Differential
Equations
& Applications
Volume 12, Number 2 (2020), 129–142
doi:10.7153/dea-2020-12-09

MULTIPLICITY OF SOLUTIONS FOR A FRACTIONAL $p-$KIRCHHOFF TYPE PROBLEM WITH SIGN–CHANGING WEIGHTS FUNCTION

YUYAN GUI

(*Communicated by S. Secchi*)

Abstract. In this paper, we consider the existence of multiple solutions for the following fractional $p-$Kirchhoff type problem

$$\begin{cases} \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dx\,dy\right)^{\theta/p} (-\Delta)_p^s u = f(x)|u|^{q-1}u + g(x)|u|^{r-1}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{0.1}$$

where Ω is an open bounded set in $\mathbb{R}^n$, $p>1$, $\theta \geqslant 0$, $0<q<\theta+p-1<r<p_s^*-1$ with $p_s^* = \frac{np}{n-ps}$ for $n>ps$ and $s\in(0,1)$ fixed, $f(x)$ and $g(x)$ are sign-changing continuous functions in Ω, $(-\triangle)_p^s u$ denotes the fractional $p-$Laplacian operator. We obtain the multiplicity of solutions to (0.1) by using fibering map analysis and the Nehari manifold approach.

1. Introduction

This paper is concerned with the existence of multiple solutions for the following problem

$$\begin{cases} \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dx\,dy\right)^{\theta/p} (-\Delta)_p^s u = f(x)|u|^{q-1}u + g(x)|u|^{r-1}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded set in $\mathbb{R}^n$, $p>1$, $\theta \geqslant 0$, $0<q<\theta+p-1<r<p_s^*-1$ with $p_s^* = np/(n-ps)$ for $n>ps$ and $s\in(0,1)$ fixed, $f(x)$ and $g(x)$ are sign-changing continuous functions in Ω, $(-\Delta)_p^s u$ denotes the fractional $p-$Laplacian operator which (up to normalization factors) can be defined as

$$(-\Delta)_p^s u = 2\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n \backslash B_\varepsilon(x)} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{n+ps}} dy, \quad x\in\mathbb{R}^n, \tag{1.2}$$

consistent, up to some normalization constant depending upon n and s, with the linear fractional Laplacian $(-\Delta)^s$ in the case $p=2$.

Mathematics subject classification (2010): 35J20, 35J92.
Keywords and phrases: Kirchhoff type problem, fractional $p-$Laplacian, fibering map analysis, Nehari manifold approach.

© ELEMENT, Zagreb
Paper DEA-12-09

This paper is motivated by some works which have been focused on the study of fractional Sobolev spaces and corresponding nonlocal equations, both from a pure mathematical point of view and for concrete applications, since they naturally arise in many different contexts. For an elementary introduction on this topic and for a quite extensive list of related references we refer to [12].

Goyal and Sreenadh [17] studied the existence and multiplicity of non-negative solutions to the following problem

$$\begin{cases} (-\Delta)_p^s u(x) = \lambda h(x)|u|^{q-1}u + b(x)|u|^{r-1}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.3}$$

where $p>1$ and $0<q<p-1<r<p_s^*-1$, $\lambda>0$ and h,b are continuous functions. They got the existence and multiplicity of solutions for $\lambda \in (0,\lambda_0)$ with $\lambda_0>0$ by minimization on the suitable subset of Nehari manifold using the fibering maps. The existence and multiplicity of non-negative solutions of (1.3) have been obtained in [18] with respect to the parameter λ, which changes according to whether $0<r<p-1$ or $p-1<r<p_s^*-1$ respectively. On the other hand, the authors in [4, 6, 17, 18, 24, 25, 26] considered the existence and multiplicity of solutions for the general fractional problems with sign-changing weight function.

Moreover, Ferrara et al. [16] studied the multiplicity of solutions for a non-homogeneous p-Kirchhoff type problem driven by a non-local integro-differential operator with concave-convex nonlinearities:

$$\left[a + b\left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\theta-1}\right](-\Delta)_p^s u$$
$$= \lambda \omega_1(x)|u|^{q-1}u + \omega_2(x)|u|^{r-1}u + h(x), \text{ in } \mathbb{R}^{\mathrm{n}},$$

where $a+b>0$ with $a>0$, $b>0$, $\lambda>0$ is a real parameter, $0<s<1$, $0<q<p-1 \leqslant \theta p-1<r<p_s^*-1$, ω_1,ω_2,h are functions which may change sign in $\mathbb{R}^n$. Under some suitable conditions, there exists $\lambda_*>0$, for $\lambda \in (0,\lambda_*)$, it has two non-trivial entire solutions by applying the mountain pass Theorem and Ekeland's variational principle.

Furthermore, the existence of solutions for fractional p-Laplacian problems has been also considered in [18], [20] and references therein. C.Brändle, E. Colorado, and A. de Pablo [3] studied the fractional Laplacian equation involving concave-convex nonlinearities for the subcritical case. The existence and multiplicity of solutions for the fractional Kirchhoff type problem have been investigated in [1, 5, 7, 9, 14, 15, 21, 23], and for fractional $p-$Laplacian system in [8, 10] and references therein.

