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OPTIMAL CONTROL TO A FACULTATIVE MUTUALISTIC MODEL WITH HARVESTING

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Abstract. In this article, we propose a general facultative mutualistic model with harvesting and investigate an associated optimal control problem. The sufficient and the necessary conditions for the existence of the optimal control are studied. Numerical simulations are carried out to show the efficiency of the proposed control.

1. Introduction

Mutualism (or interspecific cooperation) is the way two organisms of different species exist in a relationship in which each individual fitness benefits from the activity of the other [19]. Well-known mutualistic examples include the relationship between ungulates and bacteria within their intestines, the relationship between land plants and fungi, the relationship between pollinators and plants, etc.. Based on the closeness of the association, the mutualism may be classified as *facultative* or *obligate*: a species is classified as facultative when it benefits from the interaction yet can survive on its own, or obligate when it requires an interaction as it is not able to survive without the other species. The reader is referred to [13, 19] and the references therein for more details on the mutualism.

The mutualistic interactions strongly influence the structure and dynamics of ecological systems and have become an important focus of research of ecological theory. Various mathematical models have been proposed to describe the mutualistic systems. The reader is referred to [9, 10, 17] for a review of differential equation (DE) models of mutualistic systems. The dynamical behaviors of DE mutualistic models have been widely investigated. Vargas-De-León [16] proposed and studied the global stabilities of two facultative mutualistic systems. Georgescu and Zhang [7] considered a more general system and studied its global asymptotic stabilities under various assumptions by using several Lyapunov functionals. Later, Georgescu et. al [8] further applied those

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Lyapunov functionals to study the global stabilities of three types of mutualistic models where conclusions in [7] fail to apply. Recently, Maxin et. al [13] proposed a more general class of models of facultative mutualism defined by

$$\begin{cases} \dfrac{dx_1}{dt} = r_1 x_1 (a_1(x_1) - f_1(x_1, x_2)), \\ \dfrac{dx_2}{dt} = r_2 x_2 (a_2(x_2) - f_2(x_1, x_2)), \end{cases} \tag{1.1}$$

where x_1 and x_2 represent the populations of two species, and r_1 and r_2 are the corresponding species' intrinsic growth rates, respectively. Functions a_1, a_2 and f_1, f_2 are positive functions representing growth and self-limiting of the two species. System (1.1) covers a broad set of existing facultative mutualistic models. A detailed analysis for System (1.1) was performed in [13]. In particular, conditions were established that guarantee the boundedness of the solutions as well as the existence of a unique coexistence equilibrium. It was also proved therein that the unique coexistence equilibrium is globally asymptotically stable whenever it exists.

The harvesting models of mutualistic systems have been extensively studied as well. For instance, Legović and Geček [12] considered the maximum sustainable yield for a mutualistic system with harvesting. Chattopadhyay [3] studied a problem of harvesting two facultative species in the presence of a predator species which feeds on both the facultative prey species. The local stability and optimal harvesting policy were investigated both theoretically and numerically. The reader is referred to [12, 3] and the references therein for more results on the harvesting problems.

All the works cited above considered the long-term behaviors of the models, i.e. the state when $t \to \infty$. It is well known that, in practice, it is often required to reach certain goals within a finite time interval. One example is to maintain the population of the mutualists at desired levels over a fixed time interval $[0, T]$. This type of problems should be viewed as short-term problems comparing to the well studied long-term problems cited above. It is obvious that artificial interference is often needed to achieve the short-term goals. The harvesting models seem a feasible solution to describe the mutualistic systems under such artificial interference.

In addition, it is natural to consider what is the "best" way to interfere the system to achieve the desired goal. Clearly, this question may be answered by optimal control theory with appropriate objective functionals. As an important application of variational calculus, optimal control theory has been successfully applied in various areas such as engineering, ecosystems, economics, life sciences, epidemiology, etc.; see for example [1, 2, 5, 6, 11, 20]. For the mutualistic problems, the optimal control theory has been used to develop the optimal harvesting policy that maximizes the profits of harvesting agencies without any harmful effects on the system, see for example [3, 15], and to identify the trees' optimal investment strategy in a system consisting of host trees and ectomycorrhizal fungi [14].

In this paper, we study the general facultative mutualistic system (1.1) with harvesting. The desired population levels for both species are proposed. We assume that the harvesting will start once the species population is over a pre-defined desired level. In particular, we will use the optimal control theory to study how to maintain (control) the population of a facultative mutualistic system at the desired level over a finite period of time with the minimum cost. Because of the broad coverage of System (1.1), our model here obviously can be applied to many existing models as special cases.

This paper is organized as follows: after this introduction, the harvesting model, the optimal control problem, and the main results are present in Section 2. Examples are given in Section 3 to show the use of the proposed control. All the proofs are given in Section 4. The last section, Section 5, contains a summary and conclusions.

