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ANALYSIS OF STAGE–STRUCTURED MODEL WITH MIXED TYPE OF FUNCTIONAL RESPONSE AND IMPULSIVE BIOLOGICAL CONTROL

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This paper is dedicated to Professor V. Lakshmikantham
for his marvelous contribution to Theory of Impulsive Differential Equations.

(*Communicated by P. Korman*)

Abstract. The aim of this paper is to study a stage-structured pest management model with mixed type of functional response i.e., Holling type-I and Beddington-DeAngelis functional response with impulsive biological control. Stage structuring is proposed due to the fact that almost all the pests in their life pass through two stages namely, immature larva and mature adult. It is assumed that immature susceptible pests and exposed pests are attacked by a natural enemy and susceptible pests (immature and mature) are contacted by infected pests which make them exposed. Infected pests and natural enemies are infused impulsively after fixed intervals. All positive solutions are proved to be uniformly ultimately bounded. The stability analysis of pest extinction periodic solution, as well as the permanence of system, are obtained by making use of floquet's theory, small amplitude perturbation technique, and comparison theorem. The results obtained provide certain dependable theoretical findings for effective pest management. At last, theoretical findings are confirmed by means of numerical simulation.

1. Introduction

Farmers have a vast scope of pest control methods including physical control (by killing the small insects and rodents or by setting up barriers), chemical control (by spraying pesticides) and biological control (by releasing predators or infected pests). Although chemical control method controls the pest population effectively in a very short interval, however excessive use of pesticides is very harmful to a human being as well as other natural enemies of pests. Moreover, some authors indicated that prolonged use of same chemical pesticides led to the development of resistance in a number of pests with respect to that pesticide which leads to farmer's loss and forces them to use strong pesticides [1, 10]. On the contrary in biological control, pest population is suppressed by releasing the natural enemies of pests or by spreading infection among the pest population. For example, Bollworm is a pest which attacks cotton and Bacillus

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Thruingirnsis is a natural enemy of Bollworm which saves the cotton. The use of cactoblastosis beetle to control its pest cactus is the world's most monumental successful example via biological technique. As another example, Aphid pests are controlled by their natural enemy named Lady beetle. The use of one or more biological control is an alternative to chemical pesticides and a combination of two or more biological control methods with or without the use of chemical pesticides come under the category of Integrated Pest Management.

Further, the pest control models can be well constructed using impulsive differential equations, as the pest controlling agents are released periodically. A number of authors have successfully developed pest controlling models by means of impulsive differential equations [2, 13, 16, 18]. Kumar et al. [7] studied a food chain model with gestation delay in natural enemies and pests. After studying stability at equilibrium points, the authors used sensitivity analysis to study the sensitive indices of the variables in the proposed system. Kumar et al. [8] extended their paper [7] to a stage-structured plant-pest-natural enemy food chain model with gestation delays in mature pests as well as mature natural enemies. The authors studied the stability of various feasible equilibrium points and showed that larger gestation period in pests brings oscillations in the system. Recently the authors of, [15, 17] worked on susceptible-exposed-infected (SEI) pest management models. Gupta et al. [24] studied the dynamics of the plant-pest-natural enemy food chain model. However susceptible-exposed-infected-natural enemy (SEIN) models are more important as they give more significant results from the biological viewpoint. Following is the SEIN pest management model studied by Mathur and Dhar [11]:

$$\left\{\begin{array}{l}
\left.\begin{array}{l}
\frac{dS(t)}{dt} = rS(t)(1-\frac{S(t)}{K}) - \frac{\beta_1 S(t)I(t)}{1+mS(t)} - \beta_2 S(t)N(t), \\
\frac{dE(t)}{dt} = \frac{\beta_1 S(t)I(t)}{1+mS(t)} - (\alpha+\mu_1)E(t), \\
\frac{dI(t)}{dt} = \alpha E(t) - \mu_1 I(t), \\
\frac{dN(t)}{dt} = \frac{\gamma\beta_2 S(t)N(t)}{1+h\beta_2 S(t)} - \mu_2 N(t),
\end{array}\right\} t \neq nT, \\
\left.\begin{array}{l}
\Delta S(t) = 0, \\
\Delta E(t) = 0, \\
\Delta I(t) = p_1, \\
\Delta N(t) = p_2,
\end{array}\right\} yt = nT,
\end{array}\right. \tag{1}$$

where $S(t), E(t), I(t)$ and $N(t)$ are densities of susceptible pests, exposed pests, infected pests and natural enemies respectively, at any time t. Susceptible pests $S(t)$ in absence of $I(t)$ and $N(t)$ grows logistically with carrying capacity K and the intrinsic birth rate r, $\frac{\beta_1 S(t)I(t)}{1+mS(t)}$ is the Holling type II functional response at which the susceptible pests get exposed, β_2 is predation rate by natural enemy, but natural enemy does not predate on exposed and infected pests, γ is rate of conversion of pests to natural enemy, μ_1 is natural death rate of exposed and infected pests, μ_2 is natural death rate of natural enemy.

In this paper, the model (1) is extended to a stage-structured and Beddington-DeAngelis functional response model, which is more significant from the biological viewpoint.

Stage structuring of pests is proposed due to the fact that almost all pests in their life pass through two stages: immature larva and mature adult. Beddington [3] and DeAngelis et al. [5] introduced a functional response given by $F = \frac{\alpha P_1}{a+bP_1+cP_2}$ independently but simultaneously for the reason that predator has to work hard to catch prey and is named as Beddington-DeAngelis functional response, where $P_1 = P_1(t)$ and $P_2 = P_2(t)$ represents the size of the prey population and predator population respectively. The Beddington-DeAngelis functional response is constructed by improving Holling type-II functional response by means of including an extra term that describes mutual interference by the natural enemy. Negi and Gakkhar [13] studied the dynamics in a Beddington-DeAngelis prey-predator system with impulsive harvesting. (Cantrell and Cosner [4], Wang and Huang [16]) also discussed the prey-predator interactions using Beddington-DeAngelis functional response.

In this paper, it is further assumed that immature susceptible pests and exposed pests are attacked by the natural enemy and immature and mature susceptible pest are contacted by infected pest which makes them exposed. Infected pests and natural enemies are impulsively infused at the fixed moment of time.

The present paper is organized as follows. In Section 2, we develop a pest control model with stage structuring and Beddington-DeAngelis functional response with impulsive infusion of infected pests and natural enemies. Some important lemmas and boundedness of the system are established in Section 3. Using Floquet's theory, small amplitude perturbation technique and comparison principles, sufficient conditions for local stability and global attractivity of pest eradication periodic solutions are obtained in Section 4. Permanence of the proposed system is established in Section 5. In order to validate our theoretical findings, numerical simulations and discussions are done in Section 6. The last section covers the conclusion of the paper.

2. Mathematical model

Before proposing the mathematical model describing the complex dynamics, we make the following assumptions:

(A1) Susceptible pests have two life stages namely, immature larva and mature adult.

(A2) Natural enemy attacks immature pest and exposed pests.

(A3) Natural enemies consume immature pests with Beddington-DeAngelis functional response and exposed pests with Holling type-I functional response.

