesgue spaces given by Theorem 1.1 (iii)-(a) in [1].

A. Auxiliary lemmas

Let us state the following result which will be needed in the proof of the technical lemma.

LEMMA 4. *Let N be a positive integer. Let p, $q > 1$. Let $0 \leqslant \gamma < \min(N,2)$ and $0 < \rho < \min(N,2)$. Let k given by (1.3). Suppose that (1.10) and (1.11) are satisfied. Then there exists a real number r_1 satisfying the conditions:*

(i) $N\frac{pq-1}{(2-\rho)p+(2-\gamma)} < r_1$,

(ii) $Nk\frac{pq-1}{(2-\gamma)q+(2-\rho)} < r_1$,

(iii) $\frac{N}{N-\gamma}kp < r_1$,

(iv) $\frac{N}{N-\rho}q < r_1$,

(v) $\frac{N}{2-\gamma}(kp-1) < r_1$,

(vi) $\frac{N}{2-\rho}(q-k) < r_1$,

(vii) $r_1 < Nk\frac{p(pq-1)}{[2+(2-\rho)p-\gamma pq]_+}$,

(viii) $r_1 < Nk\frac{p(pq-1)}{[2+(2-\rho)p-\gamma pq]_+}$.

Proof. We will treat the cases where $2+(2-\rho)p-\gamma pq > 0$ and $2+(2-\gamma)q-\rho pq > 0$, the other cases are simple. One can easily see that r_1 exists if and only if the left-hand sides of inequalities (i)-(vi) are less than the right-hand sides of inequalities (vii) and (viii). Since $pq-1>0$ we verify easily:

(i) $N\frac{pq-1}{(2-\rho)p+(2-\gamma)} < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,

(ii) $Nk\frac{pq-1}{(2-\gamma)q+(2-\rho)} < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,

(iii) $\frac{N}{2-\gamma}(kp-1) < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,

(iv) $\frac{N}{2-\rho}(q-k) < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,

(v) $N\frac{pq-1}{(2-\rho)p+(2-\gamma)} < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$,

(vi) $Nk\frac{pq-1}{(2-\gamma)q+(2-\rho)} < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$,

(vii) $\frac{N}{2-\gamma}(kp-1) < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$,

(viii) $\frac{N}{2-\rho}(q-k) < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$.

Condition $2\alpha_1 < N$ implies that $\frac{N}{N-\gamma}kp < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$, condition $2\alpha_1 < \frac{p}{q}(N-\rho)\frac{(2-\gamma)q+(2-\rho)}{2+(2-\rho)p-\gamma pq}$ implies that $\frac{N}{N-\rho}q < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$, condition $2\alpha_2 < N$ implies that $\frac{N}{N-\rho}q < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$ and finally condition $2\alpha_2 < \frac{q}{p}(N-\gamma)\frac{(2-\rho)p+(2-\gamma)}{2+(2-\gamma)q-\rho pq}$ implies that $\frac{N}{N-\gamma}kp < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$. This finishes the proof of the lemma. □

Proof of Lemma 1. Assume the hypotheses of Lemma 1, which are the same as those of Lemma 4. Let r_1 be given by Lemma 4 and define r_2 by $r_2 = k^{-1}r_1$.

Owing to (1.9) the conditions (i)-(v) in Lemma 1 are equivalent to the conditions (i)-(viii) in Lemma 4. Finally (vi) in Lemma 1 follows by (1.6), (1.7) and (1.8). □

REMARK 3. In the case where $\gamma = \rho$ and $p = q$ it suffice to change the hypotheses (1.10) and (1.11) by the hypothesis

$$2\alpha_1 < N. \tag{A.1}$$

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REFERENCES

[1] B. BEN SLIMENE, S. TAYACHI AND F.B.WEISSLER, *Well-posedness, global existence and large time behavior for Hardy-Hénon parabolic equations*, Nonlinear Anal., **152** (2017), pp. 116–148.
[2] H. BREZIS AND T. CAZENAVE , *A nonlinear heat equation with singular initial data*, J. Analyse Math., **68** (1996), pp. 277–304.
[3] M. CANNONE, *A generalization of a theorem by Kato on Navier-Stokes equations*, Rev. Math. Ib., **13** (1997), pp. 515–541.
[4] M. CANNONE, Y. Meyer and F. Planchon, *Solutions autosimilaires des équations de Navier-Stokes*, Séminaire "Équations aux Dérivées Partielle" de l'école polytechnique, Exposé VIII, 1993-1994.
[5] T. CAZENAVE AND F. B. WEISSLER, *Asymptotically self-similar global solutions of the nonlinear Shrödinger and heat equations*, Math. Z., **228** (1998), pp. 83–120.
[6] T. CAZENAVE, F. DICKESTEIN AND F. B. WEISSLER, *An equation whose Fujita critical exponent is not given by scaling*, Nonlinear Anal., **68** (2008), pp. 862–874.
[7] Y. GIGA AND T. MIYAKAWA, *Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces*, Comm. in PDE., **14** (1989), pp. 577–618.
[8] T. KATO, *Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions*, Math. Z., **187** (1984), pp. 471–480.
[9] H. FUJITA AND T. KATO, *On the Navier-Stokes initial value problem* , Arch. Rational Mech. Anal., **16** (1964), pp. 269–315.
[10] H. FUJITA AND T. KATO, *On the non stationary Navier-Stokes system*, Rend. Sem. Math. Univ. Padova, **32** (1962), pp. 243–260.
[11] Q. H. PHAN, *Singularity and blow-up estimates via Liouville-type theorems for Hardy-Hénon parabolic equations*, J. Evol. Equ., **13** (2013), pp. 411–442.
[12] Q. H. PHAN AND PH. SOUPLET, *Liouville-type theorems and bounds of solutions of Hardy-Hénon equations*, (English) [J] J. Differential Equations, **252** (2012), pp. 2544–2562 .
[13] S. SNOUSSI and S. TAYACHI, *Asymptotic self-similar behavior of solutions for a seminlinear parabolic system*, Comm. Cont. Math., **3** (2001), pp. 363–392.
[14] S. SNOUSSI, S. TAYACHI AND F. B. WEISSLER, *Asymptotically self-similar global solutions of a semilinear parabolic equation with a nonlinear gradient term*, Proc. Royal Soc. Edinburgh Sect. A, **129** (1999), pp. 1291–1307.
[15] S. SNOUSSI, S. TAYACHI AND F. B. WEISSLER, *Asymptotically self-similar global solutions of a general semilinear heat equation*, Math. Ann., **321** (2001), pp. 131–155.
[16] S. SNOUSSI AND S. TAYACHI, *Global existence, asymptotic behavior and self-similar solutions for a class of semilinear parabolic systems*, Nonlinear Anal., **48** (2002), pp. 13–53.
[17] X. WANG, *On the Cauchy problem for reaction-diffusion equations*, Trans. Amer. Math. Soc., **337** (1993), pp. 549–590.
[18] Y. YAMAUCHI, *Blow-up results for a reaction-diffusion system*, Methods Appl. Anal., **13** (2006), pp. 337–350.

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Byrame Ben Slimene
Université de Tunis El Manar, Faculté des Sciences de Tunis
Département de Mathématiques, Laboratoire Equations aux Dérivées Partielles
LR03ES04, 2092 Tunis, Tunisia
Université Paris 13, Sorbonne Paris Cité, CNRS UMR 7539
Laboratoire Analyse, Géométrie et Applications
99, Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France
e-mail: byramebenslimene@yahoo.fr