2. Optimal control problem and main results

We consider a control problem on a fixed interval $[0,T]$ based on the facultative mutualistic model (1.1) with harvesting defined by

$$\begin{cases} \dfrac{dx_1}{dt} = F_1(x_1,x_2) - u_1\eta(x_1-N_1)x_1, \\ \dfrac{dx_2}{dt} = F_2(x_1,x_2) - u_2\eta(x_2-N_2)x_2, \\ x_1(0) = \bar{x}_1 > 0, \\ x_2(0) = \bar{x}_2 > 0, \end{cases} \tag{2.1}$$

where $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}$ are given by

$$\begin{cases} F_1(x_1,x_2) = r_1x_1(a_1(x_1) - f_1(x_1,x_2)), \\ F_2(x_1,x_2) = r_2x_2(a_2(x_2) - f_2(x_1,x_2)). \end{cases} \tag{2.2}$$

$N_i > 0$ is the pre-defined desired population, $u_i \in L^2([0,T];[0,1])$ represents the control strategy, $i = 1,2$, and $\eta \in C^1(\mathbb{R};[0,1])$ satisfies that η is increasing on $\mathbb{R}$ with $\eta(x) \equiv 0$ for $x \in (-\infty,0]$ and $\eta(x) \geqslant 0$ on $\mathbb{R}_+ = [0,\infty)$. In addition, we make the same assumptions for r_i, a_i, and $f_i, i = 1,2$ as in [13], and refer the readers to [13] for interpretations of these assumptions.

(H1) $r_i > 0$ is the intrinsic growth rate, $a_i \in C^1(\mathbb{R};(0,\infty))$ is the growth function, and $f_i \in C^1(\mathbb{R}^2;(0,\infty))$ is the self-limiting function, $i = 1,2$.

(H2) $f_1(x_1,x_2)$ is increasing in x_1 and decreasing in x_2; $f_2(x_1,x_2)$ is decreasing in x_1 and increasing in x_2.

(H3) There exists $K_1 > 0$ such that $a_1(x) - f_1(x,0) > 0$ for $x < K_1$, $a_1(x) - f_1(x,0) < 0$ for $x > K_1$ and $a_1(K_1) - f_1(K_1,0) = 0$; there exists $K_2 > 0$ such that $a_2(x) - f_2(0,x) > 0$ for $x < K_2$, $a_2(x) - f_2(0,x) < 0$ for $x > K_2$ and $a_2(K_2) - f_2(0,K_2) = 0$.

(H4) $a_1(x_1)/f_1(x_1,\alpha x_1)$ and $a_2(x_2)/f_2(\alpha x_2,x_2)$ are eventually decreasing for all $\alpha > 0$.

That is,

$$\frac{d}{dx_1}\left(\frac{a_1(x_1)}{f_1(x_1,\alpha x_1)}\right) < 0 \text{ for } x_1 > M_1,$$
$$\frac{d}{dx_2}\left(\frac{a_2(x_2)}{f_2(\alpha x_2,x_2)}\right) < 0 \text{ for } x_2 > M_2,$$

where M_1 and M_2 are positive real numbers that may depend on α.

Let $\mathbf{x} = \mathbf{x}(t) = (x_1(t),x_2(t))^T$, $\bar{\mathbf{x}} = (\bar{x}_1,\bar{x}_2)^T$, $\mathbf{u} = \mathbf{u}(t) = (u_1(t),u_2(t))^T$. Then the vector form of System (2.1) is

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x},\mathbf{u}), \\ \mathbf{x}(0) = \bar{\mathbf{x}}, \end{cases} \tag{2.3}$$

with $\mathbf{F} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{F}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} F_1(x_1,x_2) - u_1\eta(x_1 - N_1)x_1 \\ F_2(x_1,x_2) - u_2\eta(x_2 - N_2)x_2 \end{bmatrix}. \tag{2.4}$$

REMARK 1. (a) It is easy to verify that $\mathbf{F}$ defined by (2.4) satisfies the local Lipschitz condition with respect to $\mathbf{x}$ and $\mathbf{u}$. Therefore, System (2.3) has a unique Carathéodory-solution on $[0,T]$ for any $\bar{\mathbf{x}}$ and $\mathbf{u}$, see [4,18]. Hence $\mathbf{x} \in AC([0,T];\mathbb{R}^2)$ with $AC([0,T];\mathbb{R}^2)$ denoting the class of absolutely continuous vector valued functions on $[0,T]$.

(b) The term $u_i\eta(x_i - N_i) \in [0,1]$ in System (2.1) represents the harvesting rate, $i = 1,2$. The harvesting rates are designed in such a way to include into consideration the current system status. Obviously, one can see that if $x_i \leqslant N_i$, then $u_i\eta(x_i - N_i) \equiv 0$, meaning that there is no harvesting if the population x_i is below the desired level N_i. When $x_i > N_i$, the desired harvesting rate can be reached by adjusting u_i. It is notable that u_i may be different for the same $x_i > 0$ when different η is employed.

