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# STABILITY AND CONTROLLABILITY RESULTS OF EVOLUTION SYSTEM WITH IMPULSIVE CONDITION ON TIME SCALES

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*Abstract.* In this manuscript, we examine the Hyer's-Ulam stability and exact controllability results for impulsive evolution system on time scales. This manuscript has two segments: the first segment of the work is concerned with the Hyer's-Ulam type's stability analysis and the other segment is to exact controllability results. We used the Banach fixed point theorem, evolution operator theory and nonlinear functional analysis to establish these results. At last, we have presented some theoretical and numerical examples to outcome the utilization of these developed analytical results.

## 1. Introduction

Many real-world problems can be represented by the evolutionary processes which are liable to sudden change in its state because of outer unsettling influences that act instantaneously in the form of impulses. Many evolutionary processes, for example, bursting rhythm models in biology, blood flow, heartbeats, some motions of satellites and population dynamics are impulsive in nature [1,2]. Specifically, impulsive models are described by dynamical systems which are continuous in time, except at the finite number of points, where the system exhibits discrete time behaviour due to the sudden disturbances in the form of impulses. On the other hand, stability analysis of functional and differential equations becomes an important research area and various form of stabilities have been developed including Mittag-Leffler function, exponential and Lyapunov stability for dynamical equations. However, an interesting type of stability was introduced by Ulam and Hyers is known as Ulam-Hyer's stability, which is highly useful in numerical analysis and optimization for dynamical equations. As we know so far, the Hyers-Ulam stability has been employed for several dynamical equations of integer and fractional order [3, 4, 5]. Andrs and Mszros [6], examined the Hyer-Ulam's stability results for integral equations and differential equations on time scales by defining the Picard operators and it is proved the proposed results are more general than some existing works.

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Further, the time scales calculus was introduced by Hilger in [7] as a unification of the usual real calculus, the theory of difference equations and the q-calculus. Since then this theory has been widely utilized in the difference and differential equations to get a superior comprehension and a unified perspective of scientific phenomenons occurring there. More recently, several authors discussed the existence, uniqueness of periodic, antiperiodic solutions and stability of abstract equations on time scales [8, 9]. For more details on time scales one can refer the books [10, 11] and the papers [12, 13, 14, 15, 16, 17].

Eventhough controllability results of linear and nonlinear dynamical systems have turned into an essential territory of research since a very long while (see [18, 19, 20, 21] and references there in), controllability results of differential equations on time scales is a relatively newer field and only a few works have been reported [22, 23, 24, 25, 26, 27, 28, 29]. Particularly, the controllability and observability of the dynamical systems on time scales in the finite dimensional spaces is reported in [23]. In [22] Lupulescu et al. considered the dynamical system with impulsive conditions on time scales and established the necessary and sufficient conditions for the state observability and state controllability. According to as far as anyone is concerned, there is no manuscript which examined the exact controllability and Ulam's type stability analysis for an impulsive evolution dynamical systems on time scales. Motivated by the above facts, in this paper we obtain the Ulam's type stability results for the following impulsive evolution system on time scales

$$\begin{aligned} y^{\Delta}(t) &= \mathscr{A}(t)y(t) + \mathscr{M}(t, y(t)), \quad t \in I = [0, b]_{\mathbb{T}},\ t \neq t_l, \\ \Delta y(t_l) &= y(t_l^+) - y(t_l^-) = \mathscr{J}_l(t_l, y(t_l^-)), \quad l = 1, 2, \ldots, p, \\ y(0) &= y_0 \end{aligned} \tag{1.1}$$

and for the exact controllability results, we consider the following impulsive evolution system on time scales

$$\begin{aligned} y^{\Delta}(t) &= \mathscr{A}(t)y(t) + \mathscr{B}u(t) + \mathscr{M}(t, y(t)), \quad t \in I, t \neq t_l, \\ \Delta y(t_l) &= y(t_l^+) - y(t_l^-) = \mathscr{J}_l(t_l, y(t_l^-)), \quad l = 1, 2, \ldots, p, \\ y(0) &= y_0, \end{aligned} \tag{1.2}$$

where $\mathbb{T}$ is a time scale with $0, t_l, b \in \mathbb{T}$. $y(t) \in Y$ be a state function. Also, $\mathscr{A}(t)$ is a family of linear operators which generates an evolution operator $\{\mathscr{V}(t,s) : (t,s) \in \mathbb{T} \times \mathbb{T} : 0 \leqslant s \leqslant t \leqslant b\}$. Throughout the paper, it is assumed that the point of impulses $t_l$ for $l = 1, 2, \ldots, p$, are right dense with $0 \leqslant t_0 < t_1 < \ldots < t_p < t_{p+1} = b$, $y(t_l^-) = \lim_{h \to 0^+} y(t_l - h)$, $y(t_l^+) = \lim_{h \to 0^+} y(t_l + h)$ represent the left and right limit of $y(t)$ at $t = t_l$. $\mathscr{B}$ is a bounded linear operator from a Banach space $U$ to $Y$ and $u(\cdot)$ is control function given in $L^2(I, U)$. $\mathscr{M} : I \times Y \to Y$ and $\mathscr{J}_l : I \times Y \to Y$ are suitably defined functions.

Note that the problems (1.1) and (1.2) considered in this manuscript are new and start the investigation of evolution system with impulsive conditions on time scales. We trust that the acquired outcomes will be a helpful and significant contribution to the existing literature on the topic. The plan of the manuscript is as follows. In Section 2, we give some preliminaries, fundamental definitions and some useful lemmas. In the subsequent sections, main results of the paper are discussed. At last, to outcome the utilization of these obtained analytical results, an example is given.

## 2. Preliminaries and definitions

Below, we recall some notations, fundamental definitions, and lemmas which will be used to prove our main results. Let $(Y,\|.\|)$ be a Banach space and the space of all linear bounded operators from $Y$ into $Y$ is represented by $\mathbb{B}(Y)$. The space $C(I,Y)$ of all continuous functions $\tilde{f}: I \to Y$ is a Banach space with the norm $\|\tilde{f}\|_C = \sup_{t\in I}\|\tilde{f}(t)\|$. The space of functions from $I$ into $Y$ which are Lebesgue integrable are represented by $L^1(I,Y)$. $PC(I,Y) = \{y: I \to Y : y \in C((t_l,t_{l+1}]_{\mathbb{T}},Y), l = 0,1,\dots,p$ and there exists $y(t_l^+)$ and $y(t_l^-)$, $l = 1,2,\dots,p$ with $y(t_l^-) = y(t_l)\}$ denotes the space of piecewise continuous functions. One can easily find that $PC(I,Y)$ is a Banach space induced with the norm, $\|y\|_{PC} = \sup_{t\in I}\|y(t)\|$.

A non-empty closed subset of real number is called time scales $\mathbb{T}$. We set $\mathbb{T}^k = \mathbb{T}\backslash\{\max\mathbb{T}\}$ if $\max\mathbb{T}$ exists, otherwise $\mathbb{T}^k = \mathbb{T}$. A time scale interval is defined as $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$, similarly, we define $(a,b]_{\mathbb{T}}, (a,b)_{\mathbb{T}}$ etc.