(A4) Infected pest contact with immature and mature susceptible pest with Holling type-I functional response.

(A5) Natural enemy and infected pest populations are infused impulsively.

With these assumptions, the model proposed by Mathur and Dhar[11] is modified and following mathematical model is proposed:

$$\begin{cases} \left.\begin{array}{l} \frac{dS_1(t)}{dt} = r(S_1(t)+S_2(t))(1-\frac{S_1(t)+S_2(t)}{K}) - \frac{hS_1(t)N(t)}{1+\gamma_1 S_1(t)+\gamma_2 N(t)} \\ \qquad -\beta_1 S_1(t)I(t) - \alpha_1 S_1(t) - d_0 S_1(t), \\ \frac{dS_2(t)}{dt} = \alpha_1 S_1(t) - \beta_2 S_2(t)I(t) - d_0 S_2(t), \\ \frac{dE(t)}{dt} = \beta_1 S_1(t)I(t) + \beta_2 S_2(t)I(t) - (\mu+d_1)E(t) \\ \qquad -\alpha_2 E(t)N(t), \\ \frac{dI(t)}{dt} = \mu E(t) - d_2 I(t), \\ \frac{dN(t)}{dt} = \frac{\eta h S_1(t)N(t)}{1+\gamma_1 S_1(t)+\gamma_2 N(t)} + \eta\alpha_2 E(t)N(t) - d_3 N(t), \end{array}\right\} t \neq nT, \\ \left.\begin{array}{l} S_1(t^+) = S_1(t), \\ S_2(t^+) = S_2(t), \\ E(t^+) = E(t), \\ I(t^+) = I(t) + \theta_1, \\ N(t^+) = N(t) + \theta_2, \end{array}\right\} t = nT, \end{cases} \tag{2}$$

where $S_1(t), S_2(t), E(t), I(t)$ and $N(t)$ are densities of immature susceptible pest, mature susceptible pest, exposed pest, infected pest and natural enemy respectively. Susceptible pest grows logistically with carrying capacity K, and growth rate r, α_1 is the maturity rate of immature pests, h is predation rate of immature susceptible pest by natural enemy, β_1 and β_2 are conversion rates of immature and mature susceptible pests to exposed pest respectively, μ is the amount of exposed pests shifted to infected pests, α_2 is the predation rate of exposed pest by natural enemy, η is the conversion rate of predation of exposed pests by natural enemy, d_0, d_1, d_2 and d_3 are natural death rates of susceptible pests, exposed pest, infected pests and natural enemies respectively, γ_1 and γ_2 are half saturation constants of immature susceptible pest and natural enemy respectively, θ_1 is infused amount of infected pests and θ_2 is infused amount of natural enemies at $t = nT, n = 1, 2, \ldots$ and T is the period of impulse effect.

3. Preliminaries

The solution of system (2) is denoted by $Y(t) = (S_1(t), S_2(t), E(t), I(t), N(t))'$ and is a piecewise continuous function $Y : R_+ \to R_+^5$, that is, $Y(t)$ is continuous in the interval $(nT, (n+1)T], n \in Z_+$ and $Y(nT^+) = lim_{t \to nT^+} Y(t)$ exists. The global existence and uniqueness of a solution of the model (2) is guaranteed by the existence results given in [12].

Before proving the main results, we firstly state and establish boundedness of the proposed model (2).

LEMMA 1. For any solution $Y(t) = (S_1(t), S_2(t), E(t), I(t), N(t))$ of system (2) $\exists$ a constant $L > 0$ such that $S_1(t) \leqslant L, S_2(t) \leqslant L, E(t) \leqslant L, I(t) \leqslant L$ and $N(t) \leqslant L$ with t large enough.

Proof. Define $U(t) = \eta S_1(t) + \eta S_2(t) + \eta E(t) + \eta I(t) + N(t)$ and let $0 < d < \min\{d_0, d_1, d_2, d_3\}$. Then for $t \neq nT$, we obtain that

$$D^+U(t) + dU(t) \leqslant \frac{\eta rK}{4} = L_0,$$

where $D^+U(t)$ denotes the derivative of $U(t)$ as given in definition 3.1 of [9].
Now at impulsive point $t = nT, U(t^+) \leqslant U(t) + \theta_1 + \theta_2$.
By Lemma 3.1 of Jatav and Dhar [6] for $t \in (nT, (n+1)T]$, we have

$$\begin{aligned} U(t) &\leqslant U(0)exp(-dt) + \int_0^t L_0 exp(-d(t-s))ds + \sum_{0<nT<t} (\theta_1 + \theta_2)exp(-d(t-nT)) \\ &\to \frac{L_0}{d} + \frac{(\theta_1+\theta_2)exp(-dT)}{exp(dT)-1}, \text{as } t \to \infty, \end{aligned} \tag{3}$$

which shows that $U(t)$ is uniformly ultimately bounded. Thus we get a constant $L := \frac{L_0}{d} + \frac{(\theta_1+\theta_2)exp(-dT)}{exp(dT)-1}$ such that $S_1(t) \leqslant L, S_2(t) \leqslant L, E(t) \leqslant L, I(t) \leqslant L$ and $N(t) \leqslant L$ for all t large enough.
This proves that all the five populations are bounded. ☐

Now we proceed to find pest-extinction periodic solutions for the model (2). For the case of pest-extinction, we remain with the following impulsive subsystem:

$$\begin{cases} \left.\begin{aligned} \frac{dI(t)}{dt} &= -d_2 I(t), \\ \frac{dN(t)}{dt} &= -d_3 N(t), \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} I(t^+) &= I(t) + \theta_1, \\ N(t^+) &= N(t) + \theta_2, \end{aligned}\right\} t = nT. \end{cases} \tag{4}$$

Using Lemma 3.3 of Jatav and Dhar [6], we obtain that

$$I^*(t) = \frac{\theta_1 exp(-d_2(t-nT))}{1-exp(-d_2T)},\ I^*(0^+) = \frac{\theta_1}{1-exp(-d_2T)}, \tag{5}$$

and

$$N^*(t) = \frac{\theta_2 exp(-d_3(t-nT))}{1-exp(-d_3T)},\ N^*(0^+) = \frac{\theta_2}{1-exp(-d_3T)} \tag{6}$$

are positive solutions of the subsystem (4) and are globally asymptotically stable. Thus $(0,0,0,I^*(t),N^*(t))$ is the pest-extinction periodic solution of (2).

4. Stability analysis

In this section, we will discuss local and global stability of pest-extinction periodic solution using Floquet theory and small amplitude perturbation technique.