Let $\Omega = L^2([0,T];[0,1]\times[0,1])$ be the control set of System (2.1). For any $\mathbf{u} \in \Omega$, let $\mathbf{x}^{\mathbf{u}}(t) = (x_1^{\mathbf{u}}(t),x_2^{\mathbf{u}}(t))^T$ be the solution of (2.1) subject to $\mathbf{u}$ with initial value $\bar{\mathbf{x}}$. Our goal is to find a control $\mathbf{u} \in \Omega$ that makes $\mathbf{x}^{\mathbf{u}}$ stay as close to the desired population level (N_1,N_2) as possible with least control effort $\mathbf{u}$ over the interval $[0,T]$, i.e., to minimize the following objective functional

$$\begin{aligned} J[\mathbf{u}] =& \frac{1}{2}\int_0^T \gamma_1(x_1^{\mathbf{u}}(t) - N_1)^2 + (1-\gamma_1)u_1^2(t)dt \\ &+ \frac{1}{2}\int_0^T \gamma_2(x_2^{\mathbf{u}}(t) - N_2)^2 + (1-\gamma_2)u_2^2(t)dt, \end{aligned} \tag{2.5}$$

where γ_1, $\gamma_2 \in (0,1)$ are the “weight” constants.

Our first result is on the existence of optimal control of System (2.1), (2.5) which is proved in Section 4.

THEOREM 1. *Consider System* (2.1), (2.5) *on a fixed interval* $[0,T]$. *Assume that (H1)–(H4) hold. If there is* $\alpha > 0$ *such that*

$$R_1(\alpha) = \lim_{x_1\to\infty} \frac{a_1(x_1)}{f_1(x_1,\alpha x_1)} < 1 \tag{2.6}$$

and

$$R_2(1/\alpha) = \lim_{x_2\to\infty} \frac{a_2(x_2)}{f_2(\frac{x_2}{\alpha},x_2)} < 1, \tag{2.7}$$

then there exists an optimal control $\mathbf{u}^* \in \Omega$.

Next, we consider the necessary conditions for the optimal control $\mathbf{u}^*$.

THEOREM 2. *Let* $\mathbf{u}^* \in \Omega$ *be an optimal control of* (2.1), (2.5). *Then there exist an adjoint function* $\lambda^* = (\lambda_1^*,\lambda_2^*)^T$ *such that* $\mathbf{x}^*$, $\mathbf{u}^*$, λ^* *satisfy* (2.1), *and*

$$\begin{aligned}\frac{d\lambda_1^*}{dt} =&\gamma_1(N_1-x_1^*) - \lambda_2^*\frac{\partial}{\partial x_1}F_2(x_1^*,x_2^*)\\ &-\lambda_1^*\left(\frac{\partial}{\partial x_1}F_1(x_1^*,x_2^*) - u_1^*x_1^*\eta'(x_1^*-N_1) - u_1^*\eta(x_1^*-N_1)\right),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\frac{d\lambda_2^*}{dt} =&\gamma_2(N_2-x_2^*) - \lambda_1^*\frac{\partial}{\partial x_2}F_1(x_1^*,x_2^*)\\ &-\lambda_2^*\left(\frac{\partial}{\partial x_2}F_2(x_1^*,x_2^*) - u_2^*x_2^*\eta'(x_2^*-N_2) - u_2^*\eta(x_2^*-N_2)\right),\end{aligned} \tag{2.9}$$

$$\lambda_1^*(T) =0, \tag{2.10}$$

$$\lambda_2^*(T) =0, \tag{2.11}$$

where the optimal control $\mathbf{u}^* = (u_1^*(t),u_2^*(t))$ *is given by*

$$u_i^*(t) = \min\left\{1, \max\left\{0, \frac{\lambda_i^*(t)\eta(x_i^*(t)-N_i)x_i^*(t)}{1-\gamma_i}\right\}\right\}, i=1,2. \tag{2.12}$$

In addition, if $\eta \in C^2(\mathbb{R})$ *and* $F_i \in C^2(\mathbb{R}^2)$, $i=1,2$, *then the optimal control* $\mathbf{u}^*$ *is unique when* T *is small enough.*

3. Numerical simulations

In this section, we use numerical simulations to exemplify the methodology proposed in Section 2. Consider System (2.1) with η defined by

$$\eta(x) = \begin{cases} 0, & x \leqslant 0,\\ \frac{1}{2}\sin(x-\frac{\pi}{2})+\frac{1}{2}, & 0\leqslant x\leqslant \pi,\\ 1, & x\geqslant \pi,\end{cases}$$

and

$$\begin{cases} a_i(x_i) = x_i^{p_i - 1}, \ i = 1,2, \\ f_1(x_1,x_2) = \dfrac{x_1^{p_1+q_1-1}}{K_1^{q_1} + b_{12}x_2^{n_1}}, \\ f_2(x_1,x_2) = \dfrac{x_2^{p_2+q_2-1}}{K_2^{q_2} + b_{21}x_1^{n_2}}, \end{cases} \tag{3.1}$$

where $p_1 = p_2 = 1 + 10^{-4}$, $q_1 = n_1 = 3$, $q_2 = n_2 = 1$, $K_1 = 15$, $K_2 = 35$, $b_{12} = 0.75$, $b_{21} = 0.7$, $r_1 = 3 \times 10^{-5}$, $r_2 = 1.05 \times 10^{-4}$.