The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ with the substitution $\inf\{\emptyset\} = \sup\mathbb{T}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ with the substitution $\sup\{\emptyset\} = \inf\mathbb{T}$. We say $t$ is right-scattered or left-scattered if $\sigma(t) > t$ or $\rho(t) < t$. Also, $t$ is called right dense or left dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$ or $t > \inf\mathbb{T}$ and $\rho(t) = t$, respectively. The point $t$ is called the dense point if it is right and left dense at the same time. Finally, the graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$.

A function $f : \mathbb{T} \to Y$ is called regulated if its right-hand limit exists (finite) at all right-dense points in $\mathbb{T}$ and its left-hand limit exists (finite) at all left-dense points in $\mathbb{T}$. A function $\psi : \mathbb{T} \to Y$ is said to be rd-continuous, if it is regulated and it is continuous at all right-dense points. We denote $C_{rd}(\mathbb{T},Y)$ for the collections of all rd-continuous functions. Moreover, function $\psi$ is piecewise rd-continuous if it is regulated and rd-continuous at all, except possibly at finitely many, right-dense points $t \in \mathbb{T}$. The collections of all piecewise rd-continuous functions from $\mathbb{T}$ to $Y$ will be denoted by $PC_{rd}(\mathbb{T},Y)$.

DEFINITION 1. (Delta derivative, [10]) Let $\psi : \mathbb{T} \to Y$ be a function and $t \in \mathbb{T}^k$. Then the delta derivative (or $\Delta-$derivative) of $\psi$ at the point $t$ is defined to be the number $\psi^{\Delta}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood $\mathscr{U}$ of $t$ such that

$$\Big|[\psi(\sigma(t)) - \psi(s)] - \psi^{\Delta}(t)[\sigma(t) - s]\Big| \leqslant \varepsilon|\sigma(t) - s|, \ \forall\, s \in \mathscr{U}.$$

DEFINITION 2. (Delta integral, [10]) Let $\Psi$ be a function, it is called the antiderivative of $\psi : \mathbb{T} \to Y$ provided $\Psi^{\Delta}(t) = \psi(t)$, for each $t \in \mathbb{T}^k$, then the delta integral is given by

$$\int_{t_0}^{t} \psi(s)\Delta s = \Psi(t) - \Psi(t_0).$$

DEFINITION 3. ( [10]) A function $q:\mathbb{T}\to\mathbb{R}$ is said to be regressive if $\forall\ t\in\mathbb{T}, 1+\mu(t)q(t)\neq 0$ . The collections of all regressive function is represented by $\mathscr{R}$.

DEFINITION 4. ( [10]) For $q\in\mathscr{R}$, the exponential function in the sense of time scales, is given by

$$e_q(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(q(\tau))\Delta\tau\right), \quad t,s\in\mathbb{T},$$

where $\xi_{\mu(\tau)}(q(\tau))$ is the cylinder transformation given by

$$\xi_{\mu(\tau)}(\mathfrak{F}) = \begin{cases} \dfrac{1}{h}\log(1+\mathfrak{F}h), & \text{if } h\neq 0,\\ \mathfrak{F}, & \text{if } h=0. \end{cases}$$

DEFINITION 5. ( [10]) Let $p,q\in\mathscr{R}$, then:

(i) $\ominus p = \dfrac{-p}{1+\mu(t)p}$;

(ii) $p\oplus q = p+q+\mu(t)pq$;

(iii) $p\ominus q = p\oplus(\ominus q)$.

LEMMA 1. ( [10]) *If $q\in\mathscr{R}$, then:*

(i) $e_q(t,t)=1$ *and* $e_0(t,s)=1$;

(ii) $(e_{\ominus q}(t,s))^{\Delta} = \ominus q(t)e_{\ominus q}(t,s)$;

(iii) $e_q(t,s) = e_{\ominus q}(s,t)$;

(iv) $e_q(t,s)e_q(s,r) = e_q(t,r)$;

(v) $e_q(\sigma(t),s) = (1+\mu(t)q(t))e_q(t,s)$.

LEMMA 2. ( [10]) *If $t_0,t_1,a\in\mathbb{T}$ and $q\in\mathscr{R}$, then*

$$\int_{t_0}^{t_1} q(s)e_q(a,\sigma(s))\Delta s = e_q(a,t_0) - e_q(a,t_1).$$

LEMMA 3. ( [16, 8]) *If $\alpha>0$, then $e_{\ominus\alpha}(t,s)\leqslant 1$, for $t,s\in\mathbb{T}$ such that $t>s$.*

DEFINITION 6. ( [15]) A two parameter family $\mathscr{V}(t,s):\mathbb{T}\times\mathbb{T}\to\mathbb{B}(Y)$ is said to be linear evolution operator if it satisfies the conditions:

(a) $(t,s) \to \mathscr{V}(t,s)y$ is continuous mapping for any fixed $y \in Y$;

(b) $\mathscr{V}(t,t) = \tilde{I}$, where $\tilde{I}$ is the identity operator in $Y$;

(c) $\mathscr{V}(t,s)\mathscr{V}(s,r) = \mathscr{V}(t,r)$.

DEFINITION 7. An evolution operator $\mathscr{V}(t,s)$ is said to be exponentially stable if there exist two constants $K_0 \geqslant 1$ and $\nu > 0$ such that

$$\|\mathscr{V}(t,s)\| \leqslant K_0 e_{\ominus\nu}(t,s), t \geqslant s.$$

Now onwards, for the notational convenience, we set:

(i) $\bar{\mu} = \sup_{t\in I} \mu(t)$;

(ii) $K_1 = \sup_{t\in I}\{e_{\ominus\nu}(t,t_l), l = 1,2,\ldots,p\}$;

(iii) $K_2 = \max\{e_{\ominus\nu}(b,t_l), l = 1,2,\ldots,p\}$ and $K = \max\{K_1,K_2\}$.

**Assumptions:**

(A1): Function $\mathscr{M} : I \times Y \to Y$ is continuous and there exist positive constants $L_{\mathscr{M}}$ and $C_{\mathscr{M}}$ such that:

(a) $\|\mathscr{M}(t,y) - \mathscr{M}(t,z)\| \leqslant L_{\mathscr{M}}\|y-z\|, \quad \forall\, y,z \in Y,\ t \in I$;

(b) $\|\mathscr{M}(t,y)\| \leqslant C_{\mathscr{M}}, \forall\, t \in I$ and $y \in Y$.

(A2): $\mathscr{J}_l(t_l, y(t_l^-)) \in C(I \times Y, Y)$, $l = 1,2,\ldots,p$ and there exist positive constants $L_{\mathscr{J}}$ and $M_{\mathscr{J}}$ such that:

(a) $\|\mathscr{J}_l(t,y) - \mathscr{J}_l(t,z)\| \leqslant L_{\mathscr{J}}\|y-z\|, \quad \forall\, t \in I,\ y,z \in Y$;

(b) $\|\mathscr{J}_l(t,y)\| \leqslant M_{\mathscr{J}}, \forall\, t \in I$ and $y \in Y$.

(A3): The family $\{\mathscr{A}(t) : t \in \mathbb{T}\}$ of bounded linear operators in $Y$ generates an exponentially stable evolution operator $\{\mathscr{V}(t,s) : t \geqslant s\}$, i.e. there exist $K_0 \geqslant 1$ and $\nu$ such that $\|\mathscr{V}(t,s)\| \leqslant K_0 e_{\ominus\nu}(t,s)$.