THEOREM 1. The pest-extinction periodic solution $(0,0,0,I^*(t),N^*(t))$ is locally stable provided $T \leqslant T^*$, where

$$\begin{aligned} T^* = &\frac{1}{r-\alpha-d_0}\left[\frac{\beta_1\theta_1}{d_2} - \frac{h}{\gamma_2 d_3}\log\left(\frac{1-\exp(-d_3T)+\gamma_2\theta_2\exp(-d_3T)}{1-\exp(-d_3T)+\gamma_2\theta_2}\right)\right] \\ &-\frac{1}{r-\alpha-d_0}\left[\frac{\alpha r}{d_0 d_2}\log\left(\frac{\beta_2\theta_1+d_0(1-\exp(-d_2T))\exp(d_2T)}{\beta_2\theta_1+d_0(1-\exp(-d_2T))}\right)\right]. \end{aligned}$$

Proof. For the local stability of periodic solution $(0,0,0,I^*(t),N^*(t))$, we define $S_1(t) = \phi_1(t), S_2(t) = \phi_2(t), E(t) = \phi_3(t), I(t) = I^*(t) + \phi_4(t), N(t) = N^*(t) + \phi_5(t)$, where $\phi_i(t), i = 1,2,...,5$ are small amplitude perturbations of the solution respectively, then system (2) in linearized form becomes:

$$\left\{\begin{array}{l} \left.\begin{aligned} \frac{d\phi_1(t)}{dt} &= r\phi_1(t) + r\phi_2(t) - \frac{h\phi_1(t)N^*(t)}{1+\gamma_2N^*(t)} - \alpha_1\phi_1(t) \\ &\quad -\beta_1\phi_1(t)I^*(t) - d_0\phi_1(t), \\ \frac{d\phi_2(t)}{dt} &= \alpha_1\phi_1(t) - \beta_2I^*(t)\phi_2(t) - d_0\phi_2(t), \\ \frac{d\phi_3(t)}{dt} &= \beta_1I^*(t)\phi_1(t) + \beta_2I^*(t)\phi_2(t) - (\mu+d_1)\phi_3(t) \\ &\quad -\alpha_2N^*(t)\phi_3(t), \\ \frac{d\phi_4(t)}{dt} &= \mu\phi_3(t) - d_2\phi_4(t), \\ \frac{d\phi_5(t)}{dt} &= \frac{\eta hN^*(t)\phi_1(t)}{1+\gamma_2N^*(t)} + \eta\alpha_2N^*(t)\phi_3(t) - d_3\phi_5(t), \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} \phi_1(t^+) &= \phi_1(t), \\ \phi_2(t^+) &= \phi_2(t), \\ \phi_3(t^+) &= \phi_3(t), \\ \phi_4(t^+) &= \phi_4(t), \\ \phi_5(t^+) &= \phi_5(t), \end{aligned}\right\} t = nT. \end{array}\right. \tag{7}$$

Let $\Phi(t)$ be the fundamental matrix of (7), then

$$\frac{d\Phi(t)}{dt} = A\Phi(t), \tag{8}$$

where

$$\begin{aligned} A = &\begin{pmatrix} r - \frac{hN^*(t)}{1+\gamma_2N^*(t)} - \beta_1I^*(t) - \alpha - d_0 & r & 0 & 0 & 0 \\ \alpha_1 & -\beta_2I^*(t) - d_0 & 0 & 0 & 0 \\ \beta_1I^*(t) & \beta_2I^*(t) & -(\mu+d_1+\alpha_2N^*(t)) & 0 & 0 \\ 0 & 0 & \mu & -d_2 & 0 \\ \frac{\eta hN^*(t)}{1+\gamma_2N^*(t)} & 0 & \eta\alpha_2N^* & 0 & -d_3 \end{pmatrix} \\ \approx &\begin{pmatrix} r - \frac{hN^*(t)}{1+\gamma_2N^*(t)} - \beta_1I^*(t) - \alpha_1 - d_0 + \frac{\alpha_1 r}{\beta_2I^*(t)+d_0} & 0 & 0 & 0 & 0 \\ \alpha_1 & -\beta_2I^*(t) - d_0 & 0 & 0 & 0 \\ \beta_1I^*(t) & \beta_2I^*(t) & -(\mu+d_1+\alpha_2N^*(t)) & 0 & 0 \\ 0 & 0 & \mu & -d_2 & 0 \\ \frac{\eta hN^*(t)}{1+\gamma_2N^*(t)} & 0 & \eta\alpha_2N^* & 0 & -d_3 \end{pmatrix}. \end{aligned}$$

The linearization of impulsive conditions of (2) i.e. equations sixth to tenth of (2) becomes

$$\begin{pmatrix}\phi_1(t^+)\\ \phi_2(t^+)\\ \phi_3(t^+)\\ \phi_4(t^+)\\ \phi_5(t^+)\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}\phi_1(t)\\ \phi_2(t)\\ \phi_3(t)\\ \phi_4(t)\\ \phi_5(t)\end{pmatrix}.$$

Thus the monodromy matrix of (7) is

$$M=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\Phi(T).$$

From (8), we obtain that $\Phi(T)=\Phi(0)exp(\int_0^T A dt)$, where $\Phi(0)$ is identity matrix. Then the eigen values of the monodromy matrix M are:

$$\begin{aligned}
\lambda_1 &= exp\left(\int_0^T\left(r-\frac{hN^*(t)}{1+\gamma_2N^*(t)}-\beta_1I^*(t)-\alpha_1-d_0+\frac{\alpha_1 r}{\beta_2I^*(t)+d_0}\right)dt\right),\\
\lambda_2 &= exp\left(\int_0^T(-\beta_2I^*(t)-d_0)dt\right)=\exp\left(-\frac{\beta_2\theta_1}{d_2}\right)<1,\\
\lambda_3 &= exp\left(-\int_0^T(\mu+d_1+\alpha_2N^*(t)dt\right)<1,\\
\lambda_4 &= exp(-d_2T)<1,\\
\lambda_5 &= exp(-d_3T)<1.
\end{aligned}$$

Thus Floquet theory of impulsive differential equations implies that the pest extinction periodic solution of the system (2) is locally asymptotically stable if and only if $|\lambda_1|\leqslant 1$, that is $T\leqslant T^*$. Hence the result. $\square$

THEOREM 2. The pest-extinction periodic solution $(0,0,0,I^*(t),N^*(t))$ of (2) is globally attractive provided $T<T^*$.

Proof. As proved in theorem above, the system (2) is locally stable provided $T<T^*$, so we can choose $\varepsilon_1>0$ small enough such that

$$\int_{nT}^{(n+1)T}\left(r-\frac{h(N^*(t)-\varepsilon_1)}{1+\gamma_2(N^*(t)-\varepsilon_1)}-\beta_1(I^*(t)-\varepsilon_1)-\alpha_1-d_0+\frac{\alpha_1 r}{\beta_2(I^*(t)-\varepsilon_1)+d_0}\right)dt:=q<0.$$

From the fourth and ninth equations of (2), we get

$$\begin{cases}\frac{dI(t)}{dt}\geqslant -d_2I(t),\ t\neq nT,\\ I(t^+)=I(t)+\theta_1,\ t=nT.\end{cases}$$

Consider the following auxiliary system

$$\begin{cases} \frac{dw_1(t)}{dt} = -d_2 w_1(t),\ t \neq nT, \\ w_1(t^+) = w_1(t) + \theta_1,\ t = nT. \end{cases} \tag{9}$$