Inspired by the above mentioned works, our aim is to consider the multiplicity of solutions for (1.1). In order to state our result, we introduce some notations. Let Ω be an open set in $\mathbb{R}^n$, $s\in(0,1)$ and $p\in[1,+\infty)$. We define $W^{s,p}(\Omega)$, the usual fractional Sobolev space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy\right)^{1/p}. \tag{1.4}$$

Set $Q=\mathbb{R}^{2n}\backslash(\mathscr{C}\Omega\times\mathscr{C}\Omega)$ with $\mathscr{C}\Omega=\mathbb{R}^n\backslash\Omega$. We define

$$X=\left\{u \mid u:\mathbb{R}^n\to\mathbb{R} \text{ is measurable, } u|_\Omega\in L^p(\Omega), \text{ and } \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy<\infty\right\}.$$

The space X is endowed with the norm

$$\|u\|_X=\|u\|_{L^p(\Omega)}+\left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)^{1/p}. \tag{1.5}$$

The functional space X_0 denotes the closure of $C_0^\infty(\Omega)$ in X. By [17, Lemma 2.5], the space X_0 is a reflexive Banach space. For any $\phi,\psi\in X_0$, define

$$\langle\phi,\psi\rangle_{X_0}=\int_Q \frac{|\phi(x)-\phi(y)|^{p-2}(\phi(x)-\phi(y))(\psi(x)-\psi(y))}{|x-y|^{n+ps}}dxdy, \tag{1.6}$$

and the norm

$$\|u\|_{X_0}=\left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)^{1/p} \tag{1.7}$$

is equivalent to the usual one defined in (1.4). Since $u=0$ a.e. in $\mathbb{R}^n\backslash\Omega$, we have that the integral in (1.5), (1.6) and (1.7) can be extended to all $\mathbb{R}^n$. By results of [12, 17], the embedding $X_0\hookrightarrow L^r(\Omega)$ is continuous for any $r\in[1,p_s^*]$ and compact whenever $r\in[1,p_s^*)$. For further details on X and X_0 and also for their properties we refer to [12, 17], and the references therein.

In this paper, we assume that $f(x)$ and $g(x)$ are sign-changing continuous functions on Ω and satisfy

(F) $f(x)\not\equiv 0$ and $f(x)\in L^{q_1}(\Omega)$ with $q_1=\frac{p_s^*}{p_s^*-(q+1)}$;

(G) $g(x)\not\equiv 0$ and $g(x)\in L^{r_1}(\Omega)$ with $r_1=\frac{p_s^*}{p_s^*-(r+1)}$.

Let S be the best Sobolev constant for the embedding $X_0\hookrightarrow L^{p_s^*}(\Omega)$. Define

$$\Lambda_{\theta,p,q,r,S}:=\left(\frac{q+1}{\theta+p}\right)^{\frac{1}{(\theta+p-1)-q}}\left(\frac{(\theta+p-1)-q}{r-q}\right)^{\frac{1}{r-(\theta+p-1)}} \times\left(\frac{r-(\theta+p-1)}{r-q}\right)^{\frac{1}{(\theta+p-1)-q}} S^{\frac{(r-q)(\theta+p)}{(r-(\theta+p-1))((\theta+p-1)-q)}}. \tag{1.8}$$

Our result can be stated as follows.

THEOREM 1. *Let $p>1$, $\theta\geqslant 0$ and $0<q<\theta+p-1<r<p_s^*-1$ with $s\in(0,1)$. Assume that f and g are sign-changing continuous functions satisfy (F), (G) and*

$$\|f(x)\|_{L^{q_1}(\Omega)}^{\frac{1}{(\theta+p-1)-q}}\|g(x)\|_{L^{r_1}(\Omega)}^{\frac{1}{r-(\theta+p-1)}}<\Lambda_{\theta,p,q,r,S}, \tag{1.9}$$

then problem (1.1) has at least two nontrivial solutions.

REMARK 1. We use fibering map analysis and Nehari manifold approach to prove Theorem 1. In particular, $f(x)|u|^{q-1}u$ in (1.1) is *not* restricted on $q < p-1$, but the presence of non-local term

$$\left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy\right)^{\frac{\theta}{p}},$$

enables us to treat problem (1.1) as the problem involving concave-convex nonlinearity terms. The power $0 < q < \theta + p - 1 < r < p_s^* - 1$ plays important roles in our arguments.

REMARK 2. There is no any parameter on the nonlinearity terms, we find the conditions on functions $f(x)$ and $g(x)$ such that problem (1.1)exhibits multiple solutions.

When $s = 1$, problem (1.1) reduces to a $p-$Kirchhoff type problem. It has been studied in many literatures, where have proposed different methods to analyze the questions of the existence and multiplicity of solutions and related qualitative properties [2, 11, 13, 19] and references therein. In particular, the existence of solutions for $p-$Kirchhoff problem with a critical nonlinearity has been obtained in [19].

This paper is organized as follows. In Section 2, we give some preliminaries on Nehari manifold and fibering maps. Section 3 is devoted to prove Theorem 1.

2. Some preliminary results

In this section, we introduce some preliminary results. Looking for a solution of problem (1.1) is equivalent to finding a critical point of the associated Euler-Lagrange functional $J : X_0 \to \mathbb{R}$, which is defined by

$$J(u) = \frac{1}{\theta+p}\|u\|_{X_0}^{\theta+p} - \frac{1}{q+1}\int_\Omega f(x)|u|^{q+1}dx - \frac{1}{r+1}\int_\Omega g(x)|u|^{r+1}dx, \tag{2.1}$$

for all $u \in X_0$. Since $f(x)$ and $g(x)$ satisfy (F) and (G) respectively, $J(u)$ is well defined on X_0. Moreover,

$$J'(u)\phi = \|u\|_{X_0}^{\theta}\langle u,\phi\rangle_{X_0} - \int_\Omega f(x)|u|^{q-1}u\phi dx - \int_\Omega g(x)|u|^{r-1}u\phi dx$$

for any $\phi \in X_0$, where $\langle\cdot,\cdot\rangle_{X_0}$ is defined in (1.6).

DEFINITION 1. We say that $u \in X_0$ is a weak solution of problem (1.1), if u satisfies

$$\|u\|_{X_0}^{\theta}\langle u,\phi\rangle_{X_0} = \int_\Omega f(x)|u|^{q-1}u\phi dx + \int_\Omega g(x)|u|^{r-1}u\phi dx \tag{2.2}$$

for all $\phi \in X_0$.

In the sequel we will omit the term *weak* when referring to solutions that satisfy the conditions of Definition 1. In fact, every weak solution of (1.1) is in $L^\infty(\Omega)$ by the result of [22, Theorem 3.1].