(A4): The linear operator $\Pi_0^b : L^2(I,U) \to Y$ given by

$$\Pi_0^b u = \int_0^b \mathscr{V}(b,\sigma(s))\mathscr{B}u(s)\Delta s$$

has a bounded invertible operator $(\Pi_0^b)^{-1}$, which takes values in $L^2(I,U)\setminus\ker\Pi_0^b$. Also, $\mathscr{B}$ is continuous operator from $U$ to $Y$ and there exist positive constants $M_\Pi, M_{\mathscr{B}}$ such that $\left\|(\Pi_0^b)^{-1}\right\| \leqslant M_\Pi$ and $\|\mathscr{B}\| \leqslant M_{\mathscr{B}}$.

DEFINITION 8. ( [15]) A function $y \in PC(I,Y)$ is called a mild solution of the system (1.2) if it satisfies $y(0) = y_0, \Delta y(t_l) = \mathscr{J}_l(t_l, y(t_l^-))$, $l = 1,2,\ldots,p$ and the following equation

$$y(t) = \mathscr{V}(t,0)y_0 + \sum_{0<t_l<t} \mathscr{V}(t,t_l)\mathscr{J}_l(t_l,y(t_l^-)) + \int_0^t \mathscr{V}(t,\sigma(s))[\mathscr{B}u(s) + \mathscr{M}(s,y(s))]\Delta s. \tag{2.1}$$

For $\varepsilon > 0$, we consider the following inequality

$$\begin{cases} \left\| x^\Delta(t) - \mathscr{A}(t)x(t) - \mathscr{M}(t,x(t)) \right\| \leqslant \varepsilon, & t \in I,\, t \neq t_l, \\ \left\| \Delta x(t_l) - \mathscr{J}_l(t_l,x(t_l^-)) \right\| \leqslant \varepsilon, & l = 1,2,\ldots,p. \end{cases} \tag{2.2}$$

DEFINITION 9. Equation (1.1) is called Ulam-Hyer's stable if there exists a constant $H_{(L_\mathscr{M},L_\mathscr{J},p,\nu)} > 0$ such that for $\varepsilon > 0$ and for each solution $x$ of inequality (2.2) there exists a mild solution $y$ of equation (1.1) satisfies the following inequality

$$\|x(t) - y(t)\| \leqslant H_{(L_\mathscr{M},L_\mathscr{J},p,\nu)}\varepsilon, \quad \forall\, t \in I.$$

DEFINITION 10. Equation (1.1) is said to be generalized Ulam-Hyer's stable if there exists $\mathscr{H}_{(L_\mathscr{M},L_\mathscr{J},p,\nu)} \in C(\mathbb{R}^+,\mathbb{R}^+), \mathscr{H}_{(L_\mathscr{M},L_\mathscr{J},p,\nu)}(0) = 0$ such that for each solution $x$ of inequality (2.2) there exists a mild solution $y$ of equation (1.1) satisfies the following inequality

$$\|x(t) - y(t)\| \leqslant H_{(L_\mathscr{M},L_\mathscr{J},p,\nu)}(\varepsilon), \quad \forall\, t \in I.$$

REMARK 1. Definition 9 $\implies$ Definition 10.

REMARK 2. A function $x \in PC(I,Y)$ is a solution of inequality (2.2) iff there is a sequence $G_l$, for $l = 1,2,\ldots,p$, and a $G \in PC(I,Y)$ such that:

(a) $\|G(t)\| \leqslant \varepsilon, \forall\, t \in I,\, t \neq t_l$ and $\|G_l\| \leqslant \varepsilon$, $\forall\, l = 1,2,\ldots,p$;

(b) $x^\Delta(t) = \mathscr{A}(t)x(t) + \mathscr{M}(t,x(t)) + G(t),\, t \in I, \quad t \neq t_l$;

(c) $\Delta x(t_l) = \mathscr{J}_l(t_l,x(t_l^-)) + G_l, \quad l = 1,2,\ldots,p.$

From the above remark, we have

$$\begin{aligned} x^\Delta(t) &= \mathscr{A}(t)x(t) + \mathscr{M}(t,x(t)) + G(t),\, t \in I, \quad t \neq t_l, \\ \Delta x(t_l) &= \mathscr{J}_l(t_l,x(t_l^-)) + G_l, \quad l = 1,2,\ldots,p, \end{aligned}$$

then by Definition 8, we find the solution $x(t)$ of the above equation with $x(0) = y_0$ is given by

$$x(t)=\mathscr{V}(t,0)y_0+\sum_{0<t_l<t}\mathscr{V}(t,t_l)[\mathscr{J}_l(t_l,x(t_l^-))+G_l]+\int_0^t\mathscr{V}(t,\sigma(s))[\mathscr{M}(s,x(s))+G(s)]\Delta s.$$

Consequently, we get

$$\begin{aligned}&\left\|x(t)-\mathscr{V}(t,0)y_0-\sum_{0<t_l<t}\mathscr{V}(t,t_l)\mathscr{J}_l(t_l,x(t_l^-))-\int_0^t\mathscr{V}(t,\sigma(s))\mathscr{M}(s,x(s))\Delta s\right\|\\&\leqslant K_0\varepsilon\sum_{l=1}^p e_{\ominus\nu}(t,t_l)+K_0\varepsilon\int_0^t e_{\ominus\nu}(t,\sigma(s))\Delta s\\&\leqslant K_0\varepsilon\sum_{l=1}^p e_{\ominus\nu}(t,t_l)+\frac{K_0\varepsilon(1-e_{\ominus\nu}(t,0))(1+\bar{\mu}\nu)}{\nu}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\left\|x(t)-\mathscr{V}(t,0)y_0-\sum_{0<t_l<t}\mathscr{V}(t,t_l)\mathscr{J}_l(t_l,x(t_l^-))-\int_0^t\mathscr{V}(t,\sigma(s))\mathscr{M}(s,x(s))\Delta s\right\|\\&\leqslant Me_{\ominus\nu}(t,0)\varepsilon,\end{aligned}\tag{2.3}$$

where $M=K_0\left[\sum_{l=1}^p e_{\ominus\nu}(0,t_l)+\frac{(1+\bar{\mu}\nu)(e_\nu(b,0)-1)}{\nu}\right]$.

LEMMA 4. ( [15]) *If $v\in PC_{rd}(\mathbb{T},\mathbb{R}^+)$ satisfies the given inequality*

$$v(t)\leqslant\alpha+\int_a^t p(s)v(s)\Delta s+\sum_{a<t_l<t}\beta_l v(t_l),\quad\forall\, t\in\mathbb{T},$$

*then*

$$v(t)\leqslant\alpha\prod_{a<t_l<t}(1+\beta_l)e_p(t,a).$$

## 3. Ulam's type stability

In this section, we give some sufficient condition for the Ulam-Hyer's stability of the equation (1.1) in the Banach space $Y$.