Using Lemma 3.3 of Jatav and Dhar [6], we get that the system (9) has a periodic solution

$$w_1^*(t) = \frac{\theta_1 exp(-d_2(t-nT))}{1-exp(-d_2T)},\ nT < t \leqslant (n+1)T,\ n \in Z_+,$$

which is globally asymptotically stable. In view of Lemma 3.3 of Jatav and Dhar [6] and the comparison theorem of the impulsive differential equations by Lakshmikantham et al. [12], we have $I(t) \geqslant w_1(t)$ and $w_1(t) \to w_1{}^*(t)$ as $t \to \infty$. Then $\exists\, k_1 > 0$ such that

$$I(t) \geqslant w_1(t) > I^*(t) - \varepsilon_1,\ nT < t \leqslant (n+1)T,\ n > k_1. \tag{10}$$

Now from fifth and tenth equations of (2), we obtain the following subsystem

$$\begin{cases} \frac{dN(t)}{dt} \geqslant -d_3 N(t),\ t \neq nT, \\ N(t^+) = N(t) + \theta_2,\ t = nT. \end{cases}$$

Consider the following auxiliary system

$$\begin{cases} \frac{dw_2(t)}{dt} = -d_3 w_2(t),\ t \neq nT, \\ w_2(t^+) = w_2(t) + \theta_2,\ t = nT. \end{cases} \tag{11}$$

Using Lemma 3.3 of Jatav and Dhar [6], we obtain that the system (11) has a periodic solution

$$w_2^*(t) = \frac{\theta_2 exp(-d_3(t-nT))}{1-exp(-d_3T)},\ nT < t \leqslant (n+1)T,\ n \in Z_+,$$

which is globally asymptotically stable. In view of Lemma 3.3 of Jatav and Dhar [6] and the comparison theorem of the impulsive differential equations by Lakshmikantham et al. [12], we have $N(t) \geqslant w_2(t)$ and $w_2(t) \to w_2{}^*(t)$ as $t \to \infty$. Then $\exists$ positive integer $k_2 (k_2 > k_1)$ such that

$$N(t) \geqslant w_2(t) > N^*(t) - \varepsilon_1,\ nT < t \leqslant (n+1)T,\ n > k_2. \tag{12}$$

Now first equation of (2) can be written as

$$\frac{dS_1(t)}{dt} \leqslant \left(r - \alpha_1 - d_0 - \frac{h(N^*(t)-\varepsilon_1)}{1+\gamma_2(N^*(t)-\varepsilon_1)} - \beta_1(I^*(t)-\varepsilon_1) + \frac{\alpha_1 r}{\beta_2(I^*(t)-\varepsilon_1)+d_0} \right) S_1(t).$$

Integrating the above equation between the pulses, we get

$$S_1(t) \leqslant S_1(nT) exp(q),$$

where q is defined at the start of the proof.
After the successive pulse, we can obtain the following stroboscopic map

$$S_1((n+1)T^+) \leqslant S_1(nT^+) exp(q),$$

where $q < 0$ is as defined above. Thus $S_1(nT^+) \leqslant S_1(0^+)\exp(qn)$ and so $S_1(nT^+) \to 0$ as $n \to \infty$.
Therefore $S_1(t) \to 0$ as $n \to \infty$. Therefore for $\varepsilon_2 > 0$, small enough and $\exists$ positive integer $k_3(k_3 > k_2)$ such that

$$S_1(t) \ < \varepsilon_2, \forall\, nT < t \leqslant (n+1)T, \, n > k_3.$$

From second equation of system (2), we obtain

$$\frac{dS_2(t)}{dt} \leqslant \alpha_1 \varepsilon_2 - d_0 S_2(t).$$

Similarly, for $\varepsilon_2 > 0$, small enough, $\exists$ positive integer $k_4(k_4 > k_3)$ such that

$$S_2(t) \ \leqslant \varepsilon_2, \forall\, nT < t \leqslant (n+1)T, \, n > k_4.$$

From third equation of system (2) we have

$$\frac{dE(t)}{dt} \leqslant -(\beta_1 + \beta_2)\varepsilon_2 - (\mu + d_1)E(t).$$

Integrating the above equation between the pulses, sloving in a similar manner as above, we can choose $\varepsilon_3 > 0$, small enough and $\exists$ positive integer $k_5(k_5 > k_4)$ such that $E(t) < \varepsilon_3 \; \forall\, nT < t \leqslant (n+1)T, \, n > k_5$,
where

$$\varepsilon_3 = \frac{(\beta_1 + \beta_2)\varepsilon_2}{\mu + d_1}.$$

Again from fourth and ninth equation of system (2), we obtain

$$\begin{cases} \frac{dI(t)}{dt} \leqslant \mu\varepsilon_3 - d_2 I(t), \, t \neq nT, \\ I(t^+) = I(t) + \theta_1, \, t = nT. \end{cases}$$

Consider the following comparison system

$$\begin{cases} \frac{dw_3(t)}{dt} = \mu\varepsilon_3 - d_2 w_3(t), \, t \neq nT, \\ w_3(t^+) = w_3(t) + \theta_1, \, t = nT. \end{cases} \tag{13}$$

Again using Lemma 3.3 of Jatav and Dhar [6], we obtain that the system (13) has a periodic solution

$$w_3^*(t) = \frac{\mu\varepsilon_3}{d_2} + \frac{\theta_1 exp(-d_2(t-nT))}{1 - exp(-d_2 T)}, \, nT < t \leqslant (n+1)T, \, n \in Z_+,$$

which is globally asymptotically stable. In view of Lemma 3.3 of Jatav and Dhar [6] and the comparison theorem of the impulsive differential equations we have $I(t) \leqslant w_3(t)$ and $w_3(t) \to {w_3}^*(t)$ as $t \to \infty$. Then $\exists$ positive integer $k_6(k_6 \ > \ k_5)$ such that

$$I(t) \leqslant w_3(t) < w_3^* + \varepsilon_1, \, nT < t \leqslant (n+1)T, \, n > k_6. \tag{14}$$

From fifth and tenth equation of system (2), we obtain that

$$\begin{cases} \frac{dN(t)}{dt} \leqslant -(d_3 - \eta\alpha_2\varepsilon_3 - \eta h\varepsilon_1)N(t), t \neq nT, \\ N(t^+) = N(t) + \theta_2, t = nT. \end{cases}$$

In a similar manner, there exists positive integer $k_7 (k_7 > k_6)$ such that

$$N(t) \leqslant \tilde{N}(t) + \varepsilon_1,\ nT < t \leqslant (n+1)T,\ n > k_7. \tag{15}$$

where

$$\tilde{N}(t) = \frac{\theta_2 \exp(-(d_3 - \eta\alpha_2\varepsilon_3 - \eta h\varepsilon_1)(t - nT))}{1 - \exp(-(d_3 - \eta\alpha_2\varepsilon_3 - \eta h\varepsilon_1))},\ nT < t \leqslant (n+1)T,\ n \in Z_+.$$

Since $\varepsilon_1, \varepsilon_2$ and ε_3 are small enough, we have $w_1^* \to I^*(t), \tilde{N}(t) \to N^*(t)$ as $\varepsilon_3 \to 0$. Thus it is obtain that $S_1(t) \to 0, S_2(t) \to 0, E(t) \to 0, I(t) \to I^*(t)$ and $N(t) \to N^*(t)$ as $t \to \infty$. Hence $(0,0,0,I^*(t),N^*(t))$ is globally attractive. □

REMARK 1. Theorem 1 and 2 imply that necessary and sufficient condition for the pest-extinction periodic solution $(0,0,0,I^*(t),N^*(t))$ to be globally asymptotically is $T < T^*$.