We will consider critical points of the functional J on X_0. We define the Nehari manifold as follows

$$\mathcal{N} = \left\{u \in X_0\backslash\{0\} \,:\, J'(u)u = 0\right\}.$$

It it clear that all nontrivial critical points of J lie on $\mathcal{N}$ and $\mathcal{N} \subset X_0$, so we study the functional J on $\mathcal{N}$.

It is easy to see that $u \in \mathcal{N}$ if and only if

$$\|u\|_{X_0}^{\theta+p} - \int_\Omega f(x)|u|^{q+1}dx - \int_\Omega g(x)|u|^{r+1}dx = 0. \tag{2.3}$$

The Nehari manifold $\mathcal{N}$ is closely linked to the behavior of the function of the form $\varphi_u : t \mapsto J(tu)$ for $t > 0$ defined by

$$\varphi_u(t) := J(tu) = \frac{t^{\theta+p}}{\theta+p}\|u\|_{X_0}^{\theta+p} - \frac{t^{q+1}}{q+1}\int_\Omega f(x)|u|^{q+1}dx - \frac{t^{r+1}}{r+1}\int_\Omega g(x)|u|^{r+1}dx.$$

The following lemma tells us that the elements in $\mathcal{N}$ correspond to the stationary points of the maps φ_u.

LEMMA 1. *Let $u \in X_0\backslash\{0\}$, then $tu \in \mathcal{N}$ if and only if $\varphi_u'(t) = 0$.*

Proof. It is a consequence of the fact that $\varphi_u'(t) = J'(tu)u = \frac{1}{t}J'(tu)tu$. □

We observe that

$$\varphi_u'(t) = t^{\theta+p-1}\|u\|_{X_0}^{\theta+p} - t^q\int_\Omega f(x)|u|^{q+1}dx - t^r\int_\Omega g(x)|u|^{r+1}dx, \tag{2.4}$$

and

$$\varphi_u''(t) = (\theta+p-1)t^{\theta+p-2}\|u\|_{X_0}^{\theta+p} - qt^{q-1}\int_\Omega f(x)|u|^{q+1}dx - rt^{r-1}\int_\Omega g(x)|u|^{r+1}dx. \tag{2.5}$$

By Lemma 1, $u \in \mathcal{N}$ if and only if $\varphi_u'(1) = 0$. Hence for $u \in \mathcal{N}$, we have

$$\begin{aligned}
\varphi_u''(1) &= (\theta+p-1)\|u\|_{X_0}^{\theta+p} - q\int_\Omega f(x)|u|^{q+1}dx - r\int_\Omega g(x)|u|^{r+1}dx \\
&= ((\theta+p-1)-r)\int_\Omega g(x)|u|^{r+1}dx + ((\theta+p-1)-q)\int_\Omega f(x)|u|^{q+1}dx \\
&= ((\theta+p-1)-q)\,\|u\|_{X_0}^{\theta+p} - (r-q)\int_\Omega g(x)|u|^{r+1}dx \\
&= ((\theta+p-1)-r)\,\|u\|_{X_0}^{\theta+p} + (r-q)\int_\Omega f(x)|u|^{q+1}dx.
\end{aligned} \tag{2.6}$$

Thus, it is natural to split $\mathcal{N}$ into three parts corresponding to local minima, local maxima and points of inflection, i. e.

$$\mathcal{N}^{+}=\{u\in\mathcal{N}:\varphi_u''(1)>0\};\ \mathcal{N}^{-}=\{u\in\mathcal{N}:\varphi_u''(1)<0\};\ \mathcal{N}^{0}=\{u\in\mathcal{N}:\varphi_u''(1)=0\}.$$

We will prove the existence of solutions of problem (1.1) by investigating the existence of minimizers of functional J on $\mathcal{N}$. We have the following result.

LEMMA 2. *Suppose that u_0 is a local minimizer of J on $\mathcal{N}$ and $u_0\notin\mathcal{N}^0$, then u_0 is a critical point of J.*

Proof. The proof is the same as that in Brown-Zhang [6, Theorem 2.3]. We give it here for completeness. Set $I(u)=J'(u)u$. Since u_0 is a local minimizer of J under the constraint $I(u_0)=0$, by the theory of Lagrange multipliers, there exists $\sigma\in\mathbb{R}$ such that

$$J'(u_0)=\sigma I'(u_0).$$

Thus

$$J'(u_0)u_0=\sigma I'(u_0)u_0=\sigma\varphi_{u_0}''(1).$$

Since $u_0\notin\mathcal{N}^0$, $\varphi_{u_0}''(1)\neq 0$. Hence $\sigma=0$. This ends the proof. □

LEMMA 3. (i) *If $u\in\mathcal{N}^-$, then $\int_\Omega g(x)|u|^{r+1}dx>0$;*
(ii) *If $u\in\mathcal{N}^+$, then $\int_\Omega f(x)|u|^{q+1}dx>0$.*

Proof. This proof is immediate from (2.6). □

In order to understand the Nehari manifold and fibering maps, let us consider the function $\psi_u:\mathbb{R}^+\to\mathbb{R}$ defined by

$$\psi_u(t)=t^{(\theta+p-1)-q}\|u\|_{X_0}^{\theta+p}-t^{r-q}\int_\Omega g(x)|u|^{r+1}dx. \tag{2.7}$$

It is clear that for $t>0$, $tu\in\mathcal{N}$ if and only if

$$\psi_u(t)=\int_\Omega f(x)|u|^{q+1}dx. \tag{2.8}$$

Moreover,

$$\psi_u'(t)=((\theta+p-1)-q)t^{(\theta+p-1)-q-1}\|u\|_{X_0}^{\theta+p}-(r-q)t^{r-q-1}\int_\Omega g(x)|u|^{r+1}dx. \tag{2.9}$$

So we can see that if $tu\in\mathcal{N}$, then

$$t^q\psi_u'(t)=\varphi_u''(t). \tag{2.10}$$

Consequently, $tu\in\mathcal{N}^+$ (or $\mathcal{N}^-$) if and only if $\psi_u'(t)>0$ (or <0).