THEOREM 1. *If the assumptions **(A1)-(A3)** are satisfied, then the equation (1.1) is Ulam-Hyer's stable.*

*Proof.* Let $y$ be the solution of equation (1.1) and $x$ be solution of inequality (2.2). Then, by Definition 8, we have

$$y(t)=\mathscr{V}(t,0)y_0+\sum_{0<t_l<t}\mathscr{V}(t,t_l)\mathscr{J}_l(t_l,y(t_l^-))+\int_0^t\mathscr{V}(t,\sigma(s))\mathscr{M}(s,y(s))\Delta s$$

and we get

$$\begin{aligned}
&\|x(t)-y(t)\| \\
&\leqslant \left\| x(t)-\mathscr{V}(t,0)y_0-\int_0^t \mathscr{V}(t,\sigma(s))\mathscr{M}(s,y(s))\Delta s-\sum_{0<t_l<t}\mathscr{V}(t,t_l)\mathscr{J}_l(t_l,y(t_l^-))\right\| \\
&\leqslant \varepsilon M e_{\ominus\nu}(t,0)+\left\|\int_0^t \mathscr{V}(t,\sigma(s))(\mathscr{M}(s,x(s))-\mathscr{M}(s,y(s)))\Delta s\right\| \\
&\quad+\left\|\sum_{0<t_l<t}\mathscr{V}(t,t_l)(\mathscr{J}_l(t_l,x(t_l^-))-\mathscr{J}_l(t_l,y(t_l^-)))\right\| \\
&\leqslant \varepsilon M e_{\ominus\nu}(t,0)+K_0L_{\mathscr{M}}\int_0^t e_{\ominus\nu}(t,\sigma(s))\|x(s)-y(s)\|\Delta s \\
&\quad+K_0L_{\mathscr{J}}\sum_{l=1}^{p} e_{\ominus\nu}(t,t_l)\|x(t_l^-)-y(t_l^-)\|.
\end{aligned}$$

Subsequently, we get

$$\begin{aligned}
\|x(t)-y(t)\| \leqslant{}& \varepsilon M e_{\ominus\nu}(t,0)+K_0L_{\mathscr{M}}(1+\bar{\mu}\nu)\int_0^t e_{\ominus\nu}(t,s)\|x(s)-y(s)\|\Delta s \\
&+K_0L_{\mathscr{J}}\sum_{l=1}^{p} e_{\ominus\nu}(t,t_l)\|x(t_l^-)-y(t_l^-)\|.
\end{aligned}$$

Now, we set $\|x(t)-y(t)\|e_{\nu}(t,0)=\mathfrak{z}(t)$, in the above inequality, we get

$$\mathfrak{z}(t)\leqslant M\varepsilon+K_0L_{\mathscr{M}}(1+\bar{\mu}\nu)\int_0^t \mathfrak{z}(s)\Delta s+K_0L_{\mathscr{J}}\sum_{l=1}^{p}\mathfrak{z}(t_l^-).$$

Now, by using the Lemma 4, we get

$$\|x(t)-y(t)\|\leqslant M\varepsilon\prod_{l=1}^{p}(1+K_0L_{\mathscr{J}})e_{\gamma}(b,0)\leqslant H_{(L_{\mathscr{M}},L_{\mathscr{J}},p,\nu)}\varepsilon,\quad t\in I,$$

where $H_{(L_{\mathscr{M}},L_{\mathscr{J}},p,\nu)}=M\prod_{l=1}^{p}(1+K_0L_{\mathscr{J}})e_{\gamma}(b,0)>0$ and $\gamma=K_0L_{\mathscr{M}}(1+\bar{\mu}\nu)$. Hence, the equation (1.1) is Ulam-Hyer's stable. In addition, if we set $\mathscr{H}_{(L_{\mathscr{M}},L_{\mathscr{J}},p,\nu)}(\varepsilon)=H_{(L_{\mathscr{M}},L_{\mathscr{J}},p,\nu)}\varepsilon$, $\mathscr{H}_{(L_{\mathscr{M}},L_{\mathscr{J}},p,\nu)}(0)=0$, then equation (1.1) is generalized Ulam-Hyer's stable. □

## 4. Exact controllability

DEFINITION 11. The system (1.2) is called exactly controllable on $[0,b]_{\mathbb{T}}$, if for every $y_0,y_b\in Y$ there exists a control function $u\in L^2([0,b]_{\mathbb{T}},U)$ such that the mild solution (2.1) satisfies $y(b)=y_b$.

LEMMA 5. *If the assumptions **(A1)-(A4)** are satisfied, then for $t \in [0,b]_{\mathbb{T}}$, the control function*

$$u(t) = (\Pi_0^b)^{-1}\Big[y_b - \mathscr{V}(b,0)y_0 - \sum_{l=1}^{p} \mathscr{V}(b,t_l)\mathscr{J}_l(t_l, y(t_l^-)) - \int_0^b \mathscr{V}(b,\sigma(s))\mathscr{M}(s,y(s))\Delta s\Big](t) \tag{4.1}$$

*steers the state function $y(t)$ from initial state $y_0$ to final state $y_b$ at time $t=b$. Also, the estimate for the control function $u(t)$ is $M_u$ where*

$$M_u = M_\Pi\left[\|y_b\| + K_0 e_{\ominus\nu}(b,0)\|y_0\| + \frac{K_0 C_{\mathscr{M}}(1+\bar{\mu}\nu)(1-e_{\ominus\nu}(b,0))}{\nu} + mK_0K_2M_{\mathscr{J}}\right].$$

*Proof.* By substituting $t=b$, in the mild solution (2.1) of the system (1.2), we get

$$\begin{aligned}
y(b) =& \mathscr{V}(b,0)y_0 + \Pi_0^b u(t) + \sum_{l=1}^{p}\mathscr{V}(b,t_l)\mathscr{J}_l(t_l,y(t_l^-)) + \int_0^b \mathscr{V}(b,\sigma(s))\mathscr{M}(s,y(s))\Delta s\\
=& \mathscr{V}(b,0)y_0 + \sum_{l=1}^{p}\mathscr{V}(b,t_l)\mathscr{J}_i(t_l,y(t_l^-)) + \Big[y_b - \sum_{l=1}^{p}\mathscr{V}(b,t_l)\mathscr{J}_l(t_l,y(t_l^-))\\
&- \mathscr{V}(b,0)y_0 - \int_0^b \mathscr{V}(b,\sigma(s))\mathscr{M}(s,y(s))\Delta s\Big] + \int_0^b \mathscr{V}(b,\sigma(s))\mathscr{M}(s,y(s))\Delta s\\
=& y_b.
\end{aligned}$$

Hence, the control function (4.1) steers the state function $y(t)$ from initial state $y_0$ to final state $y_b$ at time $t=b$. Also,

$$\begin{aligned}
\|u(t)\| \leqslant& M_\Pi\Big[\|y_b\| + K_0e_{\ominus\nu}(b,0)\|y_0\| + K_0C_{\mathscr{M}}\int_0^b e_{\ominus\nu}(b,\sigma(s))\Delta s + K_0M_{\mathscr{J}}\sum_{l=1}^{m} e_{\ominus\nu}(b,t_l)\Big]\\
\leqslant& M_\Pi\left[\|y_b\| + K_0e_{\ominus\nu}(b,0)\|y_0\| + \frac{K_0C_{\mathscr{M}}(e_{\ominus\nu}(b,0)-1)}{\ominus\nu} + mK_0K_2M_{\mathscr{J}}\right]\\
=& M_u. \quad \square
\end{aligned}$$