5. Permanence

The populations $S_1(t), S_2(t), E(t), I(t)$ and $N(t)$ of system (2) are said to be permanent if there exist positive constants n, N and T_0 such that each positive solution of system (2) satisfies $n \leqslant S_1(t), S_2(t), E(t), I(t), N(t) \leqslant N$, $\forall\, t \geqslant T_0$. In this section permanence of the system (2) is proved.

THEOREM 3. The system (2) is permanent if $T > T^*$.

Proof. Let $(S_1(t), S_2(t), E(t), I(t), N(t))$ be any solution of (2). From the Lemma 1, we have $S_1(t) \leqslant L, S_2(t) \leqslant L, E(t) \leqslant L, I(t) \leqslant L$ and $N(t) \leqslant L\ \forall\, t \geqslant 0$.
From equations (10) and (12), it is obtained that

$$I(t) > I^*(t) - \varepsilon_1 := a_1 > 0,\ nT < t \leqslant (n+1)T,\ n \geqslant k_1,$$
$$N(t) > N^*(t) - \varepsilon_1 := a_2 > 0,\ nT \leqslant t \leqslant (n+1)T,\ n \geqslant k_2.$$

Therefore, $I(t)$ and $N(t)$ are bounded below.
Now, for the permanence of the model (2), we only need to find a_3, a_4 and a_5 such that $S_1(t) \geqslant a_3, S_2(t) \geqslant a_4$ and $E(t) \geqslant a_5$ for t being large enough.
First, we prove that $S_1(t) \geqslant a_3$ for t large enough and the result is proved in two steps.
Step I First suppose that $S_1(t) \geqslant a_3$ is not true, $\exists\, K_1 \in Z_+$, such that $S_1(t) < a_3$, $\forall\, t \geqslant K_1 T$. Using this assumption, we are left with following subsystem of (2).

$$\begin{cases} \frac{dI(t)}{dt} \leqslant \mu L - d_2 I(t),\ t \neq nT, \\ I(t^+) = I(t) + \theta_1,\ t = nT. \end{cases}$$

Consider the auxiliary system

$$\begin{cases} \frac{dv_1(t)}{dt} = \mu L - d_2 v_1(t),\ t \neq nT, \\ v_1(t^+) = v_1(t) + \theta_1,\ t = nT. \end{cases} \tag{16}$$

Using Lemma 3.3 of Jatav and Dhar [6], the system (16) has a periodic solution

$$v_1^*(t) = \frac{\mu L}{d_2} + \frac{\theta_1 exp(-d_2(t-nT))}{1 - exp(-d_2 T)},$$

which is globally asymptotically stable. Then, $\exists$ an integer $K_2 (K_2 > K_1)$ such that $I(t) \leqslant v_1^*(t) < v_1^*(t) + \varepsilon_5$, for $t \geqslant K_2 T$.
Again from the model (2), we obtain the following subsystem:

$$\begin{cases} \frac{dN(t)}{dt} \leqslant -(d_3 - \eta h a_3 - \eta \alpha_2 L) N(t),\ t \neq nT, \\ N(t^+) = N(t) + \theta_2,\ t = nT. \end{cases}$$

Again considering auxiliary system:

$$\begin{cases} \frac{dv_2(t)}{dt} = -(d_3 - \eta h a_3 - \eta \alpha_2 L) v_2(t),\ t \neq nT, \\ v_2(t^+) = v_2(t) + \theta_2,\ t = nT, \end{cases} \tag{17}$$

and using similar analysis, we get that $\exists$ an integer $K_3 (K_3 > K_2)$ such that

$$N(t) \leqslant v_2^*(t) < v_2^*(t) + \varepsilon_5,\ \text{for}\ t \geqslant K_3 T.$$

Thus, from the system (2), we get that

$$\frac{dS_1(t)}{dt} \geqslant \left(r - \alpha_1 - d_0 - h(v_2^*(t) + \varepsilon_5) - \beta_1(v_1^*(t) + \varepsilon_5) + \frac{\alpha_1 r}{\beta_2(v_1^*(t) + \varepsilon_5) + d_0} \right) S_1(t),$$

integrating the above equation between the pulses, we get

$$S_1(t) \geqslant S_1(K_2 T) \exp(\sigma_2),$$

where

$$\sigma_2 = \int_{K_2 T}^{t} \left(r - \alpha_1 - d_0 - h(v_2^*(t) + \varepsilon_5) - \beta_1(v_1^*(t) + \varepsilon_5) + \frac{\alpha_1 r}{\beta_2(v_1^*(t) + \varepsilon_5) + d_0} \right) dt.$$

After k successive pulses, we obtain the following stroboscopic map

$$S_1((K_2 + l)T) \geqslant S_1(K_2 T) \exp(\sigma_2),$$
$$S_1(K_2 T^+) \geqslant S_1(0^+) \exp(k\sigma_2).$$

As $\sigma_2 > 1$ when $T > T^*$, thus it is obtained that, $S_1(K_2 T^+) \to +\infty$ as $k \to +\infty$, which is a contradiction to the boundedness of $S_1(t)$.Therefore $S_1(t) \geqslant a_3$, for $\mathrm{t}_1 > \mathrm{K}_1\mathrm{T}$ for some $\mathrm{K}_1 \in \mathrm{Z}_+$.

Step II If $S_1(t) \geqslant a_3$, $\forall\, t \geqslant t_1$, then we are done. Otherwise, $S_1(t) < a_3$ for some $t > t_1$. Let $t^* = \inf\{t|S_1(t) < a_3, t > t_1\}$, then we have, $S_1(t) \geqslant a_3$ for $t \in [t, t^*)$ and $t^* \in (b_1T, (b_1+1)T), b_1$ is some positive integer. Continuity of $S_1(t)$ implies that $S_1(t^*) = a_3$. Assume that $T_1 = b_2T + b_3T$ where b_2 and b_3 satisfy the following inequalities:

$$b_2T > \max\left\{\frac{-\ln\frac{\varepsilon_4}{2L}}{d_2}, \frac{-\ln\frac{\varepsilon_4}{2L}}{d_3 - \eta h a_3 - \eta\alpha_2 L}\right\},$$

$$b_3\sigma_3 > \left(\frac{r}{K} + h + \beta_1 + \alpha + d_0\right) L(b_2+1)T.$$