By direct calculations, we obtain the following results.

LEMMA 4. *Suppose $u \in X_0 \backslash \{0\}$, then ψ_u satisfies the following properties:*

(a) *ψ_u has a unique critical point at $t = t_{max}(u) = \left(\frac{((\theta+p-1)-q)\|u\|_{X_0}^{(\theta+p)}}{(r-q)\int_\Omega g(x)|u|^{r+1}dx} \right)^{\frac{1}{r-(\theta+p-1)}} > 0$;*

(b) *ψ_u is strictly increasing on $(0, t_{max}(u))$ and strictly decreasing on $(t_{max}(u), +\infty)$;*

(c) $\lim_{t\to+\infty} \psi_u(t) = -\infty$.

Moreover, we have

$$\psi_u(t_{max}(u)) = \left(\left(\frac{(\theta+p-1)-q}{r-q} \right)^{\frac{(\theta+p-1)-q}{r-(\theta+p-1)}} - \left(\frac{(\theta+p-1)-q}{r-q} \right)^{\frac{r-q}{r-(\theta+p-1)}} \right) \times \frac{\|u\|_{X_0}^{\frac{(\theta+p)(r-q)}{r-(\theta+p-1)}}}{\left(\int_\Omega g(x)|u|^{r+1}dx\right)^{\frac{(\theta+p-1)-q}{r-(\theta+p-1)}}} \tag{2.11}$$

$$= \left(\frac{(\theta+p-1)-q}{r-q} \right)^{\frac{(\theta+p-1)-q}{r-(\theta+p-1)}} \left(\frac{r-(\theta+p-1)}{r-q} \right) \frac{\|u\|_{X_0}^{\frac{(\theta+p)(r-q)}{r-(\theta+p-1)}}}{\left(\int_\Omega g(x)|u|^{r+1}dx\right)^{\frac{(\theta+p-1)-q}{r-(\theta+p-1)}}}. \tag{2.12}$$

LEMMA 5. *For each $u \in \mathcal{N}^-$, we have:*

(i) *If $\int_\Omega f(x)|u|^{q+1}dx \leqslant 0$, then there exists a unique number $t_0 = t_0(u) > t_{max}$ such that $\psi_u(t_0) = 0$, $t_0 u \in \mathcal{N}^-$ and*

$$J(t_0 u) = \sup_{t\geqslant 0} J(tu);$$

(ii) *If $\int_\Omega f(x)|u|^{q+1}dx > 0$, assume that (1.9) holds, then there exists unique numbers $t_1 = t_1(u) < t_{max} < t_2 = t_2(u)$ such that $t_1 u \in \mathcal{N}^+$, $t_2 u \in \mathcal{N}^-$, and*

$$J(t_1 u) = \inf_{0\leqslant t\leqslant t_{max}} J(tu), \qquad J(t_2 u) = \sup_{t\geqslant t_{max}} J(tu).$$

Proof. (i) From the properties (a), (b) and (c) in Lemma 4, we have that there exists a unique $t_0 > t_{max}$ such that

$$\psi_u(t_0) = \int_\Omega f(x)|u|^{q+1}dx. \tag{2.13}$$

and $\psi_u'(t_0) < 0$. By (2.10), we have $\varphi_u''(t_0) = t_0^q \psi_u'(t_0) < 0$, so $t_0 u \in \mathcal{N}^-$.

Moreover, from the definition of $\psi_u(t)$, we see that

$$\frac{d}{dt}J(tu) = t^q \left(\psi_u(t) - \int_\Omega f(x)|u|^{q+1}dx \right).$$

Then we obtain

$$\frac{d}{dt}J(tu) > 0 \text{ for } t \in (0,t_0); \qquad \frac{d}{dt}J(tu) = 0 \text{ for } t = t_0; \qquad \frac{d}{dt}J(tu) < 0 \text{ for } t > t_0.$$

Thus $J(t_0u) = \sup\limits_{t\geqslant 0} J(tu)$.

(ii) By Hölder inequality and Sobolev embedding, we have

$$\int_\Omega g(x)|u|^{r+1}dx \leqslant \|g(x)\|_{L^{r_1}(\Omega)}\|u\|^{r+1}_{L^{p_s^*}(\Omega)} \leqslant \|g(x)\|_{L^{r_1}(\Omega)}S^{-(r+1)}\|u\|^{r+1}_{X_0}.$$

This together with (2.11) implies that

$$\begin{aligned}
&\psi_u(t_{max}(u)) \\
&\geqslant \left(\frac{(\theta+p-1)-q}{r-q}\right)^{\frac{(\theta+p-1)-q}{r-(\theta+p-1)}}\left(\frac{r-(\theta+p-1)}{r-q}\right)\|u\|_{X_0}^{\frac{(\theta+p)(r-q)}{r-(\theta+p-1)}} \\
&\quad\times\left(\|g(x)\|_{L^{r_1}(\Omega)}S^{-(r+1)}\|u\|^{r+1}_{X_0}\right)^{-\frac{(\theta+p-1)-q}{r-(\theta+p-1)}} \\
&= \left(\frac{(\theta+p-1)-q}{r-q}\right)^{\frac{(\theta+p-1)-q}{r-(\theta+p-1)}}\left(\frac{r-(\theta+p-1)}{r-q}\right)\|g(x)\|_{L^{r_1}(\Omega)}^{-\frac{(\theta+p-1)-q}{r-(\theta+p-1)}}S^{\frac{(r+1)((\theta+p-1)-q)}{r-(\theta+p-1)}}\|u\|^{q+1}_{X_0}.
\end{aligned} \tag{2.14}$$