THEOREM 2. *If the assumptions **(A1)-(A4)** are satisfied along with*

$$K_0\left(1 + \frac{K_0M_{\mathscr{B}}M_\Pi(1+\bar{\mu}\nu)}{\nu}\right)\left(mKL_{\mathscr{J}} + \frac{L_{\mathscr{M}}(1+\bar{\mu}\nu)}{\nu}\right) < 1,$$

*then the control system (1.2) is exactly controllable on $I$.*

*Proof.* For $\beta = K_0\|y_0\| + \dfrac{K_0(M_{\mathscr{B}}M_u + C_{\mathscr{M}})(1+\bar{\mu}\nu)}{\nu} + mK_0K_1M_{\mathscr{J}}$, consider a subset $\Omega \subseteq PC(I,Y)$ such that

$$\Omega = \{y \in PC(I,Y) : \|y\|_{PC} \leqslant \beta\}.$$

Now, define an operator $\Gamma:\Omega\to\Omega$ by

$$(\Gamma y)t=\mathscr{V}(t,0)y_0+\sum_{0<t_l<t}\mathscr{V}(t,t_l)\mathscr{J}_l(t_l,y(t_l^-))+\int_0^t\mathscr{V}(t,\sigma(s))[\mathscr{B}u(s)+\mathscr{M}(s,y(s))]\Delta s. \tag{4.2}$$

For $t\in I$ and $y\in\Omega$ we have

$$\begin{aligned}\|(\Gamma y)t\| &\leqslant K_0e_{\ominus\nu}(t,0)\|y_0\|+K_0M_{\mathscr{J}}\sum_{l=1}^{p}e_{\ominus\nu}(t,t_l)+K_0(M_{\mathscr{B}}M_u+C_{\mathscr{M}})\int_0^t e_{\ominus\nu}(t,\sigma(s))\Delta s\\ &\leqslant K_0\|y_0\|+K_0M_{\mathscr{J}}\sum_{l=1}^{p}e_{\ominus\nu}(t,t_l)+\frac{K_0(M_{\mathscr{B}}M_u+C_{\mathscr{M}})(1-e_{\ominus\nu}(t,0))(1+\bar{\mu}\nu)}{\nu}.\end{aligned}$$

Hence

$$\|\Gamma y\|_{PC}\leqslant K_0\|y_0\|+mK_0K_1M_{\mathscr{J}}+\frac{K_0(M_{\mathscr{B}}M_u+C_{\mathscr{M}})(1+\bar{\mu}\nu)}{\nu}\leqslant\beta.$$

Therefore, $\Gamma:\Omega\to\Omega$. Also, for $y,z\in\Omega$ and $t\in I$

$$\begin{aligned}&\|(\Gamma y)t-(\Gamma z)t\|\\ &\leqslant\Big\|\int_0^t\mathscr{V}(t,\sigma(\tau))\mathscr{B}(\Pi_0^b)^{-1}\Big[\int_0^b\mathscr{V}(b,\sigma(s))(\mathscr{M}(s,y(s))-\mathscr{M}(s,z(s)))\Delta s\\ &\qquad+\sum_{l=1}^{p}\mathscr{V}(b,t_l)(\mathscr{J}_l(t_l,y(t_l^-))-\mathscr{J}_l(t_l,z(t_l^-)))\Big]\Delta\tau\Big\|\\ &\quad+\Big\|\sum_{l=1}^{p}\mathscr{V}(t,t_l)(\mathscr{J}_l(t_l,y(t_l^-))-\mathscr{J}_l(t_l,z(t_l^-)))\Big\|\\ &\quad+\Big\|\int_0^t\mathscr{V}(t,\sigma(s))(\mathscr{M}(s,y(s))-\mathscr{M}(s,z(s)))\Delta s\Big\|\\ &\leqslant K_0^2M_{\mathscr{B}}M_{\Pi}\|y-z\|_{PC}\left(\frac{L_{\mathscr{M}}(e_{\ominus\nu}(b,0)-1)}{\ominus\nu}+L_{\mathscr{J}}\sum_{l=1}^{p}e_{\ominus\nu}(b,t_l)\right)\times\left(\frac{e_{\ominus\nu}(t,0)-1}{\ominus\nu}\right)\\ &\quad+\frac{K_0L_{\mathscr{M}}(e_{\ominus\nu}(t,0)-1)}{\ominus\nu}\|y-z\|_{PC}+pK_0K_1L_{\mathscr{J}}\|y-z\|_{PC}.\end{aligned}$$

Hence

$$\|\Gamma y-\Gamma z\|_{PC}\leqslant L_\alpha\|y-z\|_{PC},$$

where

$$L_\alpha=\left[\frac{K_0^2M_{\mathscr{B}}M_{\Pi}(1+\bar{\mu}\nu)}{\nu}\left(\frac{L_{\mathscr{M}}(1+\bar{\mu}\nu)}{\nu}+mL_JK_2\right)+\frac{K_0L_{\mathscr{M}}(1+\bar{\mu}\nu)}{\nu}+mK_0K_1L_{\mathscr{J}}\right].$$

Therefore, $\Gamma$ is a strict contraction operator. Thus, by means of Banach fixed point principle, $\Gamma$ has a unique fixed point on $I$. Therefore, system (1.2) has a mild solution on $I$ and hence we conclude that the control system (1.2) is exactly controllable on $I$. □

## 5. Exact controllability of integro differential system

Consider the following integro-differential equation of the form

$$\begin{aligned} y^{\Delta}(t) &= \mathscr{A}(t)y(t)+\mathscr{B}u(t)+\int_0^t \zeta(t,s)g(s,y(s))\Delta s+\mathscr{M}(t,y(t)), \quad t\in I, t\neq t_l,\\ \Delta y(t_l) &= y(t_l^+)-y(t_l^-)=\mathscr{J}_l(t_l,y(t_l^-)), \quad l=1,2,\dots,p, \\ y(0) &= y_0. \end{aligned} \tag{5.1}$$

(A5): $\zeta_b=\int_0^t|\zeta(t,s)|\Delta s$.

(A6): $g: I\times Y\to Y$ is a continuous function and there exist positive constants $L_g$ and $M_g$ such that

(a) $\|g(t,y)-g(t,z)\|\leqslant L_g\|y-z\|, \quad \forall\, y,z\in Y$, and $t\in I$.

(b) $\|g(t,y)\|\leqslant M_g$, $\forall\, t\in I$ and $y\in Y$.