We leverage System (2.1), (2.5) to maintain the system at the desired population level over a fixed interval. Note that by [13, Section 3.2], all the conditions for Theorem 1 are satisfied for (2.2) with (3.1). Hence there exists an optimal control of System (2.1), (2.5) on $[0,T]$. Then Theorem 2 and the gradient method are used to find the optimal control pair $(\mathbf{x}^*, \mathbf{u}^*)$, see [1, Chapter 3] for the details. In the following examples, the time interval is chosen as $[0,10]$ and the objective functional is defined by (2.5) with $\gamma_1 = \gamma_2 = 1/2$.

EXAMPLE 1. Let $(\bar{x}_1, \bar{x}_2) = (150, 130)$ be the initial point and $(N_1, N_2) = (60, 60)$ be the desired population level. The numerical solution without control, $\mathbf{x} = (x_1, x_2)$, and the numerical solution under the optimal control $\mathbf{u}^* = (u_1^*, u_2^*)$, $\mathbf{x}^* = (x_1^*, x_2^*)$, are computed respectively. The comparison of the solutions is given in Fig. 1 and Fig. 2. The graphs of $\mathbf{u}^*$ and the associated harvesting rates, $u_i^* \eta(x_i^* - N_i)$, $i = 1,2$, are plotted in Fig. 3 and Fig. 4.

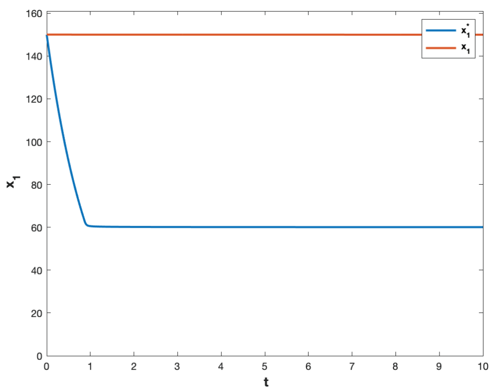

Figure 1: Comparison of the solution with control, x_1^* and the solution without control, x_1.

It is obvious that the control $\mathbf{u}^*$ drives the solution $\mathbf{x}^*$ to the desired population level. The graphs of $\mathbf{x}^*$ and harvesting rate on $[0,3]$ are given in Fig. 5 and Fig. 6 to show the detailed changes. Based on the graphs, when the population $\mathbf{x}^*$ is much higher than the desired population level, the harvesting rate should be 1, i.e. harvest as much as possible. When $\mathbf{x}^*$ is close the desired population level, the harvesting rates

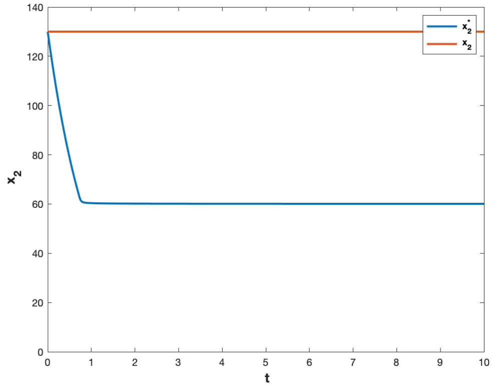

Figure 2: Comparison of the solution with control, x_2^* and the solution without control, x_2.

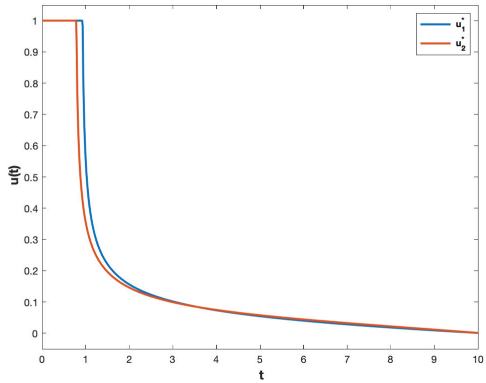

Figure 3: The graphs of u_1^* and u_2^*.

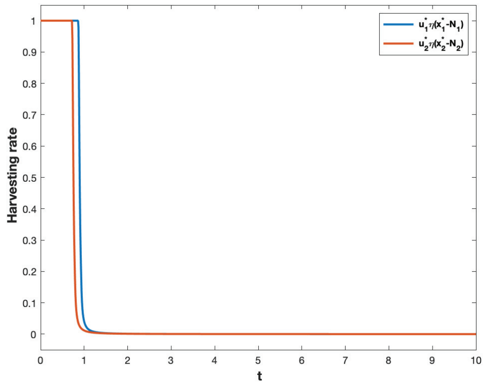

Figure 4: The graphs of the harvesting rates.

will quickly reduce due to the decrease in u^*. This simulation result is reasonable and consistent with our expectation.

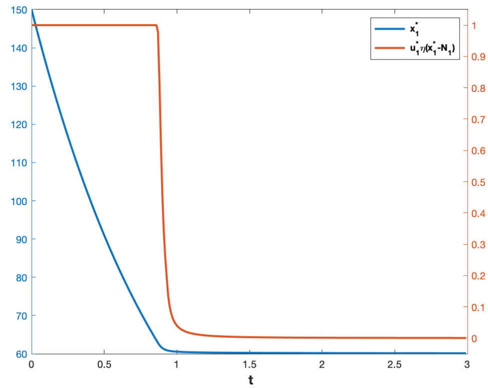

Figure 5: The graphs of x_1^* and the associated harvesting rate on $[0,3]$.