It follows from (1.8) and (1.9) that

$$\begin{aligned}
\|f(x)\|_{L^{q_1}(\Omega)} \leqslant &\left(\frac{q+1}{\theta+p}\right)\left(\frac{r-(\theta+p-1)}{r-q}\right)\left(\frac{(\theta+p-1)-q}{r-q}\right)^{\frac{(\theta+p-1)-q}{r-(\theta+p-1)}} \\
&\times\|g(x)\|_{L^{r_1}(\Omega)}^{-\frac{(\theta+p-1)-q}{r-(\theta+p-1)}}S^{\frac{(r+1)((\theta+p-1)-q)}{r-(\theta+p-1)}+(q+1)}.
\end{aligned}$$

Using the above inequality, Hölder inequality and Sobolev embedding, $q+1<\theta+p$ and (2.14), we have

$$\begin{aligned}
\psi_u(0) = 0 < \int_\Omega f(x)|u|^{q+1}dx &\leqslant \|f(x)\|_{L^{q_1}(\Omega)}S^{-(q+1)}\|u\|^{q+1}_{X_0} \\
&< \left(\frac{(\theta+p-1)-q}{r-q}\right)^{\frac{(\theta+p-1)-q}{r-(\theta+p-1)}}\left(\frac{r-(\theta+p-1)}{r-q}\right)\|g(x)\|_{L^{r_1}(\Omega)}^{-\frac{(\theta+p-1)-q}{r-(\theta+p-1)}} \\
&\quad\times S^{\frac{(r+1)((\theta+p-1)-q)}{r-(\theta+p-1)}}\|u\|^{q+1}_{X_0} \\
&\leqslant \psi_u(t_{max}(u)).
\end{aligned}$$

Thus there exist t_1 and t_2 with $t_1 < t_{max}(u) < t_2$, such that

$$\psi_u(t_1) = \psi_u(t_2) = \int_\Omega f(x)|u|^{q+1}dx, \quad \text{and} \quad \psi'_u(t_1) > 0, \quad \psi'_u(t_2) < 0,$$

this and (2.7) yield that $\varphi'_u(t_1) = \varphi'_u(t_2) = 0$. By (2.10), we have that $\varphi''_u(t_1) > 0$, $\varphi''_u(t_2) < 0$. These facts imply that the fibering map φ_u has a local minimum at t_1 and a local maximum at t_2 such that $t_1u \in \mathcal{N}^+$ and $t_2u \in \mathcal{N}^-$. As a consequence, we see that

$$J(t_1u) \leqslant J(tu) \leqslant J(t_2u) \ \text{ for all } \ t\in[t_1,t_2], \quad \text{and} \quad J(t_1u) \leqslant J(tu) \ \text{ for all } \ t\in[0,t_{max}].$$

Thus we get

$$J(t_1u) = \inf_{0\leqslant t\leqslant t_{max}} J(tu), \qquad J(t_2u) = \sup_{t\geqslant t_{max}} J(tu).$$

This completes the proof. □

3. Proof of the main result

We start with some useful lemmas.

LEMMA 6. *Assume that f and g satisfy* (1.9)*, then* $\mathcal{N}^0 = \emptyset$.

Proof. By contradiction, suppose that $u \in \mathcal{N}^0$, we have $J'(u)u = 0$ and $\varphi_u''(1) = 0$. Therefore,

$$\begin{aligned}\|u\|_{X_0}^{\theta+p} &= \frac{r-q}{r-(\theta+p-1)}\int_\Omega f(x)|u|^{q+1}dx \\ &\leqslant \frac{r-q}{r-(\theta+p-1)}\|f(x)\|_{L^{q_1}(\Omega)}\|u\|_{L^{p_s^*}(\Omega)}^{q+1} \\ &\leqslant \frac{r-q}{r-(\theta+p-1)}\|f(x)\|_{L^{q_1}(\Omega)}S^{-(q+1)}\|u\|_{X_0}^{q+1}.\end{aligned}$$

This gives that

$$\|u\|_{X_0} \leqslant \left(\frac{r-q}{r-(\theta+p-1)}\right)^{\frac{1}{(\theta+p-1)-q}} \|f(x)\|_{L^{q_1}(\Omega)}^{\frac{1}{(\theta+p-1)-q}} S^{-\frac{q+1}{(\theta+p-1)-q}}. \tag{3.1}$$

Moreover, we have

$$\begin{aligned}\|u\|_{X_0}^{\theta+p} &= \frac{r-q}{(\theta+p-1)-q}\int_\Omega g(x)|u|^{r+1}dx \\ &\leqslant \frac{r-q}{(\theta+p-1)-q}\|g(x)\|_{L^{r_1}(\Omega)}\|u\|_{L^{p_s^*}(\Omega)}^{r+1} \\ &\leqslant \frac{r-q}{(\theta+p-1)-q}\|g(x)\|_{L^{r_1}(\Omega)}S^{-(r+1)}\|u\|_{X_0}^{r+1}.\end{aligned}$$

So

$$\|u\|_{X_0} \geqslant \left(\frac{r-q}{(\theta+p-1)-q}\right)^{-\frac{1}{r-(\theta+p-1)}} \|g(x)\|_{L^{r_1}(\Omega)}^{-\frac{1}{r-(\theta+p-1)}} S^{\frac{r+1}{r-(\theta+p-1)}}. \tag{3.2}$$

From (3.1) and (3.2), we find

$$\begin{aligned}&\left(\frac{(\theta+p-1)-q}{r-q}\right)^{\frac{1}{r-(\theta+p-1)}} \left(\frac{r-(\theta+p-1)}{r-q}\right)^{\frac{1}{(\theta+p-1)-q}} S^{\frac{r+1}{r-(\theta+p-1)}+\frac{q+1}{(\theta+p-1)-q}} \\ &\leqslant \|f(x)\|_{L^{q_1}(\Omega)}^{\frac{1}{(\theta+p-1)-q}} \|g(x)\|_{L^{r_1}(\Omega)}^{\frac{1}{r-(\theta+p-1)}}.\end{aligned} \tag{3.3}$$