DEFINITION 12. A function $y\in PC(I,Y)$ is called a mild solution of the system (5.1) if it satisfies $y(0)=y_0, \Delta y(t_l)=\mathscr{J}_l(t_l,y(t_l^-))$, $l=1,2,\dots,p$ and the following equation

$$\begin{aligned} y(t) =& \mathscr{V}(t,0)y_0+\sum_{0<t_l<t}\mathscr{V}(t,t_l)\mathscr{J}_l(t_l,y(t_l^-))+\int_0^t \mathscr{V}(t,\sigma(s))\Big[\mathscr{B}u(s)+\mathscr{M}(s,y(s)) \\ &+\int_0^s \zeta(s,\tau)g(\tau,y(\tau))\Delta\tau\Big]\Delta s. \end{aligned} \tag{5.2}$$

LEMMA 6. *If the assumptions* **(A1)-(A6)** *are satisfied, then for $t\in I$, the control function*

$$\begin{aligned} u(t) = \left(\Pi_0^b\right)^{-1}\Big[& y_b-\mathscr{V}(b,0)y_0-\sum_{l=1}^{p}\mathscr{V}(b,t_l)\mathscr{J}_l(t_l,y(t_l^-)) \\ &-\int_0^b \mathscr{V}(b,\sigma(s))\left(\int_0^s \zeta(s,\tau)g(\tau,y(\tau))\Delta\tau+\mathscr{M}(s,y(s))\right)\Delta s\Big](t) \end{aligned} \tag{5.3}$$

*steers the state function $y(t)$ from initial state $y_0$ to $y_b$ at time $b$. Also, the estimate for the control function $u(t)$ is $M_u'$, where*

$$M_u' = M_{\Pi}\left[\|y_b\|+K_0e_{\ominus\nu}(b,0)\|y_0\|+mK_0K_2M_{\mathscr{J}}+\frac{K_0(C_{\mathscr{M}}+\zeta_bM_g)(1+\bar{\mu}\nu)}{\nu}\right].$$

*Proof.* By substituting $t=b$ in the mild solution (5.2) of the system (5.1), we get

$$y(b) =\mathscr{V}(b,0)y_0+\sum_{l=1}^{p}\mathscr{V}(b,t_l)\mathscr{J}_l(t_l,y(t_l^-))+\int_0^b \mathscr{V}(b,\sigma(s))\mathscr{M}(s,y(s))\Delta s$$

$$
\begin{aligned}
&+\int_0^b \mathscr{V}(b,\sigma(s))\int_0^s \zeta(s,\tau)g(\tau,y(\tau))\Delta\tau\Delta s+\Pi_0^b\left(\Pi_0^b\right)^{-1}\Bigg[y_b-\mathscr{V}(b,0)y_0\\
&-\int_0^b \mathscr{V}(b,\sigma(s))\left(\int_0^s \zeta(s,\tau)g(\tau,y(\tau))\Delta\tau+\mathscr{M}(s,y(s))\right)\Delta s\\
&-\sum_{l=1}^{p}\mathscr{V}(b,t_l)\mathscr{J}_l(t_l,y(t_l^-))\Bigg]\\
=&y_b.
\end{aligned}
$$

Hence, the control function (5.3) steers the state function $y(t)$ from initial state $y_0$ to final state $y_b$ at time $t=b$. Also,

$$
\begin{aligned}
\|u(t)\| &\leqslant M_\Pi\Bigg[\|y_b\|+K_0e_{\ominus\nu}(b,0)\|y_0\|+K_0M_{\mathscr{J}}\sum_{l=1}^{m}e_{\ominus\nu}(b,t_l)\\
&\qquad +K_0(C_{\mathscr{M}}+\zeta_bM_g)\int_0^b e_{\ominus\nu}(b,\sigma(s))\Delta s\Bigg]\\
&\leqslant M_\Pi\left[\|y_b\|+K_0e_{\ominus\nu}(b,0)\|y_0\|+mK_0M_{\mathscr{J}}+\frac{K_0(C_{\mathscr{M}}+\zeta_bM_g)(1+\bar{\mu}\nu)}{\nu}\right]\\
&=M_u'. \quad \square
\end{aligned}
$$

THEOREM 3. *If the assumptions* ***(A1)-(A6)*** *are satisfied along with*

$$
K_0\left(1+\frac{K_0M_{\mathscr{B}}M_\Pi(1+\bar{\mu}\nu)}{\nu}\right)\left(mKL_{\mathscr{J}}+\frac{L_{\mathscr{M}}(1+\bar{\mu}\nu)}{\nu}+\frac{\zeta_bL_g(1+\bar{\mu}\nu)}{\nu}\right)<1,
$$

*then the control system (5.1) is exactly controllable on $I$.*

*Proof.* Consider a subset $\Omega'\subseteq PC(I,Y)$ such that

$$
\Omega'=\{y\in PC(I,Y):\|y\|_{PC}\leqslant\beta'\},
$$

where

$$
\beta'=K_0\|y_0\|+\frac{K_0(M_{\mathscr{B}}M_u'+C_{\mathscr{M}}+\zeta_bM_g)(1+\bar{\mu}\nu)}{\nu}+mK_0K_1M_{\mathscr{J}}.
$$

Now, define an operator $\Gamma':\Omega'\to\Omega'$ by

$$
\begin{aligned}
(\Gamma'y)t=&\mathscr{V}(t,0)y_0+\int_0^t \mathscr{V}(t,\sigma(s))\left[\mathscr{B}u(s)+\mathscr{M}(s,y(s))+\int_0^s\zeta(s,\tau)g(\tau,y(\tau))\Delta\tau\right]\Delta s\\
&+\sum_{0<t_l<t}\mathscr{V}(t,t_l)\mathscr{J}_l(t_l,y(t_l^-)).
\end{aligned}
\tag{5.4}
$$

For $t\in I$ and $y\in\Omega'$, we have

$$
\|(\Gamma'y)t\|\leqslant K_0e_{\ominus\nu}(t,0)\|y_0\|+K_0(M_{\mathscr{B}}M_u'+C_{\mathscr{M}}+\zeta_bM_g)\int_0^t e_{\ominus\nu}(t,\sigma(s))\Delta s
$$

$$
\begin{aligned}
&+K_0 M_{\mathscr{J}} \sum_{l=1}^{p} e_{\ominus \nu}(t,t_l) \\
\leqslant & K_0\|y_0\| + \frac{K_0(M_{\mathscr{B}}M_u' + C_{\mathscr{M}} + \zeta_b M_g)(1+\bar{\mu}\nu)}{\nu} + K_0 M_{\mathscr{J}} \sum_{l=1}^{p} e_{\ominus \nu}(t,t_l) \\
\leqslant & \beta'.
\end{aligned}
$$

Therefore, $\Gamma': \Omega' \to \Omega'$. For $y,z \in \Omega'$ and $t \in I$

$$
\begin{aligned}
&\|(\Gamma' y)t - (\Gamma' z)t\| \\
\leqslant & \Bigg\| \int_0^t \mathscr{V}(t,\sigma(\tau))\mathscr{B}\left(\Pi_0^b\right)^{-1}\Bigg[\int_0^b \mathscr{V}(b,\sigma(s))(\mathscr{M}(s,y(s)) - \mathscr{M}(s,z(s)))\Delta s \\
&\qquad + \sum_{l=1}^{p} \mathscr{V}(b,t_l)(\mathscr{J}_l(t_l,y(t_l^-)) - \mathscr{J}_l(t_l,z(t_l^-))) \\
&\qquad + \int_0^b \mathscr{V}(b,\sigma(s)) \int_0^s \zeta(s,\eta)(g(\eta,y(\eta)) - g(\eta,z(\eta)))\Delta\eta\Delta s\Bigg]\Delta\tau\Bigg\| \\
&+ \Bigg\| \sum_{l=1}^{p} \mathscr{V}(t,t_l)(\mathscr{J}_l(t_l,y(t_l^-)) - \mathscr{J}_l(t_l,z(t_l^-)))\Bigg\| \\
&+ \Bigg\| \int_0^t \mathscr{V}(t,\sigma(s)) \int_0^s \zeta(s,\eta)(g(\eta,y(\eta)) - g(\eta,z(\eta)))\Delta\eta\Delta s\Bigg\| \\
&+ \Bigg\| \int_0^t \mathscr{V}(t,\sigma(s))(\mathscr{M}(s,y(s)) - \mathscr{M}(s,z(s)))\Delta s\Bigg\|.
\end{aligned}
$$