Now, we claim that $\exists$ a time $t' \in ((b_1+1)T, (1+b_1+b_2+b_3)T)$ such that $S_1(t') \geqslant a_3$. On contrary, let

$$S_1(t) < a_3, t \in ((b_1+1)T, (1+b_1+b_2+b_3)T).$$

If the system (16) is considered with initial value $v_1((b_1+1)T^+) = I((b_1+1)T^+)$, then using Lemma 3.3 of Jatav and Dhar [6], for $t \in (nT, (n+1)T]$ and $b_1 < n \leqslant b_1 + b_2 + b_3$, we get

$$v_1(t) = (v_1((b_1+1)T^+) - \left(\frac{\mu L}{d_2} + \frac{\theta_1}{exp(-d_2T)}\right) exp(-d_2(t-(b_1+1)T) + v_1^*(t),$$

which confirm that

$$|v_1(t) - v_1^*(t)| < 2L\exp(-d_2(t-(b_1+1)T)) < \varepsilon_5,$$

and

$$I(t) \leqslant v_1(t) < v_1^*(t) + \varepsilon_5,\ \forall\,(b_1+1)T \leqslant t \leqslant (b_1+b_2+b_3+1)T.$$

Now, we consider the system (17) with initial value $v_2((b_1+b_2+1)T^+) = N((b_1+b_2+1)T^+)$. Similarly using Lemma 3.3 of Jatav and Dhar [6], we get that

$$|v_2(t) - v_2^*(t)| < 2L\exp(-d_3 - \eta h a_3 - \eta\alpha_2 L)(t-(b_1+b_2+1)T)) < \varepsilon_5,$$

and

$$N(t) \leqslant v_2(t) < v_2^*(t) + \varepsilon_5,\ \forall\,(b_1+b_2+1)T \leqslant t \leqslant (b_1+b_2+b_3+1)T.$$

Thus, we have

$$\frac{dS_1(t)}{dt} \geqslant \left(r - \alpha_1 - d_0 - h(v_2^*(t)+\varepsilon_5) - \beta_1(v_1^*(t)+\varepsilon_5) + \frac{\alpha_1 r}{\beta_2(v_1^*(t)+\varepsilon_5)+d_0}\right) S_1(t),$$

$\forall (b_1+b_2+1)T \leqslant t \leqslant (b_1+b_2+b_3+1)T$. Integrating on $[b_1+b_2+1)T, (b_1+b_2+b_3+1)T]$, we obtain

$$S_1((b_1+b_2+b_3+1)T) \geqslant S_1((b_1+b_2+1)T)\exp(b_3\sigma_3). \tag{18}$$

Now, for $t \in (t^*, (b_1+1)T)$, two possible cases arise:
Case (a): If $S_1(t) < a_3$ for $t \in (t^*, (b_1+1)T)$ then $S_1(t) < a_3$ for $t \in (t^*, (b_1+b_2+1)T)$. Therefore, we get

$$\frac{dS_1(t)}{dt} \geqslant \left[\frac{-r}{K} - h - \beta_1 - \alpha_1 - d_0\right] LS_1(t) := \zeta S_1(t). \tag{19}$$

Integrating above inequality in the interval $[t^*, (b_1+b_2+1)T]$, we get that

$$S_1((b_1+b_2+1)T) \geqslant S_1(b_1T)\exp(\zeta(b_2+1)T). \tag{20}$$

Using the value of $S_1((b_1+b_2+1)T)$ from (20) into (18), we get

$$\begin{aligned} S_1((b_1+b_2+b_3+1)T) &\geqslant S_1(b_1T)\exp(b_3\sigma_3)\exp(\zeta(b_2+1)T) \\ &\geqslant a_3\exp(b_3\sigma_3)\exp(\zeta(b_2+1)T) > a_3, \end{aligned}$$

which is a contradiction. Let $\check{t} = \inf\{t | S_1(t) \geqslant a_3, t > t^*\}$, then $S_1(\check{t}) = a_3$. For $t \in [t^*, \check{t})$ (19) holds and integrating on $t \in [t^*, \check{t})$, we obtain

$$S_1(\check{t}) \geqslant S_1(t^*)\exp(\zeta(t-t^*)) \geqslant a_3\exp(\zeta(1+b_2+b_3)T) := \bar{a}_3.$$

Since $S_1(t) \geqslant a_3$ for $t > \check{t}$, the same argument can be obtained. Thus, we have $S_1(t) \geqslant \bar{a}_3,\ \forall\, t > t_1$.
Case (b): There exists $t^{''} \in (t^*, (b_1+1)T]$ such that $S_1(t^{''}) \geqslant a_3$. Let $\bar{t} = \inf\{t | S_1(t) \geqslant a_3, t > t^*\}$, then $S_1(t) < a_3$ for $t \in [t^*, \bar{t})$ and $S_1(\bar{t}) = a_3$. For $t \in [t^*, \bar{t})$, (19) holds. Integrating (5.4) on $[t^*, \bar{t})$, we have

$$S_1(t) \geqslant S_1(t^*)\exp(\zeta(t-t^*) \geqslant a_3\exp(\zeta T) > \bar{a}_3.$$

The same argument can be continued, since $S_1(\bar{t}) \geqslant \bar{a}_3\ \forall\, t > t_1$ Thus, in both cases, we conclude $S_1(t) \geqslant \bar{a}_3\ \forall\, t \geqslant t_1$. Therefore $S_1(t)$ is ultimately bounded below.
Now, in order to prove that $S_2(t)$ is ultimately bounded below, from system (2) we have

$$\frac{dS_2(t)}{dt} \geqslant \alpha_1\bar{a}_3 - d_0S_2(t).$$

Solving, we obtain

$$S_2(t) \geqslant \frac{\alpha_1\bar{a}_3}{d_0} := a_4.$$

It is easy to prove that $\lim_{t\to\infty}\inf S_2(t) \geqslant a_4$, where $a_4 = \frac{\alpha_1\bar{a}_3}{d_0}$.
Lastly to prove that $E(t)$ is ultimately positively bounded below. We have from system (2):

$$\frac{dE(t)}{dt} \geqslant \beta_1\bar{a}_3a_1 + \beta_2\bar{a}_4a_1 - (\mu+d_1)E(t).$$

Solving, we obtain

$$E(t) \geqslant \frac{(\beta_1\bar{a}_3+\beta_2\bar{a}_4)a_1}{\mu+d_1} := a_5.$$

It can be easily obtain that $\lim_{t\to\infty}\inf E(t) \geqslant a_5$, where $a_5 = \frac{(\beta_1\bar{a}_3+\beta_2\bar{a}_4)a_1}{\mu+d_1}$. Let $a = \min\{a_1, a_2, \bar{a}_3, a_4, a_5\}$ using Step I, Step II and Lemma 1, we get $\wedge = \{(S_1, S_2, E, I, N) : a \leqslant S_1(t), S_2(t), E(t), I(t), N(t) \leqslant L\}$. Hence the result. ☐

6. Numerical simulations and discussion

In this paper, a stage-structured pest management model with mixed type of functional responses i.e., Beddington-DeAngelis and Holling type-I functional responses with the periodic infusion of infected pest and the natural enemy is proposed and discussed. The sufficient conditions, for pest-extinction periodic solution to be locally and globally asymptotically stable, are established. Also under certain condition, the system is proved to be permanent. In this section, we shall verify the established theoretical results by performing numerical simulation on the system (2). To plot the results, we choose the values of parameters and initial values arbitrarily as given in Table 1 and Table 2 respectively. Theorem 1 and 2 imply that the pest extinction periodic solution is locally as well as globally asymptotically stable if and only if $T < T^*$. For the given values of parameters, we have $T^* = 5.8$. Fig. 1 clearly show that if $T < T^* = 5.8$ (in Fig.1a, $T = 5$ and in Fig.1b, $T = 4$), then immature pests, mature susceptible pests and exposed pests eradicate and infected pests and natural enemies exist impulsively. This verifies the results of Theorems 1 and 2.