Since $\frac{q+1}{\theta+p} < 1$, assumption (1.9) leads to a contradiction. □

LEMMA 7. *J is coercive and bounded from below on $\mathcal{N}$.*

Proof. If $u \in \mathcal{N}$, then we have

$$J(u) = \left(\frac{1}{\theta+p} - \frac{1}{r+1}\right) \|u\|_{X_0}^{\theta+p} - \left(\frac{1}{q+1} - \frac{1}{r+1}\right) \int_\Omega f(x)|u|^{q+1} dx.$$

By Hölder inequality and Sobolev embedding, we have

$$\int_\Omega f(x)|u|^{q+1} dx \leqslant \|f(x)\|_{L^{\frac{r+1}{r-q}}(\Omega)} \|u\|_{L^{r+1}(\Omega)}^{q+1} \leqslant \|f(x)\|_{L^{\frac{r+1}{r-q}}(\Omega)} S^{-(q+1)} \|u\|_{X_0}^{q+1}.$$

Therefore,

$$J(u) \geqslant \frac{r-(\theta+p-1)}{(\theta+p)(r+1)} \|u\|_{X_0}^{\theta+p} - \frac{r-q}{(q+1)(r+1)} \|f(x)\|_{L^{\frac{r+1}{r-q}}(\Omega)} S^{-(q+1)} \|u\|_{X_0}^{q+1}.$$

Since $0 < q < \theta+p-1 < r < p_s^* - 1$, we obtain J is coercive and bounded from below on $\mathcal{N}$. $\square$

By Lemmas 6 and 7, for f and g satisfying (F), (G) respectively, and (1.9), we know that $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ and J is coercive and bounded from below on $\mathcal{N}^+$ and $\mathcal{N}^-$. Therefore we may define

$$\alpha = \inf_{u \in \mathcal{N}} J(u), \qquad \alpha^+ = \inf_{u \in \mathcal{N}^+} J(u), \qquad \alpha^- = \inf_{u \in \mathcal{N}^-} J(u).$$

LEMMA 8. *Assume that f and g satisfy* (1.9)*, we have*

(i) $\alpha \leqslant \alpha^+ < 0$;

(ii) $\alpha^- \geqslant d_0 > 0$ *with some constant d_0, which depends on $p, q, r, \theta, S, \|f(x)\|_{L^{q_1}(\Omega)}$ and $\|g(x)\|_{L^{r_1}(\Omega)}$.*

Proof. (i) Since $\mathcal{N}^+ \subset \mathcal{N}$, we then have that $\alpha \leqslant \alpha^+$. For $u \in \mathcal{N}^+$, we have $\varphi_u''(1) > 0$, that is,

$$\frac{r-(\theta+p-1)}{r-q} \|u\|_{X_0}^{\theta+p} < \int_\Omega f(x)|u|^{q+1} dx.$$

Then

$$\begin{aligned} J(u) &= \left(\frac{1}{\theta+p} - \frac{1}{r+1}\right) \|u\|_{X_0}^{\theta+p} - \left(\frac{1}{q+1} - \frac{1}{r+1}\right) \int_\Omega f(x)|u|^{q+1} dx \\ &< \left(\frac{1}{\theta+p} - \frac{1}{r+1}\right) \|u\|_{X_0}^{\theta+p} - \left(\frac{1}{q+1} - \frac{1}{r+1}\right) \frac{r-(\theta+p-1)}{r-q} \|u\|_{X_0}^{\theta+p} \\ &= -\frac{(r-(\theta+p-1))((\theta+p-1)-q)}{(\theta+p)(q+1)(r+1)} \|u\|_{X_0}^{\theta+p} < 0. \end{aligned}$$

Therefore $\alpha \leqslant \alpha^+ < 0$.

(ii) Let $u \in \mathcal{N}^-$, we have $\varphi_u''(1) < 0$, then

$$\|u\|_{X_0}^{\theta+p} < \frac{r-q}{(\theta+p-1)-q}\int_\Omega g(x)|u|^{r+1}dx \leqslant \frac{r-q}{(\theta+p-1)-q}\|g(x)\|_{L^{r_1}(\Omega)}S^{-(r+1)}\|u\|_{X_0}^{r+1}.$$

Thus we get

$$\|u\|_{X_0} > \left(\frac{(\theta+p-1)-q}{r-q}\right)^{\frac{1}{r-(\theta+p-1)}}\|g(x)\|_{L^{r_1}(\Omega)}^{-\frac{1}{r-(\theta+p-1)}}S^{\frac{r+1}{r-(\theta+p-1)}},$$

and

$$\begin{aligned}
J(u) =& \left(\frac{1}{\theta+p}-\frac{1}{r+1}\right)\|u\|_{X_0}^{\theta+p} - \left(\frac{1}{q+1}-\frac{1}{r+1}\right)\int_\Omega f(x)|u|^{q+1}dx \\
\geqslant& \left(\frac{1}{\theta+p}-\frac{1}{r+1}\right)\|u\|_{X_0}^{\theta+p} - \left(\frac{1}{q+1}-\frac{1}{r+1}\right)\|f(x)\|_{L^{q_1}(\Omega)}S^{-(q+1)}\|u\|_{X_0}^{q+1} \\
=& \frac{1}{r+1}\|u\|_{X_0}^{\theta+p}\left(\frac{r-(\theta+p-1)}{\theta+p} - \frac{r-q}{q+1}\|f(x)\|_{L^{q_1}(\Omega)}S^{-(q+1)}\|u\|_{X_0}^{-((\theta+p-1)-q)}\right) \\
>& \frac{1}{r+1}\|u\|_{X_0}^{\theta+p}\left(\frac{r-(\theta+p-1)}{\theta+p}\right. \\
&-\frac{r-q}{q+1}\|f(x)\|_{L^{q_1}(\Omega)}S^{-(q+1)}\left(\frac{(\theta+p-1)-q}{r-q}\right)^{\frac{-((\theta+p-1)-q)}{r-(\theta+p-1)}}\|g(x)\|_{L^{r_1}(\Omega)}^{\frac{(\theta+p-1)-q}{r-(\theta+p-1)}} \\
&\left.\times S^{-\frac{(r+1)((\theta+p-1)-q)}{r-(\theta+p-1)}}\right) \\
=& \frac{\|u\|_{X_0}^{\theta+p}}{r+1}\frac{r-q}{q+1}\left(\frac{(\theta+p-1)-q}{r-q}\right)^{\frac{-((\theta+p-1)-q)}{r-(\theta+p-1)}}S^{-(q+1)}S^{-\frac{(r+1)((\theta+p-1)-q)}{r-(\theta+p-1)}} \\
&\times\left(\frac{q+1}{\theta+p}\frac{r-(\theta+p-1)}{r-q}\left(\frac{(\theta+p-1)-q}{r-q}\right)^{\frac{((\theta+p-1)-q)}{r-(\theta+p-1)}}S^{(q+1)}S^{\frac{(r+1)((\theta+p-1)-q)}{r-(\theta+p-1)}}\right. \\
&\left.-\|f(x)\|_{L^{q_1}(\Omega)}\|g(x)\|_{L^{r_1}(\Omega)}^{\frac{(\theta+p-1)-q}{r-(\theta+p-1)}}\right) \\
\geqslant& d_0 > 0
\end{aligned}$$