Hence

$$
\|\Gamma' y - \Gamma' z\|_{PC} \leqslant L'_\alpha \|y - z\|_{PC},
$$

where

$$
\begin{aligned}
L'_\alpha = \Bigg[ & \frac{K_0^2 M_{\mathscr{B}} M_\Pi (1+\bar{\mu}\nu)}{\nu}\left(\frac{L_{\mathscr{M}}(1+\bar{\mu}\nu)}{\nu} + mL_J K_2 + \zeta_b L_g \frac{(1+\bar{\mu}\nu)}{\nu}\right) \\
& + \frac{K_0 L_{\mathscr{M}}(1+\bar{\mu}\nu)}{\nu} + mK_0K_1L_{\mathscr{J}} + \zeta_b L_g \frac{(1+\bar{\mu}\nu)}{\nu}\Bigg].
\end{aligned}
$$

Therefore, $\Gamma'$ is a strict contraction operator. Thus, by means of Banach fixed point principle, $\Gamma'$ has a unique fixed point on $I$. Therefore, the system (5.1) has a mild solution on $I$ and hence we conclude that the system (5.1) is exactly controllable on $I$. □

REMARK 3. Under some suitable conditions and adopting the strategies of Theorem 2, one can set up the exact controllability results for the following non-local system:

$$
\begin{aligned}
y^\Delta(t) &= \mathscr{A}(t)y(t) + \mathscr{B}u(t) + \mathscr{M}(t,y(t)), \quad t \in I, t \neq t_l, \\
\Delta y(t_l) &= y(t_l^+) - y(t_l^-) = \mathscr{J}_l(t_l,y(t_l^-)), \quad l = 1,2,\ldots,p, \qquad (5.5)
\end{aligned}
$$

$$y(0) = r(y) + y_0,$$

where $r : Y \to Y$ is a continuous function.

REMARK 4. Under some suitable conditions and by adopting the strategies of Theorem 3, Remark 3, one can set up the exact controllability results for the following non-local integro-differential equation:

$$\begin{aligned} y^{\Delta}(t) &= \mathscr{A}(t)y(t) + \mathscr{B}u(t) + \int_0^t \zeta(t,s)g(s,y(s))\Delta s + \mathscr{M}(t,y(t)), \quad t \in I, t \neq t_l, \\ \Delta y(t_l) &= y(t_l^+) - y(t_l^-) = \mathscr{J}_l(t_l, y(t_l^-)), \quad l = 1,2,\ldots,p, \\ y(0) &= r(y) + y_0, \end{aligned} \tag{5.6}$$

where $r : Y \to Y$ is a continuous function.

## 6. Illustrative examples

EXAMPLE 1. We consider the partial differential equation on time scale $\mathbb{T}$ in the following form

$$\begin{aligned} \frac{\partial}{\Delta_1 t}(Z(t,\eta)) &= c(t,\eta)\frac{\partial^2}{\Delta_2 \eta^2}(Z(t,\eta)) + b(\eta)W(t,\eta) + G(t,Z(t,\eta)), \\ &\qquad t \in [0,b]_{\mathbb{T}}, t \neq t_l, \eta \in [0,\pi]_{\mathbb{T}}, \\ Z(t,\pi) &= Z(t,0) = 0, \quad t \in [0,b]_{\mathbb{T}}, \\ \Delta Z(t_l,\eta) &= Z(t_l^+,\eta) - Z(t_l^-,\eta) = \mathscr{J}_l(t_l, Z(t_l^-,\eta)), \quad l = 1,2,\ldots,p, \\ Z(0,\eta) &= Z_0, \quad \eta \in [0,\pi]_{\mathbb{T}}, \end{aligned} \tag{6.1}$$

where $c(t,\eta)$ is a continuous function. Let $Y = L^2[0,\pi]_{\mathbb{T}}$. Define an operator $\mathscr{A}(t)$ by $\mathscr{A}(t)y = c(t,\eta)\dfrac{\partial^2}{\Delta_2\eta^2}y, \forall\ y \in D(\mathscr{A}) = \left\{y \in H_0^1[0,\pi]_{\mathbb{T}} \cap H^2[0,\pi]_{\mathbb{T}}\right\}$. Further, it is known that $\mathscr{A}(t)$ generates an evolution operator $\{\mathscr{V}(t,s) : t \geqslant s\}$ (please see [15]) which satisfies $\mathscr{V}(t,s) \leqslant K_0 e_{\ominus\nu}(t,s)$, $\forall\ (t,s)\ (t \geqslant s)$ with $K_0 = 1$ and $\nu = \frac{1}{2}$. Define $\mathscr{B} \in \mathbb{B}(U,Y)$ by $\mathscr{B}u(t)(\eta) = b(\eta)W(t,\eta)$, $\eta \in [0,\pi]_{\mathbb{T}}$, $b(\eta) \in L^2[0,\pi]_{\mathbb{T}}$. With the above formulations, the equation (6.1) can be rewritten as the following abstract equation $Y = L^2[0,\pi]_{\mathbb{T}}$,

$$\begin{aligned} y^{\Delta}(t) &= \mathscr{A}(t)y(t) + \mathscr{B}u(t) + \mathscr{M}(t,y(t)), \quad t \in [0,b]_{\mathbb{T}}, \\ \Delta y(t_l) &= y(t_l^+) - y(t_l^-) = \mathscr{J}_l(t_l, y(t_l^-)), \quad l = 1,\ldots,p, \\ y(0) &= y_0, \end{aligned} \tag{6.2}$$

where $y(t) = Z(t,.)$ that is $y(t)(\eta) = Z(t,\eta)$ and $\mathscr{M}(t,y(t))\eta = G(t,Z(t,\eta)), \eta \in [0,\pi]_{\mathbb{T}}$. Suppose that the functions $\mathscr{M}(t,y)$ and $\mathscr{J}_l(t,y)$ satisfy the conditions of Theorem 2. Therefore, based on the Theorem 2, it can be concluded that the equation (6.1) is controllable.

EXAMPLE 2. Let us consider the following non-linear system with impulsive condition when $Y=\mathbb{R}$

$$\begin{aligned} y^{\Delta}(t) &= \mathscr{A}(t)y(t)+\mathscr{M}(t,y(t)), \quad t\in\mathbb{I}=[0,3]_{\mathbb{T}}\setminus t_1,\\ \Delta y(t_1) &= \mathscr{J}_1(t_1,y(t_1^-)),\\ y(0) &= 1, \end{aligned} \tag{6.3}$$

where $A(t)=\dfrac{-2}{(1+2\mu(t))}$, $t_0=0$, $t_1=1/3$, $t_2=b=3$, $\mathscr{M}(t,y(t))=\dfrac{3+\sin(y(t))}{(t+4)^2}$, $\mathscr{J}_1(t,y(t_1^-))=\dfrac{3+\sin(y(t_1^-))}{(t_1+3)^2}$. Here $\mathscr{V}(t,s)=e_{\ominus 2}(t,s)$. Also, $\|\mathscr{V}(t,s)\|\leqslant e_{\ominus 2}(t,s)$, i.e. it is exponentially stable with $K_0=1$ and $\nu=2$.