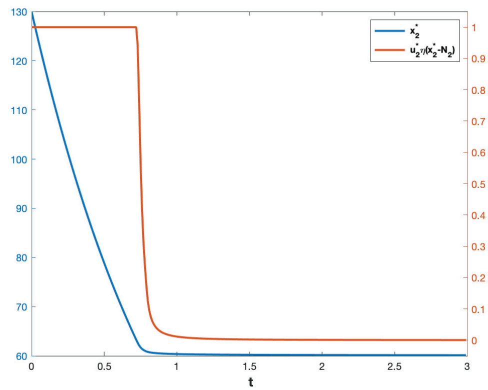

Figure 6: The graphs of x_2^* and the associated harvesting rate on $[0,3]$.

EXAMPLE 2. Let $(\bar{x}_1, \bar{x}_2) = (10, 100)$ be the initial point and $(N_1, N_2) = (70, 50)$ be the desired population level. The numerical solutions with and without controls are given in Fig. 7 and Fig. 8. The graphs of $\mathbf{u}^*$ and the corresponding harvesting rates, $u_i^* \eta(x_i^* - N_i)$, $i = 1, 2$, are plotted in Fig. 9 and Fig. 10. Fig. 11 shows the graphs of x_2^*, u_2^*, and the associated harvesting rate on $[0,3]$.

Note that $x_1(t) < N_1$ on $[0,10]$, see Fig. 7. Furthermore, by the proof of Lemma 2 in Section 4, $x_1^{\mathbf{u}}(t) \leqslant x_1(t)$ on $[0,10]$ for any $\mathbf{u} \in \Omega$. Therefore no control is needed for x_1, i.e. $u_1^*(t) \equiv 0$ and $x_1^*(t) \equiv x_1(t)$ on $[0,10]$. As a result, the harvesting rate $u_1^*(t)\eta(x_1^*(t) - N_1) \equiv 0$ on $[0,10]$. On the other hand, since $\bar{x}_2 > N_2$, a control u_2^* is needed to drive x_2^* to N_2. These conclusions are consistent with the simulation results shown in Fig. 9 – 11.

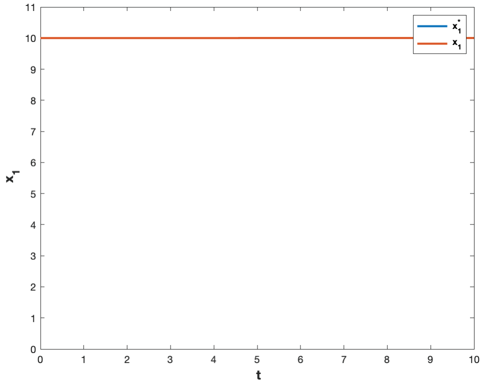

Figure 7: Comparison of the solution with control, x_1^* and the solution without control, x_1.

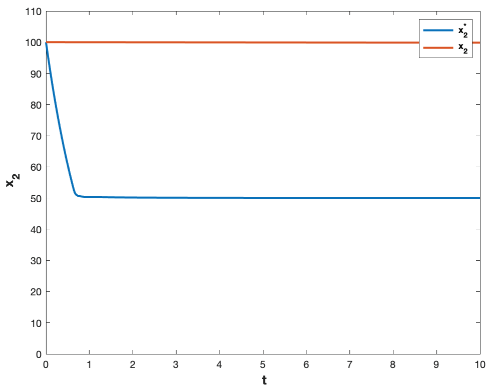

Figure 8: Comparison of the solution with control, x_2^* and the solution without control, x_2.

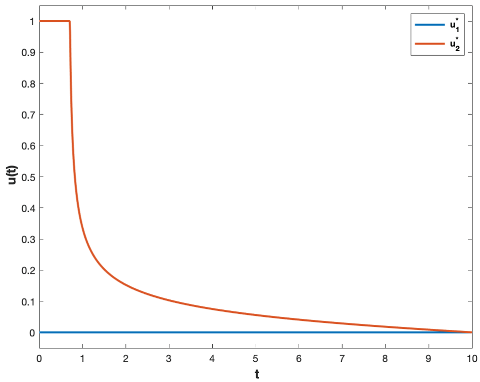

Figure 9: The graphs of u_1^* and u_2^*.

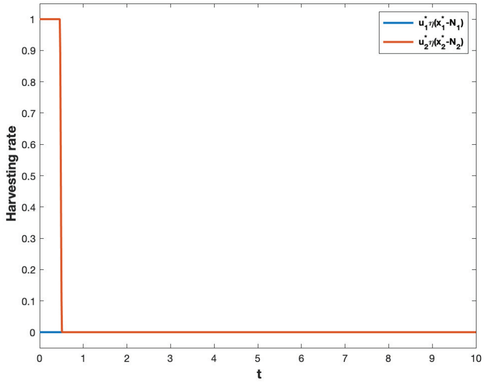

Figure 10: The graphs of the harvesting rates.