for f and g satisfying (1.9), where $d_0 > 0$ depends on $p, q, r, \theta, S, \|f(x)\|_{L^{q_1}(\Omega)}$ and $\|g(x)\|_{L^{r_1}(\Omega)}$. □

We have the following result.

PROPOSITION 1. If f and g satisfy (1.9), then the functional J has a minimizer u_1 in $\mathcal{N}^+$ and satisfies

(1) $J(u_1) = \inf_{u\in\mathcal{N}^+} J(u) < 0$;

(2) u_1 is a solution of problem (1.1).

Proof. Since J is bounded from below on $\mathcal{N}^+$, there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}^+$ such that

$$\lim_{k\to\infty} J(u_k) = \inf_{u\in\mathcal{N}^+} J(u).$$

By Lemma 7, the sequence $\{u_k\}$ is bounded in X_0. Since $(X_0, \|\cdot\|_{X_0})$ is a reflexive Banach space, then there exists $u_1 \in X_0$, such that, up to a subsequence denote by itself, $u_k \to u_1$ weakly in X_0 as $k \to \infty$. Moreover, $u_k \to u_1$ strongly in $L^m(\Omega)$ as $k \to \infty$ for any $m \in [1, p_s^*)$.
First, we claim that $\int_\Omega f(x)|u_1|^{q+1}dx > 0$. If not, we can conclude that

$$\int_\Omega f(x)|u_k|^{q+1}dx \to \int_\Omega f(x)|u_1|^{q+1}dx \leqslant 0, \quad \text{as} \quad n \to \infty.$$

We know

$$J(u_k) = \left(\frac{1}{\theta+p} - \frac{1}{r+1}\right)\|u_k\|^{\theta+p} - \left(\frac{1}{q+1} - \frac{1}{r+1}\right)\int_\Omega f(x)|u_k|^{q+1}dx,$$

this contradicts $J(u_k) < 0$ as $n \to \infty$.

Since $\int_\Omega f(x)|u_1|^{q+1}dx > 0$, then by Lemma 5 (ii), there exist unique numbers $t_1 = t_1(u_1) < t_{max} < t_2 = t_2(u_1)$ such that $t_1u_1 \in \mathcal{N}^+$, $t_2u_1 \in \mathcal{N}^-$.

Next we prove that $u_k \to u_1$ strongly in X_0, as $k \to \infty$. If not, then $\|u_1\|_{X_0} < \liminf_{k\to\infty} \|u_k\|$. For $u_k \in \mathcal{N}^+$, we get

$$\begin{aligned}\lim_{k\to\infty} \varphi'_{u_k}(t_1) &= \lim_{k\to\infty}\left(t_1^{\theta+p-1}\|u_k\|_{X_0}^{\theta+p} - t_1^q\int_\Omega f(x)|u_k|^{q+1}dx - t_1^r\int_\Omega g(x)|u_k|^{r+1}dx\right)\\ &> t_1^{\theta+p-1}\|u_1\|_{X_0}^{\theta+p} - t_1^q\int_\Omega f(x)|u_1|^{q+1}dx - t_1^r\int_\Omega g(x)|u_1|^{r+1}dx = \varphi'_{u_1}(t_1) = 0.\end{aligned}$$

That is, $\varphi'_{u_k}(t_1) > 0$ for k large enough. Since $u_k = 1\cdot u_k \in \mathcal{N}^+$, and we can see that $\varphi'_{u_k}(t) < 0$ for $t \in (0,1)$ and $\varphi'_{u_k}(1) = 0$ for all k. Then we must have $t_1 > 1$. On the other hand, $\varphi_{u_1}(t)$ is decreasing on $(0, t_1)$, and so

$$J(t_1u_1) \leqslant J(u_1) < \lim_{k\to\infty} J(u_k) = \inf_{u\in\mathcal{N}^+} J(u),$$

which is a contradiction. Hence $u_k \to u_1$ strongly in X_0. This implies

$$J(u_k) \to J(u_1) = \inf_{u\in\mathcal{N}^+} J(u) < 0 \quad \text{as } k \to \infty.$$

Namely, u_1 is a minimizer of J on $\mathcal{N}^+$. From Lemma 2, u_1 is a solution of (1.1). □

We next establish the existence of a local minimum for J on $\mathcal{N}^-$.

PROPOSITION 2. If f and g satisfy (1.9), then the functional J has a minimizer u_2 in $\mathcal{N}^-$ and satisfies

(1) $J(u_2) = \inf_{u\in\mathcal{N}^-} J(u) > 0$;

(2) u_2 is a solution of problem (1.1).