From a biological point of view, the complete eradication of pest populations is neither desired nor safer. We only need to control the pests up to a level that it may not destroy the crop. Theorem 3 implies that if $T > T^*$ then all the populations coexist and the system becomes permanent. Fig. 2 verifies the result stated that when $T > T^* = 5.8$ (in Fig.2a, $T = 7$ and in Fig.2b, $T = 20$) then all the populations will coexist and hence the system becomes permanent.

Further extensive simulation is performed to analyze the effect of impulsive releasing of infectious pest as well as natural enemies on eradication and permanence of system (2). In Fig. 3 time series of immature pests, mature pests and exposed pests population is plotted for different pulse releasing amounts (i.e., θ_1 and θ_2). It is analyzed that higher the value of pulse releasing the immature pests, mature pests and exposed pests population becomes eradicated.

In Fig. 4 the bifurcating behavior of immature pests population is plotted with respect to infectious pests and natural enemies respectively. It is seen that phase portrait changes from one periodic → two periodic → chaos. In Fig. 5 the bifurcation of mature pests population is seen with respect to infectious pests. Again, it is seen that phase portrait changes from one periodic → two periodic → chaos. In Fig. 6 the bifurcating behavior of exposed pests population is plotted with respect to infectious pests and natural enemies respectively.

Table 1 Parametric values chosen for simulation.

Parameter	Description	Value per week
r	Growth rate of immature susceptible pest	1.7
K	Carring capacity	5
d_0	Natural death rate of immature and mature susceptible pest	0.05
d_1	Natural death rate of exposed pest	0.01
d_2	Natural death rate of infected pest	0.3
d_3	Natural death rate of natural enemy	0.2
α_1	Maturity rate of immature pest	0.7
α_2	Predation rate of exposed pest by natural enemy	0.7
β_1	Conversion rate of immature susceptible pest by infected pest to exposed pest	2
β_2	Conversion rate of mature susceptible pest by infected pest to exposed pest	2.5
h	Predation rate of immature susceptible pest by natural enemy	0.5
μ	Amount of exposed pest shifted to infected pest	0.01
η	Conversion rate of predation by natural enemy	0.2
γ_1	Half saturation constant for immature susceptible pest	0.1
γ_2	Half saturation constant for natural enemy	0.1
θ_1	Impulsive releasing amount of infected pest	1.1
θ_2	Impulsive releasing amount of natural enemy	0.7

Table 2 Initial values chosen for simulation.

Population	Description	Inital value
$S_1(0^+)$	Intial value of immature susceptible pest	0.5
$S_2(0^+)$	Intial value of mature susceptible pest	0.5
$E(0^+)$	Intial value of exposed pest	1.5
$I(0^+)$	Intial value of infected pest	0.8
$N(0^+)$	Intial value of natural enemy	0.5

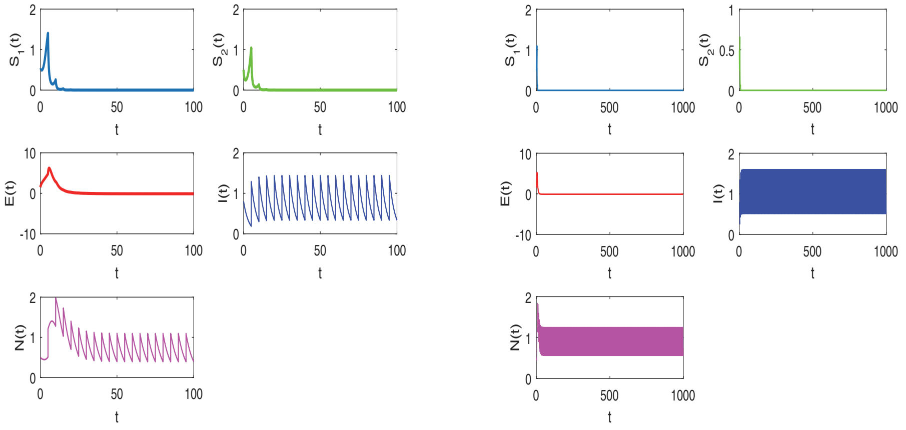

Fig.1a Fig.1b

Figure 1: The pest extinction periodic solution $(0,0,0,I^*(t),N^*(t))$ for $(a)T=5<T^*=5.8$ and $(b)T=4<T^*=5.8$

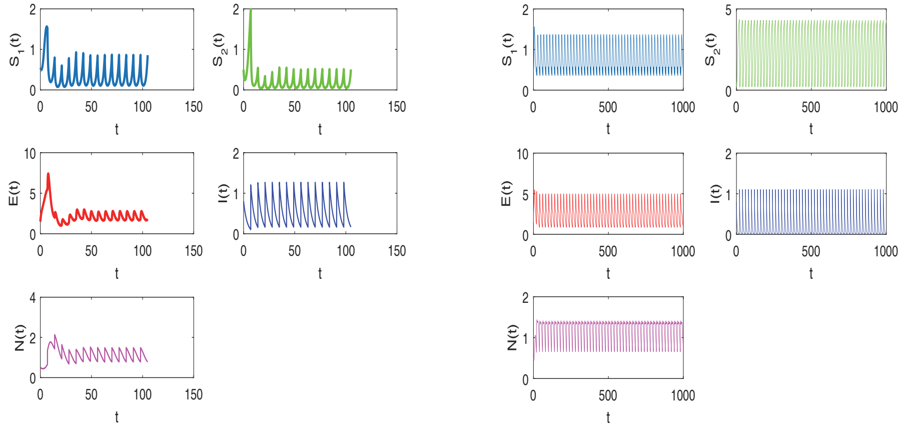

Fig.2a Fig.2b

Figure 2: The time series plot of all populations $S_1(t),S_2(t),E(t),I(t)$ and $N(t)$ for $(a)T=7>T^*=5.8$ and $(b)T=20>T^*=5.8$ showing permanence of the system.