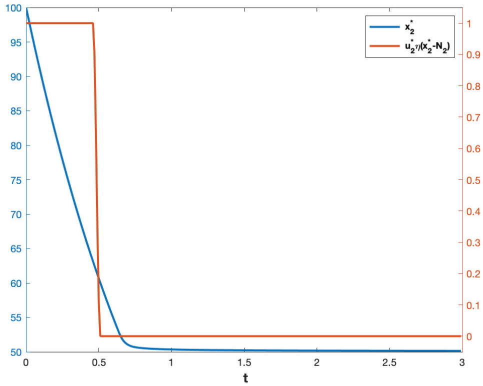

Figure 11: The graphs of x_2^* and the associated harvesting rate on $[0,3]$.

4. Proofs

In this section, we will prove our main results given in Section 2. The following result is proved in [13] and we will use it in our proof.

LEMMA 1. *[13, Theorem 2.2] Suppose that Assumptions (H1)–(H4) hold. If there is* $\alpha > 0$ *such that*

$$R_1(\alpha) = \lim_{x_1 \to \infty} \frac{a_1(x_1)}{f_1(x_1, \alpha x_1)} < 1$$

and

$$R_2(1/\alpha) = \lim_{x_2 \to \infty} \frac{a_2(x_2)}{f_2(\frac{x_2}{\alpha}, x_2)} < 1,$$

then solutions of System (1.1) *are uniformly bounded in a compact region of the positive quadrant and there is a coexistence equilibrium* $E^* = (E_1^*, E_2^*)$.

We first consider the boundedness of solutions of System (2.1).

LEMMA 2. *Assume all the conditions of Theorem 1 hold. For a given positive initial condition $\bar{\mathbf{x}}$, the solution of System (2.3) (or (2.1)), $\mathbf{x}^{\mathbf{u}}$, is uniformly bounded in the first quadrant for any control $\mathbf{u} \in \Omega$.*

Proof. We will first show that for any $\mathbf{u} \in \Omega$ and $\bar{\mathbf{x}} \in \mathbb{R}^2$ with $\bar{x}_1 > 0$ and $\bar{x}_2 > 0$, the solution $\mathbf{x}^{\mathbf{u}}$ is positive, i.e. $x_i^{\mathbf{u}}(t) > 0$, $i = 1,2$.

By (2.1), (H2), and (H3), when $0 < x_1^{\mathbf{u}}(t) < \min\{K_1, N_1\}$ and $x_2^{\mathbf{u}}(t) > 0$,

$$\frac{dx_1^{\mathbf{u}}}{dt} = r_1 x_1^{\mathbf{u}}(a_1(x_1^{\mathbf{u}}) - f_1(x_1^{\mathbf{u}}, x_2^{\mathbf{u}})) > r_1 x_1^{\mathbf{u}}(a_1(x_1^{\mathbf{u}}) - f_1(x_1^{\mathbf{u}}, 0)) > 0.$$

This implies $x_1^{\mathbf{u}}(t) > 0$. Similarly, when $0 < x_2^{\mathbf{u}}(t) < \min\{K_2, N_2\}$ and $x_1^{\mathbf{u}}(t) > 0$,

$$\frac{dx_2^{\mathbf{u}}}{dt} = r_2 x_2^{\mathbf{u}}(a_2(x_2^{\mathbf{u}}) - f_2(x_1^{\mathbf{u}}, x_2^{\mathbf{u}})) > r_2 x_2^{\mathbf{u}}(a_2(x_2^{\mathbf{u}}) - f_2(0, x_2^{\mathbf{u}})) > 0.$$

Hence $x_2^{\mathbf{u}}(t) > 0$.

Next, we claim that for any $\mathbf{u} \in \Omega$ and $\bar{x}_1 > 0, \bar{x}_2 > 0$, $x_i^{\mathbf{u}}(t) \leqslant x_i(t)$, $i = 1,2$, where $x_i(t)$ is the solution to system (1.1). In fact, by (2.1) and the assumption for η and $\mathbf{u}$, we have

$$\frac{dx_i^{\mathbf{u}}}{dt} - F_i(x_1^{\mathbf{u}}, x_2^{\mathbf{u}}) - u_i \eta(x_i^{\mathbf{u}} - N_i) x_i^{\mathbf{u}} \leqslant \frac{dx_i}{dt} - F_i(x_1, x_2), \, i = 1,2.$$

Then by the Comparison Theorem [18, P112, Comparison Theorem (b)], $x_i^{\mathbf{u}}(t) \leqslant x_i(t)$, $i = 1,2$. Therefore, Lemma 1 implies that $\mathbf{x}^{\mathbf{u}}$ is uniformly bounded for all $\mathbf{u} \in \Omega$.

The existence of the optimal control of System (2.1), (2.5) is proved by a scheme given in [1, Section 2.1] and the following lemma is needed. The reader is referred to [1] for details.

LEMMA 3. *[1, Theorem A.5] Let $\{\mathbf{u}_n\}$ be a bounded sequence in a real Hilbert space $\mathscr{H}$. Then there exists a subsequence $\{\mathbf{u}_{n_r}\} \subset \{\mathbf{u}_n\}$ that is weakly convergent to an element of $\mathscr{H}$.*

Now we are ready to prove our first main result, Theorem 1.