Now, we consider the following two cases.

**Case A:** When we take $\mathbb{T}=\mathbb{R}$, then $\mathbb{I}=[0,3]_{\mathbb{R}}=[0,3]$, $e_p(t,s)=e^{p(t-s)}$ and $\bar{\mu}=0$. Now, we choose $y(3)=5$. The trajectory of the system (6.3) is given in Figure 1.

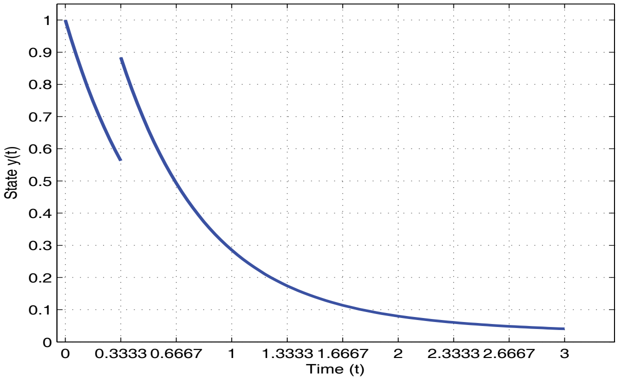


Figure 1: Trajectory of the system (6.3), when $\mathbb{T}=\mathbb{R}$.

Clearly, we can see that the trajectory reaches to the desire point $y(3)=5$. But if we add a function $u(t)$ with $\mathscr{B}=1$ in the system (6.3), then the system (6.3) becomes

$$\begin{aligned} y^{\Delta}(t) &= \mathscr{A}(t)y(t)+\mathscr{M}(t,y(t))+u(t), \quad t\in\mathbb{I}=[0,3]_{\mathbb{T}}\setminus t_1,\\ \Delta y(t_1) &= \mathscr{J}_1(t_1,y(t_1^-)),\\ y(0) &= 1. \end{aligned} \tag{6.4}$$

Also, we have

$$\Pi_0^3=\int_0^3 e_{\ominus 2}(3,\sigma(s))\Delta s=\frac{1}{2}(1-e_{\ominus 2}(3,0))=\frac{1}{2}(1-e^{-6}).$$

Hence, $\Pi_0^3$ is invertible. Thus, (A1)-(A4) are satisfied with $L_\alpha=0.1779$. Thus, from Theorem 2, we conclude that the system (6.3) is controllable and the controlled trajectory is shown in Figure 2.

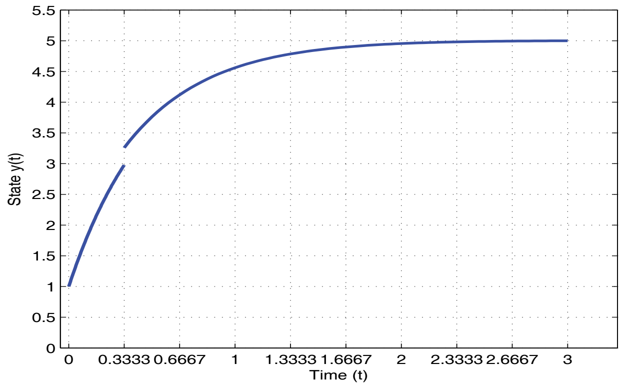


Figure 2: Trajectory of the controlled system (6.4), when $\mathbb{T}=\mathbb{R}, y(3)=5$.

**Case B:** When we take $\mathbb{T}=\mathbb{P}_{1,1}=\cup_{i=0}^{\infty}[2i,2i+1]$, then $\mathbb{I}=[0,3]_{\mathbb{T}}=[0,1]\cup[2,3]$ and $e_p(t,0)=(1-p)^l e^{p(t-l)}, \forall t\in[2l,2l+1]$, $l=0,1,\ldots$. Now, we choose $y(3)=2$. The trajectory of the system (6.3) is shown in Figure 3. Clearly, we can see that the trajectory does not reach to the desire point $y(3)=5$. But if we add a function $u(t)$ with $\mathscr{B}=1$ in the system (6.3), then the system (6.3) becomes

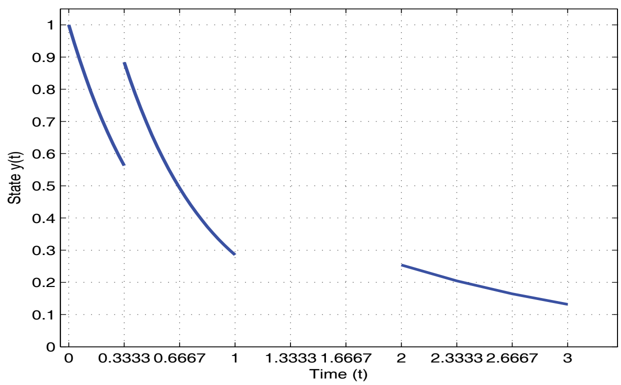


Figure 3: Trajectory of the system (6.3), when $\mathbb{T}=\mathbb{P}_{1,1}$.

$$
\begin{aligned}
y^{\Delta}(t) &= \mathscr{A}(t)y(t)+\mathscr{M}(t,y(t))+u(t), \quad t\in\mathbb{I}=[0,3]_{\mathbb{T}}\setminus t_1,\\
\Delta y(t_1) &= \mathscr{J}_1(t_1,y(t_1^-)),\\
y(0) &= 1.
\end{aligned}
\tag{6.5}
$$

Also, we have

$$\Pi_0^3 = \int_0^3 e_{\ominus 2}(3,\sigma(s))\Delta s = \frac{3}{2}\left(1-\frac{1}{3}e^{-2}\right).$$

Hence, $\Pi_0^3$ is invertible. Thus, (A1)-(A4) are satisfied with $L_\alpha = 0.6657$. Thus, from Theorem 2, we can conclude that the system (6.3) is controllable and the controlled trajectory is shown in Figure 4.

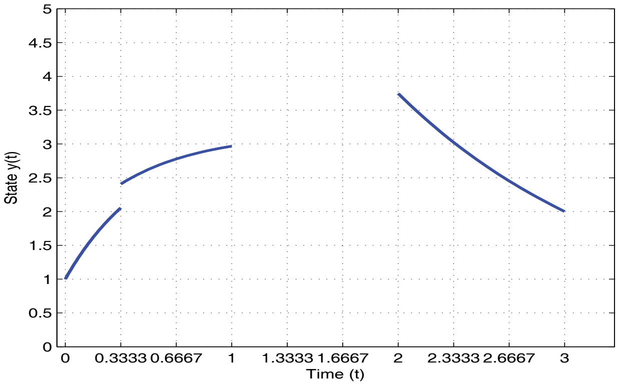


Figure 4: Trajectory of the controlled system (6.5), when $\mathbb{T} = \mathbb{P}_{1,1}, y(3) = 2$.

## 7. Conclusion

In this paper, we have studied the Hyer's-Ulam stability and exact controllability results for the system (1.1) and (1.2), respectively. Also, we have studied the exact controllability results for the integro-differential system (5.1). We used the Banach fixed point principle, evolution operator theory and non-linear functional analysis to examine these results. At last, we have given some theoretical and numerical examples to validate the developed analytical results.

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