7. Conclusion

In this paper, the effect of impulsive perturbations on stage-structured pest management model with mixed type of functional responses is investigated. The theoretical

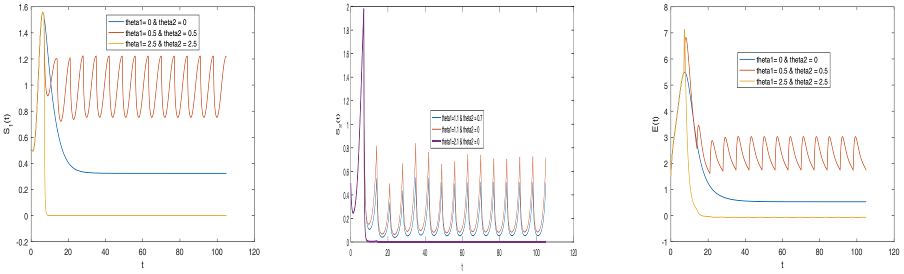

Fig.3a Fig.3b Fig.3c

Figure 3: Time series plots for immature susceptible pests, mature susceptible pests and exposed pests populations for different values of pulse releasing amount (i.e., $(a)S_1(t)$ vs t, $(b)S_2(t)$ vs t and $(c)E(t)$ vs t)

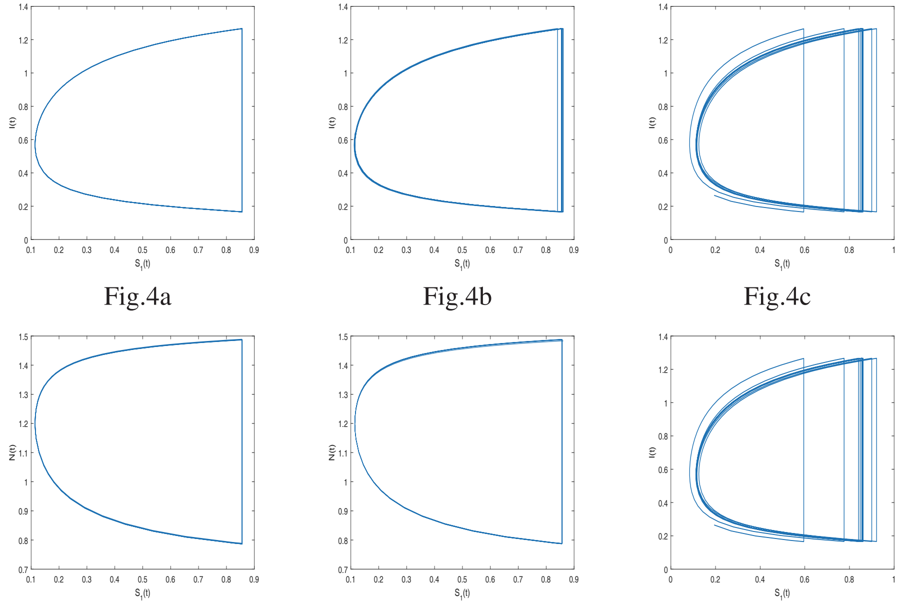

Fig.4a Fig.4b Fig.4c

Fig. 4d Fig.4e Fig.4f

Figure 4: Dynamical behavior of Fig.$4a-4c$ immature susceptible pests verses infectious pests (i.e., $S_1(t)$ vs $I(t)$) and Fig.$4d-4f$ immature susceptible pests verses natural enemies population (i.e., $S_1(t)$ vs $N(t)$)

results obtained are verified numerically. It is concluded that impulsive control strategy plays an important role in the extinction and permanence of species. Our results analyze that immature-mature susceptible pests and exposed pests eradicate completely when the impulsive period is less than some critical value and all the pest populations

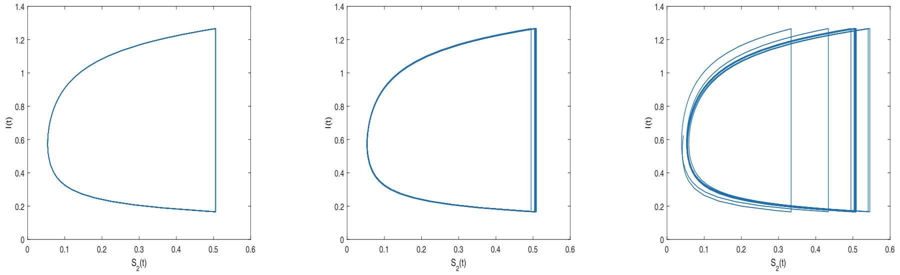

Fig.5a Fig.5b Fig.5c

Figure 5: Dynamical behavior of Fig. $5a - 5c$ mature susceptible pests verses infectious pests (i.e., $S_2(t)$ vs $I(t)$)

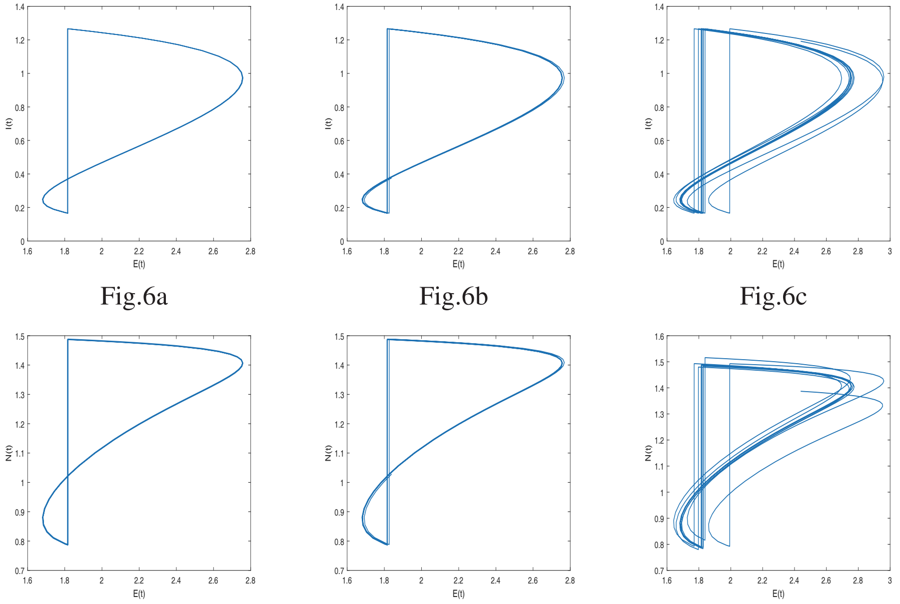

Fig.6a Fig.6b Fig.6c

Fig.6d Fig.6e Fig.6f

Figure 6: Dynamical behavior of Fig. $6a - 6c$ exposed pests verses infectious pests (i.e., $E(t)$ vs $I(t)$) and Fig. $6d - 6f$ exposed pests verses natural enemies population (i.e., $E(t)$ vs $N(t)$)

with natural enemies will coexist when the impulsive period is greater than the critical value. In other words, we can say that the eradication of immature/mature susceptible and exposed pests depends upon the pulse releasing amount as well impulsive period. The analysis also shows that the biological techniques are efficient and sufficient to control the pests.

Key features and future scope

From the environment and farmers point of view, our analysis has the following key features:

1. Found under which condition all the pest populations and natural enemies will co-exist, without harming crops.
2. Found that pulse releasing amount and impulsive period is responsible for eradication/permanence of the system.

It is further stated that, the proposed model can be extended by incorporating maturation time (i.e., delay) from immature pest to mature pest. We will keep on studying these problems in future and try our best to find some nice results.

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