Proof of Theorem 1. By Remark 1 (a), for any $T > 0$, System (2.1) has a unique solution $\mathbf{x}^{\mathbf{u}}$ for each $\mathbf{u} \in \Omega$ on $[0,T]$. Hence there exists a mapping $\Gamma : \Omega \to AC([0,T];\mathbb{R}^2)$ such that

$$\mathbf{x}^{\mathbf{u}} = \Gamma \mathbf{u}.$$

Let X be the Banach space $C([0,T];\mathbb{R}^2)$ with the maximum norm. Lemma 2 implies that $\Gamma\Omega$ is uniformly bounded in X and there exists $M > 0$ such that

$$\|\Gamma \mathbf{u}\| \leqslant M \text{ for any } \mathbf{u} \in \Omega. \tag{4.1}$$

By (2.1), for any $t \in [0,T]$ and $\mathbf{u} \in \Omega$,

$$\begin{cases} x_1^{\mathbf{u}}(t) = \bar{x}_1 + \int_0^t (F_1(x_1^{\mathbf{u}}(s), x_2^{\mathbf{u}}(s)) - u_1(s)\eta(x_1^{\mathbf{u}}(s) - N_1)x_1^{\mathbf{u}}(s))ds, \\ x_2^{\mathbf{u}}(t) = \bar{x}_2 + \int_0^t (F_2(x_1^{\mathbf{u}}(s), x_2^{\mathbf{u}}(s)) - u_2(s)\eta(x_2^{\mathbf{u}}(s) - N_2)x_2^{\mathbf{u}}(s))ds. \end{cases} \tag{4.2}$$

By the assumptions for r_i, a_i, f_i, u_i, and η, we know that all functions a_i, f_i and η are C^1 functions. One can easily show that there exist constants $A_i > 0$ such that, for $t \in [0,T]$

$$|r_i a_i(x_i(t)) - r_i f_i(x_1(t), x_2(t)) - u_i(t)\eta(x_i(t) - N_i)| < A_i,\ i = 1,2.$$

Then it follows that, for any $t_1, t_2 \in [0,T]$, by (4.2), (2.2) and (4.1),

$$\begin{aligned}|x_i^{\mathbf{u}}(t_1) - x_i^{\mathbf{u}}(t_2)| &= \left|\int_{\min\{t_1,t_2\}}^{\max\{t_1,t_2\}} (F_i(x_1^{\mathbf{u}}(s), x_2^{\mathbf{u}}(s)) - u_i(s)\eta(x_i^{\mathbf{u}}(s) - N_i)x_i^{\mathbf{u}}(s))ds\right| \\ &\leqslant \int_{\min\{t_1,t_2\}}^{\max\{t_1,t_2\}} |x_i^{\mathbf{u}}(s)||r_i a_i(x_i^{\mathbf{u}}(t)) - r_i f_i(x_1^{\mathbf{u}}(s), x_2^{\mathbf{u}}(s)) - u_i(s)\eta(x_i^{\mathbf{u}}(s) - N_i)|\,ds \\ &\leqslant MA_i|t_1 - t_2|.\end{aligned}$$

Therefore, $\Gamma\Omega$ is equicontinuous in X. Then by Arzelà-Ascoli Theorem, $\Gamma : \Omega \to X$ is completely continuous on Ω.

Let $d = \inf_{\mathbf{u}\in\Omega} J[\mathbf{u}]$, with J defined by (2.5). It is easy to see that $d \in \mathbb{R}_+$. Let $\{\mathbf{u}_n\} \subset \Omega$ be a sequence with

$$d \leqslant J[\mathbf{u}_n] < d + \frac{1}{n},\ n = 1,2,\ldots. \tag{4.3}$$

Since Ω is bounded in $L^2([0,T];\mathbb{R}^2)$, then by Lemma 3, there exists a subsequence $\{\mathbf{u}_{n_r}\} \subset \{\mathbf{u}_n\}$ with

$$\mathbf{u}_{n_r} \stackrel{*}{\rightharpoonup} \mathbf{u}^* \in L^2([0,T];\mathbb{R}^2).$$

In addition, since Ω is closed convex in $L^2([0,T];\mathbb{R}^2)$, Ω is weakly closed in $L^2([0,T];\mathbb{R}^2)$. Therefore, $\mathbf{u}^* \in \Omega$.

Let

$$\mathbf{x}^{\mathbf{u}_{n_r}} = \Gamma\mathbf{u}_{n_r},\ r = 1,2,....$$

Then there exists a subsequence of $\{\mathbf{x}^{\mathbf{u}_{n\tau}}\}$, we use the same notation for convenience, such that

$$\mathbf{x}^{\mathbf{u}_{n\tau}} \to \mathbf{x}^* \in X.$$

By (4.2),

$$\begin{cases} x_1^*(t) = \bar{x}_1 + \displaystyle\int_0^t F_1(x_1^*(s), x_2^*(s)) - u_1^*(s)\eta(x_1^*(s) - N_1)x_1^*(s)ds, \\ x_2^*(t) = \bar{x}_2 + \displaystyle\int_0^t F_2(x_1^*(s), x_2^*(s)) - u_2^*(s)\eta(x_2^*(s) - N_2)x_2^*(s)ds. \end{cases} \tag{4.4}$$

So $\mathbf{x}^* \in AC([0,T];\mathbb{R}^2)$ and satisfies System (2.1) with $\mathbf{u}^*$. In addition (4.3) implies

$$J[\mathbf{u}^*] = d.$$

Therefore $(\mathbf{x}^*, \mathbf{u}^*)$ is an optimal control pair of System (2.1), (2.5).