Proof. Since J is bounded from below on $\mathcal{N}^-$, there exists a minimizing sequence $\{\tilde{u}_k\} \subset \mathcal{N}^-$ such that

$$\lim_{k\to\infty} J(\tilde{u}_k) = \inf_{u\in\mathcal{N}^-} J(u).$$

By the same arguments given in the proof of Proposition 1, there exists $u_2 \in \mathcal{N}^-$ such that, up to a subsequence, $\tilde{u}_k \to u_2$ strongly in X_0 as $k \to \infty$. Then we can get $J(u_2) = \inf_{u\in\mathcal{N}^-} J(u) > 0$. This yields u_2 is a solution of (1.1). □

Proof of Theorem 1. By Propositions 1 and 2 and Lemma 2, we get that problem (1.1) has two solutions $u_1 \in \mathcal{N}^+$ and $u_2 \in \mathcal{N}^-$ in X_0. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, then these two solutions are distinct. We finish the proof. □

REFERENCES

[1] G. Autuori, A. Fiscella, P. Pucci, *Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity*, Nonlinear Anal. **125**, (2015), 699–714.

[2] G. Autuori, F. Colasuonno, P. Pucci, *On the existence of stationary solutions for higher-order p-Kirchhoff problems*, Commun. Contemp. Math. **16**, (2014), no. 5, 1450002, 43 pp.

[3] C. Brändle, E. Colorado, A. de Pablo, U. Sánchez, *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A, **143**, (01) (2013), 39–71.

[4] C. Bai, *Multipicity results for a fractional Kirchhoff equation involving sign-chaging weight function*, Bound. value probl. **2016**, (1)2016, 1–16.

[5] C. M. Chu, J. J. Sun, Z. P. Cai, *Multiple solutions for a Kirchhoff-type problem involving nonlocal fractional p-Laplacian and concave-convex nonlinearities*, Rocky Mountain J.Math. **47**,(2017), no. 6, 1803–1823.

[6] K. Brown, Y. Zhang, *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*, J. Diff. Equations, **193**, (2003), 481–499.

[7] W. Chen, S. Deng, *Existence of solution for a Kirchhoff type problem involving the fractional p−Laplace operator*, Electron. J. Qual. Theory Differ. Equ., (2015), No. 87, 1–8.

[8] W. Chen, S. Deng, *The Nehari manifold for a fractional p-Laplacian system involving concave-convex nonlinearities*, Nonlinear Anal. Real World Appl. **27**,(2016), 80–92.

[9] W. Chen, Y. Gui, *Multiple solutions for a fractional p-Kirchhoff problem with Hardy nonlinearity*, Nonlinear Analysis, **188**, (2019), 316–338.

[10] W. Chen, M. Squassina, *Critical Nonlocal Systems with Concave-Convex Powers*, Advance Nonlinear Studies, **16**, (4)(2016), 821–842.

[11] F. Colasuonno, P. Pucci, *Multiplicity of solutions for $p(x)$−polyharmonic elliptic Kirchhoff equations*, Nonlinear Anal. **74**, (17)(2011), 5962–5974.

[12] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. math., **136**, (2012), 521–573.

[13] M. Dreher, *The Kirchhoff equation for the p-Laplacian*, Rend. Semin. Mat. (Torino), **64**, (2006), 217–238.

[14] A. Fiscella, P. K. Mishra, *The Nehari manifold for fractional Kirchhoff problems involving singular and critical terms*, Nonlineae Anal. **186**, (2019), 6–32.

[15] A. Fiscella, E. Valdinoci, *A critical Kirchhoff type problem involving a nonlocal operator*, Nonlinear Anal. **94**, (2014), 156–170.

[16] M. Ferrara, B. Zhang, M. Xiang, *Multiplicity results for the nonhomogeneous fractional p-Kirchhoff equations with concave-convex nonlinearities*, Proc.R.Soc. A 471: 20150034.

[17] S. Goyal, K. Sreenadh, *Nehari manifold for non-local elliptic operator with concave-convex nonlinearities and sign-changing weight functions*, Proceedings Mathematical Sciences, **125**, (4)(2015), 545–558.

[18] S. GOYAL, K. SREENADH, *Existence of multiple solutions of p-fractional Laplace operator with sign-changing weight function*, Adv. Nonlinear Anal. **4**, (1)(2015), 37–58.

[19] A. OURRAOUI, *On a p-Kirchhoff problem involving a critical nonlinearity*, C. R. Acad. Sci. Paris, Ser. I, **352**, (2014), 295–298.

[20] R. LEHRER, L. A. MAIA, M. SQUASSINA, *On fractional p-Laplacian problems with weight*, Differential & Integral Equations, **28**, (2015), 15–28.

[21] J. LIAO, Y. PU, X. KE, C. TANG, *Multiple positive solutions for Kirchhoff type problems involving concave-convex nonlinearities*, Commun. Pure Appl. Anal. **16**, (2017), no. 6, 2157–2175.

[22] A. IANNIZZOTTO, S. LIU, K. PERERA, M. SQUASSINA, *Existence results for fractional p-Laplacian problems via Morse theory*, Adv. Calc. Var. **9**, (2)(2016), 101–125.

[23] G. SUN, K. TENG, *Existence and multiplicity of solutions for a class of fractional Kirchhoff-type problem*, Math. Commun., **19**, (2014), 183–194.

[24] T. F. WU, *Multiplicity results for a semilinear elliptic equation involving sign-chaging weight function*, Rocky Mountain J. Math. **39**, (2009), no.3, 995–1011.

[25] Y. WEI, C. CHEN, H. SONG, *Multiplicity of solutions for a class of fractional p-Kirchhoff system with sgin-changing weight functions*, Bound. Value Problem. **2018**, (1)2018, 1–18.

[26] J. ZOU, T. AN, L. REN, *The Nehari manifold for a fractional p-Kirchhoff system involving sign-changing weight function and concave-convex nonlinrarities*, J. Founct. Spaces 2019, Art. Id 7624373,9.

(Received June 20, 2019)

Yuyan Gui
School of Mathematics and Statistics
Southwest University
Chongqing 400715, People's Republic of China
e-mail: yuyangui@126.com