We will use the minimum principle to derive the necessary conditions of the optimal control pair $(\mathbf{x}^*, \mathbf{u}^*)$. Consider a control system

$$\begin{cases} \mathbf{x}'(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad 0 < t \leqslant T, \\ \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{u} \in \Omega, \end{cases} \tag{4.5}$$

with the objective functional

$$J[\mathbf{u}] = \int_0^T L(\mathbf{x}(t), \mathbf{u}(t))dt, \tag{4.6}$$

where $\mathbf{f}: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $L: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Let $H: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$H(\mathbf{x}, \mathbf{u}, \lambda) = L(\mathbf{x}, \mathbf{u}) + \lambda \cdot \mathbf{f}(\mathbf{x}, \mathbf{u}). \tag{4.7}$$

The following minimum principle for the fixed-terminal-time problem is taken from [2].

LEMMA 4. *[2, Theorem 5-10] Let $\mathbf{u}^*$ be an admissible control and $\mathbf{x}^*$ be the trajectory corresponding to $\mathbf{u}^*$. In order that $\mathbf{u}^*$ to be optimal, it is necessary that there exists λ^* such that*

$$\begin{aligned} \frac{d\mathbf{x}^*}{dt} &= \frac{\partial H}{\partial \lambda}[\mathbf{x}^*, \mathbf{u}^*, \lambda^*], \\ \frac{d\lambda^*}{dt} &= -\frac{\partial H}{\partial \mathbf{x}}[\mathbf{x}^*, \mathbf{u}^*, \lambda^*], \\ \mathbf{x}^*(0) &= \mathbf{x}_0, \\ \lambda^*(T) &= 0, \\ \min_{\mathbf{u} \in \Omega} H(\mathbf{x}^*, \mathbf{u}, \lambda^*) &= H(\mathbf{x}^*, \mathbf{u}^*, \lambda^*). \end{aligned} \tag{4.8}$$

To prove the uniqueness of the optimal control, we also need the following lemma.

LEMMA 5. *[11, Theorem 2.3] Consider the following two-point boundary value problem*

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{p}(t, \mathbf{x}, \mathbf{y}), \\ \dfrac{d\mathbf{y}}{dt} = \mathbf{q}(t, \mathbf{x}, \mathbf{y}), \\ \mathbf{x}(0) = \bar{\mathbf{x}}, \\ \mathbf{y}(T) = \bar{\mathbf{y}}, \end{cases} \tag{4.9}$$

where $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$, and $\mathbf{p}: \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{q}: \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Assume that $\mathbf{p}$ and $\mathbf{q}$ are bounded and satisfy a Lipschitz condition relative to $\mathbf{x}$ and $\mathbf{y}$ with constant $C > 0$. Then solutions of System (4.9) are unique if T is sufficiently small.

Proof of Theorem 2. Let H be defined by

$$
\begin{aligned}
H(\mathbf{x},\mathbf{u},\lambda) = &\frac{\gamma_1(x_1-N_1)^2+(1-\gamma_1)u_1^2+\gamma_2(x_2-N_2)^2+(1-\gamma_2)u_2^2}{2} \\
&+\lambda_1(r_1x_1(a_1(x_1)-f_1(x_1,x_2))-u_1\eta(x_1-N_1)x_1) \\
&+\lambda_2(r_2x_2(a_2(x_2)-f_2(x_1,x_2))-u_2\eta(x_2-N_1)x_2).
\end{aligned} \tag{4.10}
$$

Then System (2.1), (2.8) – (2.11) can be obtained by direct computation using Lemma 4. By (4.10),

$$\frac{\partial H}{\partial \mathbf{u}} = 0$$

implies that

$$\left(\frac{\lambda_1(t)\eta(x_1-N_1)x_1(t)}{1-\gamma_1}, \frac{\lambda_2(t)\eta(x_2-N_2)x_2(t)}{1-\gamma_2}\right)$$

is the unique critical point of H. Then (2.12) follows from (4.8).

It is clear that (2.1), (2.8) – (2.11) is a system with both initial and terminal conditions defined at $t=0$ and $t=T$ respectively. Furthermore, by Lemma 5, System (2.1), (2.8) – (2.11) has a unique solution when T is sufficiently small. Therefore, $\mathbf{u}^*$ is unique when T is small enough.

5. Conclusion and discussion

In this paper, a general facultative mutualistic model with harvesting is proposed. Instead of the long-term dynamical behavior, a type of short-term optimal feedback control problem is investigated. The existence and the necessary conditions of the optimal control are proved. Numerical simulations are carried out to demonstrate the applications as well. Our results show that it is feasible to interfere a facultative mutualistic system with minimal harvesting efforts. The proposed model covers many existing facultative mutualistic models as special cases. The results on the associated optimal control problem provide a generic solution to develop optimal harvesting strategies for the models in the literature.

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